Palestine Journal of Mathematics

Vol 13(3)(2024) , 223–232

FRACTIONAL CALCULUS PERTAINING TO MULTIVARIABLE A-FUNCTION

D. Kumar and F.Y. Ayant

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 33C05; Secondary 33C60.

Keywords and phrases: Multivariable A-function, A-function, Fractional integral, general class of multivariable polynomials, Mellin transform, multivariable H-function, Srivastava-Daoust polynomial.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

The author Dinesh Kumar would like to thank Agriculture University Jodhpur for supporting and encouraging this work.

Abstract In this paper we study a pair of unified and extended fractional integral operator involving the multivariable A-function, A-function and general class of multivariable polynomials. During this study, we establish five theorems pertaining to Mellin transforms of these operators. Furthers, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of (known and new) fractional integral operators involved simpler functions can be obtained. We will quote the particular cases concerning multivariable H-function and the Srivastava-Daoust polynomial.

1 Introduction and Preliminaries

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. Recently, it has turned out many phenomena in physics, mechanics, chemistry, biology and other sciences; and can be described very successfully by models using mathematical tools from fractional calculus. Chaurasia and Srivastava [2], Choi et al. [3], Kumar and Daiya [11], Kumar et al. [12], and others have studied the fractional calculus pertaining to multivariable H-function defined by Srivastava and Panda [22].
The multivariable A-function defined by Gautam et al. [6] is an extension of the multivariable H-function [22]. It is defined and represented in the following manner:

$$A\left(z_1,\ldots,z_r\right)=$$

$$A^{m,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left(\begin{array}{c|c} z_1 & \left(a_j;A_j^{(1)},\ldots,A_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r} \\ \cdot & \\ \cdot & \\ z_r & \left(b_j;B_j^{(1)},\cdots,B_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{array}\right) \tag{1.1}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\left(s_1,\ldots,s_r\right)\prod_{i=1}^{r}\theta_i\left(s_i\right)\,z_i^{s_i}\,\mathrm{d}s_1\ldots\mathrm{d}s_r, \tag{1.2}$$

where $\phi\left(s_1,\ldots,s_r\right)$ and $\theta_i\left(s_i\right)$, $i=1,\ldots,r$ are given by

$$\phi\left(s_1,\ldots,s_r\right)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\sum_{i=1}^{r}B_j^{(i)}s_i\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}A_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}A_j^{(i)}s_j\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}B_j^{(i)}s_j\right)}, \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+C_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-D_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-C_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+D_j^{(i)}s_i\right)}. \tag{1.4}$$

Here $m, n, p, m_i, n_i, p_i, c_i \in \mathbb{N}^*$ $(i = 1, \ldots, r)$; $a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$. The multiple integral defining A-function of r variables converges absolutely if

$$|\arg(\Omega_i) z_k| < \frac{1}{2}\eta_k\pi, \ \xi^* = 0, \ \eta_i > 0, \tag{1.5}$$

$$\Omega_i = \prod_{j=1}^{p}\left\{A_j^{(i)}\right\}^{A_j^{(i)}}\prod_{j=1}^{q}\left\{B_j^{(i)}\right\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\left\{D_j^{(i)}\right\}^{D_j^{(i)}}\prod_{j=1}^{p_i}\left\{C_j^{(i)}\right\}^{-C_j^{(i)}} \quad (i = 1, \ldots, r), \tag{1.6}$$

$$\xi_i^* = Im\left(\sum_{j=1}^{p}A_j^{(i)} - \sum_{j=1}^{q}B_j^{(i)} + \sum_{j=1}^{q_i}D_j^{(i)} - \sum_{j=1}^{p_i}C_j^{(i)}\right) \quad (i = 1, \ldots, r), \tag{1.7}$$

$$\eta_i = Re\left(\sum_{j=1}^{n}A_j^{(i)} - \sum_{j=n+1}^{p}A_j^{(i)} + \sum_{j=1}^{m}B_j^{(i)} - \sum_{j=m+1}^{q}B_j^{(i)} + \sum_{j=1}^{m_i}D_j^{(i)} \right.$$
$$\left. - \sum_{j=m_i+1}^{q_i}D_j^{(i)} + \sum_{j=1}^{n_i}C_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\right) \text{ with } i = 1, \ldots, r. \tag{1.8}$$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable A-function. We shall note

$$A = \left(a_j; A_j^{(1)}, \ldots, A_j^{(r)}\right)_{1,p}; \quad B = \left(b_j; B_j^{(1)}, \ldots, B_j^{(r)}\right)_{1,q}. \tag{1.9}$$

$$C = \left(c_j^{(1)}, C_j^{(1)}\right)_{1,p_1}; \ldots; \left(c_j^{(s)}, C_j^{(r)}\right)_{1,p_r}; \quad D = \left(d_j^{(1)}, D_j^{(1)}\right)_{1,q_1}; \cdots; \left(d_j^{(r)}, D_j^{(r)}\right)_{1,q_r}. \tag{1.10}$$

The A- function [6], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral, as given by

$$A(z) = A_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_j, \alpha_i)_{n,n+1}, (a_j, \alpha_i)_p \\ (b_j, \beta_j)_{m,m+1}, (b_j, \beta_j)_q \end{matrix} \right.\right) = \frac{1}{2\pi\omega}\int_L \Omega_{p,q}^{m,n}(s)\, z^{-s}\mathrm{d}s, \tag{1.11}$$

for all z different to 0 and

$$\Omega_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)}. \tag{1.12}$$

Srivastava and Garg [21] introduced and defined a general class of multi-variable polynomials as follows

$$S_L^{h_1,\cdots,h_s}[z_1, \ldots, z_s] = \sum_{R_1,\cdots,R_s=0}^{h_1R_1+\ldots h_sR_s \leqslant L}(-L)_{h_1R_1+\cdots+h_sR_s}\, B(L; R_1, \ldots, R_s)\, \frac{z_1^{R_1}\ldots z_s^{R_s}}{R_1!\ldots R_s!}. \tag{1.13}$$

The coefficients $B(L; R_1, \ldots, R_s)$ are arbitrary constants, real or complex.

2 Definitions

The pair of new extended fractional integral operators are defined by the following equations

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right] = t\,x^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta}\,A\left(\begin{array}{c|c}\gamma_1\upsilon_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n\upsilon_n & B:D\end{array}\right)$$

$$\times\prod_{j=1}^{k}A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\right]$$

$$\times\prod_{j=1}^{r}S_L^{l_1^{(j)},\ldots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}}\end{array}\right)\psi\left(\frac{y^t}{x^t}\right)f(y)\,\mathrm{d}y, \quad (2.1)$$

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = t\,x^{\rho}\int_x^{\infty} y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta}\,A\left(\begin{array}{c|c}\gamma_1\mu_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n\mu_n & B:D\end{array}\right)$$

$$\times\prod_{j=1}^{k}A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\right]$$

$$\times\prod_{j=1}^{r}S_L^{l_1^{(j)},\ldots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right)\psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y, \quad (2.2)$$

where $\upsilon_i = \left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu_i = \left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$; and $t, u_i, v_i, g_i^{(j)}, h_i^{(j)}, a_j, b_j$ are positive numbers.

The kernels $\psi\left(\frac{y^t}{x^t}\right)$ and $\psi\left(\frac{x^t}{y^t}\right)$ appearing in (2.1) and (2.2) respectively, are assumed to be continuous functions such the integrals make sense for wide classes of function $f(x)$.
The conditions for existence of these operators are as follows.

(a). $f(x)\in L_p(0,\infty)$, (b). $1\leqslant p,q<\infty,\ p^{-1}+q^{-1}=1$.

(c). $Re\left(\alpha+ta_j\frac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^{n}u_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)>-q^{-1}$.

(d). $Re\left(\beta+tb_j\frac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^{n}v_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)>-q^{-1}$.

(e). $Re\left(\rho+ta_j\frac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^{n}u_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right)>-p^{-1}$,

where $j=1,\ldots,k;\ j'=1,\ldots,M_j'$.
Condition (a) ensures that both operators defined in (2.1) and (2.2) exist and belong to. These operators are extensions of fractional integral operators defined and studied by several authors like Erdélyi [5], Love [13], Saigo et al. [15], Saxena and Kiryakova [16], Baleanu et al. [1], Kumar and Daiya [11], Kumar et al. [12], Ram and Kumar [14], Saxena and Kumbhat [18, 19], etc.

3 Main results

Theorem 3.1. *If $f(x) \in L_p(0,\infty)$, $1 \leqslant p \leqslant 2$; or $f(x) \in L_p(0,\infty)$, $p > 2$, also following conditions satisfied:*

$$p^{-1}+q^{-1} = 1,$$

$$Re\left(\alpha + ta_j \frac{b_{j'j}}{\beta_{j'j}}\right) + t\sum_{i=1}^{n} u_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -q^{-1},$$

$$Re\left(\beta + tb_j \frac{b_{j'j}}{\beta_{j'j}}\right) + t\sum_{i=1}^{n} v_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -q^{-1},$$

and the integrals are absolutely convergent, then

$$M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = M\left\{f(x)\right\}\ R_{\gamma_n}^{\alpha-s+1,\beta}[1], \tag{3.1}$$

where $M_p(0,\infty)$ stands for the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p > 2$, which are inverse Mellin-transforms of the function of $L_p(-\infty,\infty)$.

Proof. By making Mellin transform of (2.1), we get

$$M\left\{Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\right\} = \int_0^{\infty} x^{s-1} \left\{ tx^{-\alpha-t\beta-1} \int_0^x y^{\alpha}\left(x^t - y^t\right)^{\beta} A\left(\begin{array}{c|c} \gamma_1 \upsilon_1 & A:C \\ \cdot & \cdot \\ \cdot & \cdot \\ \gamma_n \upsilon_n & B:D \end{array}\right)\right.$$
$$\times \prod_{j=1}^{k} A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\right]$$
$$\left.\times \prod_{j=1}^{r} S_L^{l_1^{(j)},\ldots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}} \\ \cdot \\ \cdot \\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}} \end{array}\right) \psi\left(\frac{x^t}{y^t}\right) f(y)\,\mathrm{d}y \right\} \mathrm{d}x. \tag{3.2}$$

On interchanging the order of integration, which is permissible under the conditions, result (3.1) follows in view of (2.2). □

Theorem 3.2. *If $f(x) \in L_p(0,\infty)$, $1 \leqslant p \leqslant 2$; or $f(x) \in L_p(0,\infty)$, $p > 2$, also following conditions satisfied:*

$$p^{-1}+q^{-1} = 1,$$

$$Re\left(\beta + tb_j \frac{b_{j'j}}{\beta_{j'j}}\right) + t\sum_{i=1}^{n} v_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -q^{-1},$$

$$Re\left(\rho + ta_j \frac{b_{j'j}}{\beta_{j'j}}\right) + t\sum_{i=1}^{n} u_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -p^{-1},$$

and the integrals are absolutely convergent, then we have

$$M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\} = M\left\{f(x)\right\} Q_{\gamma_n}^{\rho+s-1,\beta}[1], \tag{3.3}$$

where $M_p(0,\infty)$ stands for the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p > 2$, which are inverse Mellin-transforms of the function $L_p(-\infty,\infty)$.

Proof. By making Mellin transform of (2.2), we get

$$M\left\{R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]\right\}=\int_0^\infty x^{s-1}\left\{tx^\rho\int_x^\infty y^{-\rho-t\beta-1}\left(y^t-x^t\right)^\beta A\left(\begin{array}{c|c}\gamma_1\mu_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n\mu_n & B:D\end{array}\right)\right.$$
$$\times\prod_{j=1}^k A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\right]$$
$$\left.\times\prod_{j=1}^r S_L^{l_1^{(j)},\dots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right)\psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y\right\}\mathrm{d}x. \tag{3.4}$$

□

Theorem 3.3. *If* $f(x)\in L_p(0,\infty)\,,\ v(x)\in L_p(0,\infty)$, *also satisfy*

$p^{-1}+q^{-1}=1,$

$Re\left(\alpha+ta_j\dfrac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n u_i\min\limits_{1\leqslant j\leqslant m_i}Re\left(\dfrac{d_j^{(i)}}{D_j^{(i)}}\right)>\max\left\{p^{-1},q^{-1}\right\},$

$Re\left(t\beta+tb_j\dfrac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n v_i\min\limits_{1\leqslant j\leqslant m_i}Re\left(\dfrac{d_j^{(i)}}{D_j^{(i)}}\right)>0,$

and the integrals are absolutely convergent, then have

$$\int_0^\infty v(x)Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]\mathrm{d}x=\int_0^\infty f(x)R_{\gamma_n}^{\alpha,\beta}\left[v(x)\right]\mathrm{d}x. \tag{3.5}$$

Proof. The result of (3.5) can be obtained in view of (2.1) and (2.2). □

4 Inversion formulas

Theorem 4.1. *If* $f(x)\in L_p(0,\infty)\,,\ 1\leqslant p\leqslant 2$; *or* $f(x)\in L_p(0,\infty)\,,\ p>2$, *also satisfied following conditions:*

$p^{-1}+q^{-1}=1,$

$Re\left(\alpha+ta_j\dfrac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n u_i\min\limits_{1\leqslant j\leqslant m_i}Re\left(\dfrac{d_j^{(i)}}{D_j^{(i)}}\right)>-q^{-1},$

$Re\left(\beta+tb_j\dfrac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n v_i\min\limits_{1\leqslant j\leqslant m_i}Re\left(\dfrac{d_j^{(i)}}{D_j^{(i)}}\right)>-q^{-1},$

and the integrals are absolutely convergent, also

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=v(x), \tag{4.1}$$

then we have

$$f(x)=\int_0^\infty y^{-1}\left[v(y)\right]\left[h\left(\frac{x}{y}\right)\right]\mathrm{d}y, \tag{4.2}$$

where

$$h(x)=\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}y^{-1}\frac{x^{-s}}{R(s)}\,\mathrm{d}s, \tag{4.3}$$

$$R(s)=R_{\gamma_n}^{\alpha-s+1,\beta}[1]. \tag{4.4}$$

Proof. On taking Mellin transform of (4.1) and then applying Theorem 3.1, we get

$$M\left\{f(x)\right\} = \frac{M\left\{v(x)\right\}}{R(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M\left\{v(x)\right\}}{R(s)}\,\mathrm{d}s = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{R(s)}\left\{\int_0^\infty [v(y)]\,\mathrm{d}y\right\}\mathrm{d}s.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{v(y)}{y}\left\{\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^s\frac{1}{R(s)}\mathrm{d}s\right\}\mathrm{d}y.$$

Now in view of (4.3), we obtain the desired result (4.2). □

Theorem 4.2. *If* $f(x) \in L_p(0,\infty)$, $1 \leqslant p \leqslant 2$*; or* $f(x) \in L_p(0,\infty)$, $p > 2$*, also following conditions satisfied:*

$p^{-1}+q^{-1} = 1,$

$$Re\left(\beta + tb_j\frac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n v_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -q^{-1},$$

$$Re\left(\rho + ta_j\frac{b_{j'j}}{\beta_{j'j}}\right)+t\sum_{i=1}^n u_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{D_j^{(i)}}\right) > -p^{-1},$$

and the integrals are absolutely convergent, also

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right] = w(x), \tag{4.5}$$

then, we have

$$f(x) = \int_0^\infty y^{-1}\left[w(y)\right]\left[g\left(\frac{x}{y}\right)\right]\mathrm{d}y, \tag{4.6}$$

where

$$g(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y^{-1}\frac{x^{-s}}{T(s)}\,\mathrm{d}s, \tag{4.7}$$

$$T(s) = Q_{\gamma_n}^{\rho+s-1,\beta}[1]. \tag{4.8}$$

Proof. On taking Mellin transform of (4.1) and then applying Theorem 3.2, we get

$$M\left\{f(x)\right\} = \frac{M\left\{w(x)\right\}}{T(s)},$$

which on inverting leads to

$$f(x) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{M\left\{w(x)\right\}}{T(s)}\,\mathrm{d}s = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{T(s)}\left\{\int_0^\infty [w(y)]\,\mathrm{d}y\right\}\mathrm{d}s.$$

Interchanging the order of integration, we obtain

$$f(x) = \int_0^\infty \frac{w(y)}{y}\left\{\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^s\frac{1}{T(s)}\,\mathrm{d}s\right\}\mathrm{d}y.$$

Now in view of (4.7), we obtain the desired result (4.6). □

5 General properties

The properties given below are consequences of the definitions (2.1) and (2.2).

$$x^{-1}Q_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=R_{\gamma_n}^{\alpha,\beta}\left[f(x)\right], \tag{5.1}$$

$$x^{-1}R_{\gamma_n}^{\rho,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=Q_{\gamma_n}^{\rho,\beta}\left[f(x)\right], \tag{5.2}$$

$$x^{\mu}Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=Q_{\gamma_n}^{\alpha-\mu,\beta}\left[x^{\mu}f(x)\right], \tag{5.3}$$

$$x^{\mu}R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]=R_{\gamma_n}^{\rho+\mu,\beta}\left[x^{\mu}f(x)\right], \tag{5.4}$$

The properties given below express the homogeneity of operator Q and R respectively.

$$\text{If } Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=v(x) \text{ then } Q_{\gamma_n}^{\alpha,\beta}\left[f(cx)\right]=v(cx). \tag{5.5}$$

$$\text{If } R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]=w(x) \text{ then } R_{\gamma_n}^{\rho,\beta}\left[f(cx)\right]=w(cx). \tag{5.6}$$

6 Multivariable H-function

If $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}$ and $m=0$, the multivariable A-function reduces to multivariable H-function defined by Srivastava and Panda [22], then we obtain two following operators:

$$Q_{\gamma_n}^{\alpha,\beta}\left[f(x)\right]=tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}\left(x^t-y^t\right)^{\beta} H\left(\begin{array}{c|c}\gamma_1 v_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n v_n & B:D\end{array}\right)$$
$$\times\prod_{j=1}^{k}A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\right]$$
$$\times\prod_{j=1}^{r}S_L^{l_1^{(j)},\ldots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}}\end{array}\right)\psi\left(\frac{y^t}{x^t}\right)f(y)\,\mathrm{d}y, \tag{6.1}$$

under the same notations and conditions that (2.1) with $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}$ and $m=0$.

$$R_{\gamma_n}^{\rho,\beta}\left[f(x)\right]=tx^{\rho}\int_x^{\infty} y^{-\rho-t\beta-1}\left(y^t-x^t\right)^{\beta} H\left(\begin{array}{c|c}\gamma_1 \mu_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n \mu_n & B:D\end{array}\right)$$
$$\times\prod_{j=1}^{k}A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\right]$$
$$\times\prod_{j=1}^{r}S_L^{l_1^{(j)},\ldots,l_s^{(j)}}\left(\begin{array}{c} z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right)\psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y, \tag{6.2}$$

under the same notations and conditions that (2.2) with $A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R}$ and $m=0$.
We can obtain similar theorems and properties concerning these operators as given in section 3.

7 Srivastava-Daoust polynomial

If

$$B^{(j)}(L;R_1,\dots,R_s)=\frac{\prod_{i=1}^{\bar{A^{(j)}}}\left(a_i^{(j)}\right)_{R_1^{(j)}\theta_i^{(1j)}+\dots+R_s^{(j)}\theta_i^{(sj)}}\prod_{i=1}^{B^{(1j)}}\left(b_i^{(1j)}\right)_{R_1^{(j)}\phi_i^{(1j)}}\cdots\prod_{i=1}^{B^{(sj)}}\left(b_i^{(sj)}\right)_{R_s^{(j)}\phi_i^{(sj)}}}{\prod_{i=1}^{\bar{C^{(j)}}}\left(c_i^{(j)}\right)_{m_1^{(j)}\psi_i^{(1j)}+\dots+m_s^{(j)}\psi_i^{(sj)}}\prod_{i=1}^{D^{(1j)}}\left(d_i^{(1j)}\right)_{R_1^{(j)}\delta_i^{(1j)}}\cdots\prod_{i=1}^{D^{(sj)}}\left(d_i^{(sj)}\right)_{R_s^{(j)}\delta_i^{(sj)}}},\tag{7.1}$$

we obtain two following operators concerning the Srivastava-Daoust polynomial [20]:

$$\begin{aligned}Q_{\gamma_n}^{\alpha,\beta}[f(x)]&=tx^{-\alpha-t\beta-1}\int_0^x y^\alpha\left(x^t-y^t\right)^\beta A\left(\begin{array}{c|c}\gamma_1 v_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n v_n & B:D\end{array}\right)\\&\quad\times\prod_{j=1}^k A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a_j}\left(1-\frac{y^t}{x^t}\right)^{b_j}\right]\\&\quad\times\prod_{j=1}^r F_{\bar{C^{(j)}}:D^{(1j)};\dots;D^{(uj)}}^{1+\bar{A^{(j)}}:B^{(1j)};\dots;B^{(uj)}}\left(\begin{array}{c|}z_1^{(j)}\left(\frac{y^t}{x^t}\right)^{g_1^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{y^t}{x^t}\right)^{g_s^{(j)}}\left(1-\frac{y^t}{x^t}\right)^{h_s^{(j)}}\end{array}\right.\\&\quad\left.\begin{array}{c}\left[\left(-L^{(j)}\right);R_1^{(j)},\dots,R_s^{(j)}\right]\left[(a);\theta^{(1j)},\dots,\theta^{(sj)}\right]:\left[\left(b^{(1j)}\right);\phi^{(1j)}\right];\dots;\left[\left(b^{(sj)}\right);\phi^{(sj)}\right]\\ \left[(c);\psi^{(1j)},\dots,\psi^{(sj)}\right]:\left[\left(d^{(1j)}\right);\delta^{(1j)}\right];\dots;\left[\left(d^{(sj)}\right);\delta^{(sj)}\right]\end{array}\right)\\&\quad\times\psi\left(\frac{y^t}{x^t}\right)f(y)\,\mathrm{d}y,\end{aligned}\tag{7.2}$$

under the same notations and conditions that (2.1).

$$\begin{aligned}R_{\gamma_n}^{\rho,\beta}[f(x)]&=tx^{\rho}\int_x^\infty y^{-\rho-t\beta-1}\left(y^t-x^t\right)^\beta A\left(\begin{array}{c|c}\gamma_1 \mu_1 & A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \gamma_n \mu_n & B:D\end{array}\right)\\&\quad\times\prod_{j=1}^k A_{M_j'',N_j''}^{M_j',N_j'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a_j}\left(1-\frac{x^t}{y^t}\right)^{b_j}\right]\\&\quad\times\prod_{j=1}^r F_{\bar{C^{(j)}}:D^{(1j)};\dots;D^{(uj)}}^{1+\bar{A^{(j)}}:B^{(1j)};\dots;B^{(uj)}}\left(\begin{array}{c|}z_1^{(j)}\left(\frac{x^t}{y^t}\right)^{g_1^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_1^{(j)}}\\ \cdot\\ \cdot\\ z_s^{(j)}\left(\frac{x^t}{y^t}\right)^{g_s^{(j)}}\left(1-\frac{x^t}{y^t}\right)^{h_s^{(j)}}\end{array}\right.\\&\quad\left.\begin{array}{c}\left[\left(-L^{(j)}\right);R_1^{(j)},\dots,R_s^{(j)}\right]\left[(a);\theta^{(1j)},\dots,\theta^{(sj)}\right]:\left[\left(b^{(1j)}\right);\phi^{(1j)}\right];\dots;\left[\left(b^{(sj)}\right);\phi^{(sj)}\right]\\ \left[(c);\psi^{(1j)},\dots,\psi^{(sj)}\right]:\left[\left(d^{(1j)}\right);\delta^{(1j)}\right];\dots;\left[\left(d^{(sj)}\right);\delta^{(sj)}\right]\end{array}\right)\\&\quad\times\psi\left(\frac{x^t}{y^t}\right)f(y)\,\mathrm{d}y,\end{aligned}\tag{7.3}$$

under the same notations and conditions that (2.2).
We can obtain similar theorems and properties concerning these operators as given in section 3.

8 Conclusion

We studied a pair of unified and extended fractional integral operator involving the multivariable A-function, A-function and general class of multivariable polynomials. The functions involved in the given results are unified and general nature, hence a large number of known results follows as special cases of our main findings. Further, on suitable specifications of the involved parameters, many new results involving simpler functions may also be derived.

9 Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] D. Baleanu, D. Kumar and S. D. Purohit, *Generalized fractional integrals of product of two H-functions and a general class of polynomials*, Int. J. Comput. Math., **93**(8), 1320–1329, (2016).

[2] V.B.L. Chaurasia and A. Srivastava, *A unified approach to fractional calculus pertaining to H-functions*, Soochow J. Math., **33**(2), 211–221, (2007).

[3] J. Choi, J. Daiya, D. Kumar and R. K. Saxena, *Fractional differentiation of the product of Appell function F_3 and multivariable H-function*, Commun. Korean Math. Soc., **31**(1), 115–129, (2016).

[4] J. Daiya, J. Ram and D. Kumar, *The multivariable H-function and the general class of Srivastava polynomials involving the generalized Mellin-Barnes contour integrals*, FILOMAT, **30**(6), 1457–1464, (2016).

[5] A. Erdélyi, *On some functional transformations*, Univ. Politec. Torino, Rend. Sem. Mat., **10**, 217-234, (1950-51).

[6] B.P. Gautam, A.S. Asgar and A.N. Goyal, *On the multivariable A-function*, Vijnana Parishad Anusandhan Patrika, **29**(4), 67–81, (1986).

[7] D. Kumar and F.Y. Ayant, *Fractional calculus operators pertaining to multivariable Aleph-function*, Bol. Soc. Parana. Mat., **40**, 1–10, (2022).

[8] D. Kumar, F.Y. Ayant and J. Choi, *Application of product of the multivariable A-function and the multivariable Srivastava polynomials*, East Asian Math. J., **34**(3), 295–303, (2018).

[9] D. Kumar, F.Y. Ayant, J. Daiya and K.S. Nisar, *Finite summation formulas for the multivariable A-function*, Bol. Soc. Parana. Mat., **41**, xxx–xxx, (2023).

[10] D.Kumar, F.Y. Ayant and Naresh, *Integral involving the product of multivariable Aleph-function, general class of Srivastava polynomials and Aleph-function of one variable*, Kragujev. J. Math., **49**(6), 863–871, (2025).

[11] D. Kumar and J. Daiya, *Fractional calculus pertaining to generalized H-functions*, Glob. J. Sci. Front. Res. (F), **14**(3), 25–35, (2014).

[12] D. Kumar, S. D. Purohit and J. Choi, *Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials*, J. Nonlinear Sci. Appl., **9**, 8–21, (2016).

[13] E.R. Love, *Some integral equations involving hypergeometric functions*, Proc. Edinb. Math. Soc., **15** (3), 169–198, (1967).

[14] J. Ram and D. Kumar, *Generalized fractional integration involving Appell hypergeometric function of the product of two H-functions*, Vijanana Parishad Anusandhan Patrika, **54**(3), 33–43, (2011).

[15] M. Saigo, R.K. Saxena and J. Ram, *On the fractional calculus operator associated with the H-function*, Ganita Sandesh, **6**(1), 36–47, (1992).

[16] R.K. Saxena and V.S. Kiryakova, *On relation between the two-dimensional H-transforms in terms of Erdélyi-Kober operators*, Math. Balkanica, **6**, 133–140, (1992).

[17] R.K. Saxena and D. Kumar, *Generalized fractional calculus of the Aleph-function involving a general class of polynomials*, Acta Math. Sci. Ser. B Engl. Ed., **35**(5), 1095–1110, (2015).

[18] R.K. Saxena and R.K. Kumbhat, *Integral operators involving H-function*, Indian J. Pure Appl. Math., **5**, 1–6, (1974).

[19] R.K. Saxena and R.K. Kumbhat, *Some properties of generalized Kober operators*, Vijnana Parishad Anusandhan Patrika, **18**, 139–150, (1975).

[20] H.M. Srivastava and M.C. Daoust, *Certain generalized NEUMANN expansions associated with the Kampé de Fériet function*, Nederl. Akad. Wetensch. Proc. Ser. A, **72** = Indag. Math., **31**, 449-–457, (1969).

[21] H.M.Srivastava and M. Garg, *Some integral involving a general class of polynomials and multivariable H-function*, Rev. Roumaine Phys., **32**, 685–692, (1987).

[22] H.M. Srivastava and R. Panda, *Some bilateral generating functions for a class of generalized hypergeometric polynomials*, J. Reine Angew. Math., **283/284**, 265–274, (1976).

Author information

D. Kumar, Department of Applied Sciences, College of Agriculture–Jodhpur,
Agriculture University Jodhpur, Jodhpur–342304, India.
E-mail: dinesh_dino03@yahoo.com

F.Y. Ayant, [1]Collége Jean L'herminier, Allée des Nymphéas, 83500 La Seyne-sur-Mer;
[2]Six-Fours-les-Plages 83140, Department of Var, France, France.
E-mail: fredericayant@gmail.com

Received: 2023-03-06
Accepted: 2024-08-08