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SOME RESULTS ON HYPONORMAL OPERATORS

Messaoud Guesba

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Abstract In this paper we introduce some results on hyponormal operators acting on a complex Hilbert space $\mathcal{H}$. We give sufficient conditions for which the sum and product of two hyponormal operators is a hyponormal operator. Also, we study the sum direct and tensor product of this class.

1 Introduction and Preliminaries

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes to the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. The symbol I stands for the identity operator on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ then, T^* is its adjoint and $T = A + iB$ is its Cartesian decomposition. Let $\ker(T)$ and $\Re(T)$ denote the kernel and range of $T \in \mathcal{B}(\mathcal{H})$, respectively. For any operator T in $\mathcal{B}(\mathcal{H})$ set as usual,

$$|T| = (T^*T)^{\frac{1}{2}}.$$

Many classes of operators are defined according to the relation betwen T and T^*, for example T is normal if $TT^* = T^*T$; self-adjoint or hermitian if $T^* = T$; skew-adjiont if $T^* = -T$; positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and seky-normal if $T^2 = {T^*}^2$; quasinormal if $TT^*T = T^*T^2$; projection if $T^2 = T = T^*$; idempotent if $T^2 = T$. For an operator $T \in \mathcal{B}(\mathcal{H})$, if $\|Tx\| = \|x\|$ for all $x \in \mathcal{H}$ (or equivalently $T^*T = I$), then T is called an isometry; T is called unitary if $TT^* = T^*T = I$. An operator $T \in \mathcal{B}(\mathcal{H})$ is a partial isometry if $\|Tx\| = \|x\|$ for all $x \in (\ker(T))^{\perp}$. For more facts about these and other classes of operators, we refer the reader to [2, 3, 4, 6, 7, 8, 9, 10, 14, 15] and the references therein.

Recall that An operator $T \in \mathcal{B}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker(U) = \ker(|T|) = \ker(T)$ and $\ker(U^*) = \ker(T^*)$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called hyponormal if $TT^* \leq T^*T$, which is equivalent to the condition $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$. The notation of hyponormality was first introduced in [11]. Further, the class of hyponormal operators has been studied by many authors. In recent years this class has been generalized, in some sense, to the larger sets of so called m-hyponormal, p-hyponormal etc see [1, 13, 17, 18].

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m-hyponormal, if there exists a positive number m, such that

$$m^2 (A - \lambda I)^* (A - \lambda I) \geq (A - \lambda I)(A - \lambda I)^*, \text{ for, all } \lambda \in \mathbb{C}$$

and T is called p-hyponormal for $p > 0$ if

$$(TT^*)^p \leq (T^*T)^p.$$

Recall that any $T \in \mathcal{B}(\mathcal{H})$ is expressible as $T = A + iB$ where $A, B \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators. Besides,

$$A = \operatorname{Re} T = \frac{T + T^*}{2},$$

is the real part of T, and

$$B = \operatorname{Im} T = \frac{T - T^*}{2i},$$

the imaginary part of T.

It is to check that T is normal if and only if $AB = BA$.

It is well known that a self-adjoint operator can be characterized in the following way an operator T in $\mathcal{B}(\mathcal{H})$ is self-adjoint if and only if $\langle Tx, x \rangle$ is real. In [4] the authors gave another characterization involving inequalities. We denote by (WN) the class of operators in $\mathcal{B}(\mathcal{H})$ satisfying the following inequality:

$$(\operatorname{Re} T)^2 \leq |T|^2 .$$

This class has been introduced by Fong and Istratescu [4]. Notice that this class contains the class of hyponormal operators. Indeed, T is hyponormal implies that

$$(\operatorname{Re} T)^2 + (\operatorname{Im} T)^2 \leq |T|^2 .$$

Since $(\operatorname{Im} T)^2$ is a positive operator, then

$$(\operatorname{Re} T)^2 \leq |T|^2 ,$$

that is $T \in (WN)$.

The well-known Fuglede-Putnam theorem asserts that if S and T are normal and $SX = XT$ for some operator $X \in \mathcal{B}(\mathcal{H})$, then $S^*X = XT^*$ (see [5]). There have been many generalizations of the Fuglede-Putnam theorem (see for example [19] and the references therein). Resently, in [16] the author proved that if $S, T \in \mathcal{B}(\mathcal{H})$ are hyponormal operators, such that $T^*S = ST^*$ then, the sum and product of S and T are hyponormal.

Our aim in this paper is to give new classical results on hyponormal operators in a complex Hilbert space. More precisely, we show that well-known results related to normal operators hold true also for those operators.

2 Main Results

In this section, we present our results. We first state the following lemma.

Lemma 2.1. *[12, Löwner-Heinz's inequality] Let $A, B \in \mathcal{B}(\mathcal{H})$. If $0 \leq A \leq B$ and $\delta \in (0, 1]$, then*

$$0 \leq A^\delta \leq B^\delta.$$

Our first result is stated as follows.

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an ivertible operator with polar decomposition $T = U|T|$. Then, T is hyponormal if and only if*

$$|T| \geq U |T| U^*.$$

Proof. Let T be a hyponormal, then $|T|^2 \geq |T^*|^2$ it follows that

$$U^* |T|^2 U \geq |T|^2 \geq U |T|^2 U^*.$$

By Lemma 2.1, we obtain

$$\left(|T|^2\right)^{\frac{1}{2}} \geq \left(U |T|^2 U^*\right)^{\frac{1}{2}},$$

i.e., $|T| \geq U |T| U^*$.

To prove the reverse implication, let $|T| \geq U |T| U^*$ implies that

$$|T|^2 \geq U |T|^2 U^*,$$

then $|T|^2 \geq |T^*|^2$. Hence, T is hyponormal. ∎

Our next result reads as follows.

Proposition 2.3. *Let $T = U|T|$ be an invertible hyponormal operator. Then, $U|T|^{\frac{1}{2}}$ is also hyponormal.*

Proof. By setting $S = U|T|^{\frac{1}{2}}$, we observe that

$$\begin{aligned} SS^* &= U|T|U^* = |T^*| \\ &\leq |T| = |T|^{\frac{1}{2}} U^*U |T|^{\frac{1}{2}} \\ &= S^*S. \end{aligned}$$

Thus, S is hyponormal. ∎

Proposition 2.4. *Let $T = U|T|$ be an invertible hyponormal operator. Then, $U|T|^p$ is also hyponormal, for all $p \in (0, 1]$.*

Proof. By setting $S = U|T|^p$, we observe that

$$\begin{aligned} SS^* &= U|T|^{2p} U^* = |T^*|^{2p} \\ &\leq |T|^{2p} = |T|^p U^*U |T|^p \\ &= S^*S. \end{aligned}$$

Thus, S is hyponormal. ∎

Remark 2.5. The hyponormality of T (i.e., $TT^* \leq T^*T$) implies $|T^*| \leq |T|$. But $|T^*| \leq |T|$ does not necesarily imply the hyponormality of T.

Example 2.6. Consider the 2×2 matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $A \leq B$ and $A^2 \nleq B^2$.

We set

$$T = \begin{pmatrix} 0 & & & & & \\ A & 0 & & & & \\ & B & 0 & & & \\ & & B & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \end{pmatrix}.$$

Then, we compute

$$TT^* = \begin{pmatrix} 0 & & & & & \\ & A^2 & & & & \\ & & B^2 & & & \\ & & & B^2 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix} \text{ and } T^*T = \begin{pmatrix} A^2 & & & & & \\ & B^2 & & & & \\ & & B^2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}.$$

Therefore, $|T^*| \leq |T|$, but T is not hyponormal (because of $A^2 \nleq B^2$).
One of the main results of this paper reads as follows.

Theorem 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator and $T = U|T|$ be the polar decomposition of T such that $U^{n_0} = U^*$ for some positive integer n_0. Then T is normal.*

Proof. Assume that T is hyponormal. Hence

$$|T|^2 \geq |T^*|^2 = U\,|T|^2\,U^*.$$

By multiplying both sides of this inequality by U and U^*, we get $U\,|T|^2\,U^* \geq U\,\left|^2 T\right|^2 U^{*^2}$ whence

$$|T|^2 \geq U\,|T|^2\,U^* \geq U\,\left|^2 T\right|^2 U^{*^2}.$$

By repeating this process, we reach the following sequence of operators inequalities

$$|T|^2 \geq |T^*|^2 = U\,|T|^2\,U^* \geq U\,\left|^2 T\right|^2 U^{*^2} \geq ... \geq U^{n_0+1}\,|T|^2\,U^{(n_0+1)*} \geq ... \tag{2.1}$$

Because of $U^{n_0} = U^*$ we can observe that $U^{n_0+1} = U^*U = U^{(n_0+1)^*}$ is the projection onto $\overline{\Re\left(|T|\right)}$.

Hence $U^{n_0+1}\,|T|^2\,U^{(n_0+1)*} = |T|^2$, from which and inequalities (2.1) we obtain $|T|^2 = |T^*|^2$ this completes the proof. ∎

We are now in a position to state the following theorem.

Theorem 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator and $T = U\,|T|$ be the polar decomposition of T such that $U^{*n} \to I$ or $U^n \to I$ as $n \to \infty$, where limits are taken in the strong operator. Then, T is normal.*

Proof. We assume that $U^{*n}x \to x$ as $n \to \infty$ for all $x \in \mathcal{H}$. Let T be a hyponormal operator, it follows from (2.1) that

$$\||T|\,x\| \geq \||T^*|\,x\| = \||T|\,U^*x\| \geq \left\||T|\,U^{2*}x\right\| \geq ... \geq \left\||T|\,U^{n*}x\right\| \geq .... \tag{2.2}$$

Since

$$\begin{aligned} \left|\left\||T|\,U^{n*}x\right\| - \||T|\,x\|\right| &\leq \left\||T|\,U^{n*}x - |T|\,x\right\| \\ &\leq \||T|\|\,\left\|U^{n*}x - x\right\| \to 0, \end{aligned}$$

as $n \to \infty$, we have $\left\||T|\,U^{n*}x\right\| \to \||T|\,x\|$.

Hence by (2.2), we get

$$\||T|\,x\| = \||T^*|\,x\|\,,$$

so $|T|^2 = |T^*|^2$.

Thus, T is normal. ∎

Proposition 2.9. *Let $T = A + iB \in \mathcal{B}(\mathcal{H})$ be the Cartesian decomposition of the operator T. Let $BA = C + iD$ be the Cartesian decomposition of BA, then T is hyponormal if and only of $D \geq 0$.*

Proof. Since T is hyponormal, we have

$$T^*T - TT^* = 2i\,(AB - BA) \geq 0.$$

Thus if $BA = C + iD$, then the last equation implies that

$$AB = (BA)^* = C - iD,$$

it follows that

$$T^*T - TT^* = 2D \geq 0.$$

This completes the proof. ∎

The following corollary is an immediate consequence of above proposition.

Corollary 2.10. *If $D = 0$, then T is normal.*

In order to prove our next result, we need the following lemma.

Lemma 2.11. *[19, Therorem 2] Let $T \in \mathcal{B}(\mathcal{H})$ be a p-hyponormal operator and $L \in \mathcal{B}(\mathcal{H})$ be a self-adjoint which satisfies $TL = LT^*$. Then*

$$T^*L = LT.$$

The next theorem explain what conditions imply normality for hyponormal operators.

Theorem 2.12. *Let $T = A + iB \in \mathcal{B}(\mathcal{H})$ be the Cartesian decomposition of T, with AB is hyponormal. If A is idempotent, then T is normal.*

Proof. Assume that A is idempotent, we have

$$(AB)\,A = A\,(BA)\,.$$

Let $S = AB$, then $SA = AS^*$. Then, it follows from Lemma 2.11, that $S^*A = AS$.
Thus,

$$BA^2 = A^2B.$$

Since A is idempotent ($A^2 = A$) then $AB = BA$. Thus, T is normal. ∎

Theorem 2.13. *Let $T = A + iB \in \mathcal{B}(\mathcal{H})$ be the Cartesian decomposition of T. If AB is hyponormal, then T is normal.*

Proof. Let $W = AB$ then $WA = AW^*$. Then, by Lemma 2.11, we get $W^*A = AW$ i.e., $BA^2 = A^2B$. Since T is hyponormal, we have

$$T^*T - TT^* = 2i\,(AB - BA) \geq 0.$$

Let $X = 2i\,(AB - BA)$, then $X \geq 0$ and $XA = -AX$.
Now,

$$\begin{aligned} X^2A &= X\,(XA) \\ &= X\,(-AX) \\ &= -\,(XA)\,X \\ &= AX^2. \end{aligned}$$

Since X is positive, then

$$XA = -AX = 0.$$

Hence

$$A\,(AB - BA) = (AB - BA)\,A = 0,$$

implies that

$$\sigma\,(AB - BA) = \{0\}\,.$$

Therefore, $AB - BA$ is quasi-nilpotent and skew-hermitian.
Thus,

$$AB - BA = 0.$$

So, T is normal. ∎

In the end of this paper, we study the sum direct and tensor product of such operators.

Theorem 2.14. *Let T_1, T_2,....., T_m be hyponormal operators in $\mathcal{B}(\mathcal{H})$.*

Then $(T_1 \oplus T_2 \oplus \ldots\ldots \oplus T_m)$ and $(T_1 \otimes T_2 \otimes \ldots\ldots \otimes T_m)$ are hyponormal operators.

Proof. Since

$$\begin{aligned}(T_1 \oplus T_2 \oplus ...... \oplus T_m)(T_1 \oplus T_2 \oplus ...... \oplus T_m)^* &= (T_1 \oplus T_2 \oplus ...... \oplus T_m)(T_1^* \oplus T_2^* \oplus ...... \oplus T_m^*) \\ &= T_1T_1^* \oplus .... \oplus T_mT_m^* \\ &\leq T_1^*T_1 \oplus .... \oplus T_m^*T_m \\ &= (T_1^* \oplus T_2^* \oplus ...... \oplus T_m^*)(T_1 \oplus T_2 \oplus ...... \oplus T_m) \\ &= (T_1 \oplus T_2 \oplus ...... \oplus T_m)^*(T_1 \oplus T_2 \oplus ...... \oplus T_m).\end{aligned}$$

Then, $(T_1 \oplus T_2 \oplus ...... \oplus T_m)$ is a hyponormal operator.

Now, for $x_1, ..., x_m \in \mathcal{H}$

$$\begin{aligned}&(T_1 \otimes T_2 \otimes ...... \otimes T_m)(T_1 \otimes T_2 \otimes ...... \otimes T_m)^*(x_1 \otimes ...... \otimes x_m) \\ =\ &(T_1 \otimes T_2 \otimes ...... \otimes T_m)(T_1^* \otimes T_2^* \otimes .... \otimes T_m^*)(x_1 \otimes ...... \otimes x_m) \\ =\ &T_1T_1^*x_1 \otimes .... \otimes T_mT_m^*x_m \\ \leq\ &T_1^*T_1x_1 \otimes .... \otimes T_m^*T_mx_m \\ =\ &(T_1^* \otimes T_2^* \otimes ...... \otimes T_m^*)(T_1 \otimes T_2 ...... \otimes T_m)(x_1 \otimes ...... \otimes x_m) \\ =\ &(T_1 \otimes T_2 \otimes ...... \otimes T_m)^*(T_1 \otimes T_2 \otimes ...... \otimes T_m)(x_1 \otimes ...... \otimes x_m).\end{aligned}$$

So,

$$(T_1 \otimes T_2 \otimes ...... \otimes T_m)(T_1 \otimes T_2 \otimes ...... \otimes T_m)^* \leq (T_1 \otimes T_2 \otimes ...... \otimes T_m)^*(T_1 \otimes T_2 \otimes ...... \otimes T_m).$$

Thus, $(T_1 \otimes T_2 \otimes ...... \otimes T_m)$ is a hyponormal operator. ∎

3 Conclusion

This paper aims is to obtain some results on hyponormal operators acting on a complex Hilbert space $\mathcal{H}$. We gave sufficient conditions for which the sum and product of two hyponormal operators is a hyponormal operator. Also, the sum direct and tensor product of this class has been studied.

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Author information

Messaoud Guesba, Faculty of Exact Sciences, Department of Mathematics El Oued University, 39000, Algeria.
E-mail: guesbamessaoud2@gmail.com, guesba-messaoud@univ-eloued.dz

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