Palestine Journal of Mathematics

Vol 13(3)(2024) , 240–245

RICCI SOLITONS ON SEQUENTIAL WARPED PRODUCT MANIFOLDS

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Communicated by Harikrishnan Panackal

MSC 2010 Classifications: Primary 53C21; Secondary 53C25.

Keywords and phrases: Ricci solitons, warped product manifold, Killing vector fields, concurrent vector fields.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract *In this work, we study Ricci solitons on a new type of warped product manifolds called sequential warped products. We investigate some hereditary properties and provide some conditions to find Einstein manifolds by considering the potential field as Killing and conformal vector fields. Also, we derive some results when the potential field is a concurrent vector field.*

1 Introduction

Ricci solitons are a natural generalization of Einstein manifolds. They correspond to self-similar solutions of the Ricci flow which is defined by Hamilton [14, 15]. Perelman's solution to the Poincaré Conjecture [19, 20] led to a growing interest in the Ricci flow and the Ricci solitons. There have been many studies involving the Ricci solitons from several different perspectives [2, 3, 5, 6, 10, 12, 18, 21, 22, 23, 25].

There have been studies involving Ricci solitons and their generalizations (e.g., quasi-Einstein manifolds, Ricci-harmonic solitons) in different geometric spaces. These studies attract the attention of mathematicians and physicists as well. Lately, (singly-multiply) warped product manifolds with this kind of structures studied intensively [1, 11, 13, 16, 26].

A Riemannian manifold (M, g) is said to be Ricci soliton if there exists a smooth vector field X so that the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1.1}$$

is satisfied for some constant λ, and denoted by (M, g, X, λ). Here, Ric *and $\mathcal{L}$ denote Ricci tensor and Lie derivative, and the vector field $X \in \mathfrak{X}(M)$ is called potential field. If the potential field is gradient of a smooth function u on M, then $(M, g, \nabla u, \lambda)$ is called a gradient Ricci soliton and the equation (1.1) turns into*

$$\mathrm{Ric} + \mathrm{Hess}u = \lambda g. \tag{1.2}$$

Warped product manifolds, defined by O'Neill and Bishop [4] to construct manifolds with negative curvature, are studied since the concept has an important role in both geometry and physics, where warped product spaces are used in general relativity to model the spacetime. Doubly and multiply warped product manifolds are generalizations of the warped product manifolds[28, 27]. A recent study [8] introduces a new generalization, the sequential warped product manifolds.

In [8], the authors introduced the sequential warped products and calculated curvature tensor, Ricci and scalar curvature. Then they reached some characterizations by using concircular and Killing vector fields. They also show two space-time applications of the sequential warped products. In [17], the authors found some necessary conditions for a sequential warped product to be a quasi-Einstein manifold. They also acquired some conditions for an application of sequential warped products to standard static space-time. In [24], the author introduced sequential warped product submanifolds of Kaehler manifolds and provided some examples. In [9], the authors studied Ricci soliton warped product manifold and found relations to its factors. Motivating from the above studies, in this paper, we inquire about Ricci solitons on this new space, sequential warped product manifolds. We investigate some hereditary properties and provide some conditions to find Einstein manifolds by considering the potential field as Killing, conformal and concurrent vector fields.

2 Preliminaries

First, we give the definition of the sequential warped product manifolds.

Definition 2.1. Let M_i be Riemannian manifolds with metrics g_i for $1 \leq i \leq 3$ and $f : M_1 \longrightarrow \mathbb{R}^+$ and $h : M_1 \times M_2 \longrightarrow \mathbb{R}^+$ be two smooth positive functions. The sequential warped product manifold M is the triple product manifold $M = (M_1 \times_f M_2) \times_h M_3$ equipped with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Here the functions f, h are called warping functions.

From now on, (M, g) will be considered as sequential warped product manifold where $M^n = (M_1^{n_1} \times_f M_2^{n_2}) \times_h M_3^{n_3}$ with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Here, the warped product manifold $(\overline{M} = M_1 \times_f M_2, \overline{g} = g_1 \oplus f^2 g_2)$ is the *base of warped product manifold $M = \overline{M} \times_h M_3$. The restriction of the warping function $h : \overline{M} = M_1 \times M_2 \longrightarrow \mathbb{R}$ to $M_1 \times \{0\}$ is $h^1 = h|_{M_1 \times \{0\}}$.*

We denote ∇, ∇^i; Ric, Ric^i; Hess, Hess^i; Δ, Δ^i; $\mathcal{L}$, $\mathcal{L}^i$ for the Levi-Civita connection, Ricci tensor, Hessian, Laplacian and Lie derivative of the M, and M_i, respectively. Hessian of the $\overline{M}$ is denoted as $\overline{\text{Hess}}$. A smooth vector field X can be expressed as $X = X_1 + X_2 + X_3$ where $X_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$.

The following propositions on sequential warped product manifolds are needed to prove our results.

Proposition 2.2. *[8] Let (M, g) be a sequential warped product and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. Then,*

(i) $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$,

(ii) $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(\ln f) X_2$,

(iii) $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \nabla^1 f$,

(iv) $\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(\ln h) X_3$,

(v) $\nabla_{X_2} X_3 = \nabla_{X_3} X_2 = X_2(\ln h) X_3$,

(vi) $\nabla_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) \nabla h$.

Proposition 2.3. *[8] Let (M, g) be a sequential warped product and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. Then,*

(i) $\text{Ric}(X_1, Y_1) = \text{Ric}^1(X_1, Y_1) - \frac{n_2}{f} \text{Hess}^1 f(X_1, Y_1) - \frac{n_3}{h} \overline{\text{Hess}} h(X_1, Y_1)$,

(ii) $\text{Ric}(X_2, Y_2) = \text{Ric}^2(X_2, Y_2) - f^\sharp g_2 (X_2, Y_2) - \frac{n_3}{h} \overline{\text{Hess}} h(X_2, Y_2)$,

(iii) $\text{Ric}(X_3, Y_3) = \text{Ric}^3(X_3, Y_3) - h^\sharp g_3 (X_3, Y_3)$,

(iv) $\text{Ric}(X_i, X_j) = 0$ *when* $i \neq j$

where $f^\sharp = f \Delta^1 f + (n_2 - 1) \left\|\nabla^1 f\right\|^2$ and $h^\sharp = h \Delta h + (n_3 - 1) \|\nabla h\|^2$.

Proposition 2.4. *[8] Let (M, g) be a sequential warped product manifold. A vector field $X \in \mathfrak{X}(M)$ satisfies the equation*

$$\begin{aligned}\mathcal{L}_X g(Y, Z) &= \left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) + f^2 \left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) + h^2 \left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) \\ &\quad + 2 f X_1(f) g_2(Y_2, Z_2) + 2h(X_1 + X_2)(h) g_3(Y_3, Z_3)\end{aligned}$$

for $Y, Z \in \mathfrak{X}(M)$.

Lastly, a vector field V on a Riemannian manifold (M, g) is conformal if there exists a smooth function on M satisfying the equation

$$\mathcal{L}_V g = 2\rho g.$$

If $\rho = 0$, then V is a Killing vector field.

3 Main Results

In this section, we will study Ricci solitons on sequential warped product manifolds. First, we will examine the inheritance property as follows.

Proposition 3.1. *Let (M, g, X, λ) be a sequential warped product Ricci soliton. Then,*

(i) $(M_1, g_1, X_1 - n_2 \nabla^1 (\ln f) - n_3 \nabla^1 (\ln h^1), \lambda)$ *is a Ricci soliton.*

(ii) If $\overline{Hess} h = \varphi g$, $(M_2, g_2, f^2 X_2, \lambda f^2 + f^\sharp + \frac{n_3}{h} \varphi f^2 - f X_1(f))$ is a Ricci soliton when $\lambda f^2 + f^\sharp + \frac{n_3}{h} \varphi f^2 - f X_1(f)$ is a constant.

(iii) $(M_3, g_3, h^2 X_3, \lambda h^2 + h^\sharp - h(X_1 + X_2)(h))$ is a Ricci soliton when $\lambda h^2 + h^\sharp - h(X_1 + X_2)(h)$ is a constant.

Proof. Assume that (M, g, X, λ) be a sequential warped product Ricci soliton. For $Y, Z \in \mathfrak{X}(M)$,

$$\text{Ric}(Y, Z) + \frac{1}{2} \mathcal{L}_X g(Y, Z) = \lambda g(Y, Z)$$

is satisfied. From Proposition 2.3 and Proposition 2.4 , we may write

$$\begin{aligned}&\text{Ric}^1(Y_1, Z_1) - \frac{n_2}{f} \text{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h} \overline{\text{Hess}} h(Y_1, Z_1) + \text{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) \\ &- \frac{n_3}{h} \overline{\text{Hess}} h(Y_2, Z_2) + \text{Ric}^3(Y_3, Z_3) - h^\sharp g_3(Y_3, Z_3) + \frac{1}{2} \left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) + \frac{1}{2} f^2 \left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) \\ &+ \frac{1}{2} h^2 \left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) + f X_1(f) g_2(Y_2, Z_2) + h(X_1 + X_2)(h) g_3(Y_3, Z_3) \\ &= \lambda g_1(Y_1, Z_1) + \lambda f^2 g_2(Y_2, Z_2) + \lambda h^2 g_3(Y_3, Z_3) \qquad (3.1)\end{aligned}$$

It is noted that

$$\begin{aligned}
\frac{1}{2}\mathcal{L}_X g(Y_1, Z_1) &- \frac{n_2}{f}\text{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h}\overline{\text{Hess}}h(Y_1, Z_1) \\
&= \frac{1}{2}g_1(\nabla^1_{Y_1}X_1, Z_1) - \frac{n_2}{2f}g_1(\nabla^1_{Y_1}\nabla^1 f, Z_1) - \frac{n_3}{2h}g_1(\nabla^1_{Y_1}\nabla^1 h^1, Z_1) \\
&\quad + \frac{1}{2}g_1(Y_1, \nabla^1_{Z_1}X_1) - \frac{n_2}{2f}g_1(Y_1, \nabla^1_{Z_1}\nabla^1 f) - \frac{n_3}{2h}g_1(Y_1, \nabla^1_{Z_1}\nabla^1 h^1) \\
&= \frac{1}{2}g_1\left(\nabla^1_{Y_1}(X_1 - n_2\nabla^1(\ln f) - n_3\nabla^1(\ln h^1)), Z_1\right) \\
&\quad + \frac{1}{2}g_1\left(Y_1, \nabla^1_{Z_1}(X_1 - n_2\nabla^1(\ln f) - n_3\nabla^1(\ln h^1))\right) \\
&= \frac{1}{2}\left(\mathcal{L}^1_{X_1 - n_2\nabla^1(\ln f) - n_3\nabla^1(\ln h^1)}g_1\right)(Y_1, Z_1).
\end{aligned}$$

Thus, equation 3.1 may be rewritten as

$$\begin{aligned}
&\text{Ric}^1(Y_1, Z_1)\frac{1}{2}\left(\mathcal{L}^1_{X_1 - n_2\nabla^1(\ln f) - n_3\nabla^1(\ln h^1)}g_1\right)(Y_1, Z_1) \\
&+\text{Ric}^2(Y_2, Z_2) + \frac{1}{2}f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2, Z_2) + \text{Ric}^3(Y_3, Z_3) + \frac{1}{2}h^2\left(\mathcal{L}^3_{X_3}g_3\right)(Y_3, Z_3) \\
&= \lambda g_1(Y_1, Z_1) + (\lambda f^2 + f^\sharp - fX_1(f) + \frac{n_3}{h}\overline{\text{Hess}}h(Y_2, Z_2))g_2(Y_2, Z_2) \\
&\quad +(\lambda h^2 + h^\sharp - h(X_1 + X_2)(h))g_3(Y_3, Z_3)
\end{aligned}$$

and hence, when the arguments are restricted to the factor manifolds, the proof is completed. □

Remark 3.2. Let (M, g, X, λ) be a sequential warped product Ricci soliton. When the potential field X is Killing vector field on M, then

(i) $\left(M_1, g_1, -n_2\nabla^1(\ln f) - n_3\nabla^1(\ln h^1), \lambda\right)$ is a gradient Ricci soliton.

(ii) If $\overline{Hess}h = \varphi g$, then M_2 is Einstein.

(iii) M_3 is Einstein.

This result coincides with the characterization in [8], Theorem 2.4 since the assumption on X causes (M, g) to be an Einstein manifold.

In the next theorem, we will provide some conditions for a sequential warped product Ricci soliton to be an Einstein manifold and examine the potential function of a gradient Ricci soliton. In [26], the authors proved that potential function of a gradient Ricci soliton on a warped product manifold is defined on the base or the warping function is constant. Inspiring from this point of view, we provide a similar result for sequential warped product gradient Ricci solitons.

Theorem 3.3. *Let (M, g, X, λ) be a sequential warped product Ricci soliton.*

(a) (M, g) is Einstein if one of the following conditions holds.

(i) $X = X_3$ and X_3 is a Killing vector field on M_3.

(ii) X_1 is a Killing vector field on M_1, X_2 and X_3 are conformal vector fields on M_2 and M_3 with factors $-2X_1(\ln f)$ and $-2(X_1 + X_2)(\ln h)$, respectively.

(iii) $X = X_2 + X_3$ so that X_2 and X_3 are Killing on M_2 and M_3, respectively and $X_2(h) = 0$.

(b) When $X = \nabla u$ for a smooth function u on M, then the warping function h is constant or u is defined on $\overline{M} = M_1 \times_f M_2$. Moreover, if u is defined on $\overline{M}$, then the warping function f is constant or u is defined on M_1.

Proof. Proof of (a) is omitted here. For (b), assume that $(M, g, \nabla u, \lambda)$ is a gradient Ricci soliton. From Theorem 1.1 of [26], we know that the warping function h is constant or potential function u is defined on $\overline{M}$. Suppose that the potential function u is defined on $\overline{M}$, then the potential field can be written as $\nabla u = (\nabla u)^T + (\nabla u)^\perp \in \mathfrak{X}(M_1 \times_f M_2)$.

For $Y_1 \in \mathfrak{X}(M_1)$ and $Z_2 \in \mathfrak{X}(M_2)$, the equation (1.2) can be written as

$$\text{Ric}(Y_1, Z_2) + \text{Hess}u(Y_1, Z_2) = \lambda g(Y_1, Z_2).$$

Since $\text{Ric}(Y_1, Z_2) = 0$ and $g(Y_1, Z_2) = 0$, we conclude $\text{Hess}u(Y_1, Z_2) = 0$. Hence,

$$\begin{aligned}
0 = \text{Hess}u(Y_1, Z_2) &= g(\nabla_{Y_1}\nabla u, Z_2) \\
&= g(\nabla_{Y_1}(\nabla u)^T, Z_2) + g(\nabla_{Y_1}(\nabla u)^\perp, Z_2) \\
&= g_1(\nabla^1_{Y_1}(\nabla u)^T, Z_2) + g(\frac{Y_1(f)}{f}(\nabla u)^\perp, Z_2) \\
&= fY_1(f)g_2((\nabla u)^\perp, Z_2)
\end{aligned}$$

which means f is constant or u is defined on M_1 because Y_1, Z_2 are arbitrary. □

4 Sequential Warped Product Ricci Solitons with Concurrent Potential Fields

In [7], concurrent vector fields and Ricci solitons are studied in details. A vector field $X \in \mathfrak{X}(M)$ is called a concurrent vector field if

$$\nabla_V X = V$$

for all $V \in \mathfrak{X}(M)$.

We firstly examine concurrent vector fields on sequential warped product manifolds.

Lemma 4.1. *X is a concurrent vector field on M^n if and only if X_1 is concurrent on $M_1^{n_1}$ and one of the following conditions holds.*

(i) X_2 and X_3 are concurrent on $M_2^{n_2}$ and $M_3^{n_3}$, respectively and the functions f, h are constant.

(ii) X_2 is concurrent on M_2, $X_3 = 0$, f is constant and $(X_1 + X_2)(h) = h$.

(iii) $X_2 = 0$, X_3 is concurrent on M_3, $X_1(f) = f$ and h is constant.

(iv) $X_2 = X_3 = 0$, $X_1(f) = f$ and $X_1(h) = h$.

Proof. Let X be a concurrent vector field on M^n and $\{\partial_i\}_{i=1}^n$ be a local basis for $\mathfrak{X}(M)$. It is enough to check base vectors on M_1 for showing X_1 is concurrent, i.e., for $1 \leq i \leq n_1$,

$$\partial_i = \nabla_{\partial_i} X = \nabla^1_{\partial_i} X_1 + \frac{\partial_i(f)}{f} X_2 + \frac{\partial_i(h)}{h} X_3.$$

Here X_1 must be a concurrent vector field on M_1 and the remaining term is zero.
Case 1: When $\partial_i(f) = \partial_i(h) = 0$ for $1 \leq i \leq n_1$, f and h^1 are constant. For $n_1 + 1 \leq j \leq n_1 + n_2$, i.e., $\partial_j \in \mathfrak{X}(M_2)$,

$$\begin{aligned} \partial_j &= \nabla_{\partial_j} X = \frac{X_1(f)}{f}\partial_j + \nabla^2_{\partial_j} X_2 - f g_2(\partial_j, X_2)\nabla^1 f + \frac{\partial_j(h)}{h} X_3 \\ &= \nabla^2_{\partial_j} X_2 + \frac{\partial_j(h)}{h} X_3 \end{aligned}$$

since f is constant. Hence, h must be constant or $X_3 = 0$. For $n_1 + n_2 + 1 \leq k \leq n$, i.e., $\partial_k \in \mathfrak{X}(M_3)$,

$$\partial_k = \nabla_{\partial_k} X = \frac{X_1(h)}{h}\partial_k + \frac{X_2(h)}{h}\partial_k + \nabla^3_{\partial_k} X_3 - h g_3(\partial_k, X_3)\nabla h$$

so there are two options which vanish the last term: if h is constant, then X_3 is concurrent; if $X_3 = 0$, then $(X_1+X_2)(h) = h$ which proves (i) and (ii).
Case 2: When $\partial_i(f) = 0$ for $1 \leq i \leq n_1$ and $X_3 = 0$, (ii) can be proved by following the same path above.
Case 3: When $X_2 = 0$ and $\partial_i(h) = 0$ for $1 \leq i \leq n_1$, h^1 is constant. For $n_1 + 1 \leq j \leq n_1 + n_2$,

$$\begin{aligned} \partial_j &= \frac{X_1(f)}{f}\partial_j + \nabla^2_{\partial_j} X_2 - f g_2(\partial_j, X_2)\nabla^1 f + \frac{\partial_j(h)}{h} X_3 \\ &= \frac{X_1(f)}{f}\partial_j + \frac{\partial_j(h)}{h} X_3 \end{aligned}$$

since $X_2 = 0$. Therefore $X_1(f) = f$ and $\partial_j(h) = 0$ or $X_3 = 0$.

For $n_1 + n_2 + 1 \leq k \leq n$,

$$\begin{aligned} \partial_k &= \frac{X_1(h)}{h}\partial_k + \frac{X_2(h)}{h}\partial_k + \nabla^3_{\partial_k} X_3 - h g_3(\partial_k, X_3)\nabla h \\ &= \frac{X_1(h)}{h}\partial_k + \nabla^3_{\partial_k} X_3 - h g_3(\partial_k, X_3)\nabla h \end{aligned}$$

since $X_2 = 0$. If h is constant, then X_3 is concurrent. If $X_3 = 0$, then $X_1(h) = h$ which proves (iii) and (iv).
Case 4: When $X_2 = X_3 = 0$, (iv) is again satisfied.

For the converse, assume that (i) holds and X_1 is concurrent. For $1 \leq i \leq n_1$,

$$\nabla_{\partial_i} X = \nabla^1_{\partial_i} X_1 + \frac{\partial_i(f)}{f} X_2 + \frac{\partial_i(h)}{h} X_3 = \nabla^1_{\partial_i} X_1 = \partial_i,$$

for $n_1 + 1 \leq j \leq n_1 + n_2$,

$$\nabla_{\partial_j} X = \frac{X_1(f)}{f}\partial_j + \nabla^2_{\partial_j} X_2 - f g_2(\partial_j, X_2)\nabla^1 f + \frac{\partial_j(h)}{h} X_3 = \nabla^2_{\partial_j} X_2 = \partial_j$$

and for $n_1 + n_2 + 1 \leq k \leq n$,

$$\nabla_{\partial_k} X = \frac{X_1(h)}{h}\partial_k + \frac{X_2(h)}{h}\partial_k + \nabla^3_{\partial_k} X_3 - h g_3(\partial_k, X_3)\nabla h = \nabla^3_{\partial_k} X_3 = \partial_k$$

so X is concurrent. The remaning items could be shown similarly.

□

In the next theorem, we provide a characterization for the sequential warped product Ricci solitons when the potential field is concurrent and also show the relation between the potential field and the warping functions f and h.

Theorem 4.2. *Let (M, g) be a sequential warped product manifold and X be a concurrent vector field on $M = (M_1 \times_f M_2) \times_h M_3$.*

(i) Let (M, g, X, λ) be a sequential warped product Ricci soliton. If X_2 and X_3 are nonzero, then $M_i (1 \leq i \leq 3)$ is Ricci flat gradient Ricci soliton with $\lambda = 1$ and so is M.

(ii) The warping functions f and h are constant if and only if each X_i is a nonzero concurrent vector field on M_i for $1 \leq i \leq 3$.

Proof. Suppose that (M, g) is a sequential warped product manifold and X is a concurrent vector field on M.

Proof of (i): Assume that (M, g, X, λ) is a Ricci soliton. For $Y, Z \in \mathfrak{X}(M)$,

$$\mathcal{L}_X g(Y, Z) = 2g(Y, Z)$$

and the equation (1.1) becomes

$$\mathrm{Ric}(Y, Z) = (\lambda - 1) g(Y, Z). \tag{4.1}$$

Taking $Y = Y_3$ and $Z = Z_3$ in the above equation and using Proposition 2.3 we have

$$\mathrm{Ric}^3(Y_3, Z_3) - \left(h\Delta h + (n_3 - 1)\|\nabla h\|^2\right) g_3(Y_3, Z_3) = (\lambda - 1) h^2 g_3(Y_3, Z_3).$$

From hypothesis and (i) of Lemma 4.1, $h = c$ is constant and the above equation becomes

$$\mathrm{Ric}^3(Y_3, Z_3) = (\lambda - 1) c^2 g_3(Y_3, Z_3) \tag{4.2}$$

which means M_3 is an Einstein manifold with the factor $\mu = (\lambda - 1)c^2$. Since this equation holds for any vector fields in $\mathfrak{X}(M_3)$, taking $Y_3 = Z_3 = X_3$ we get

$$\mathrm{Ric}^3(X_3, X_3) = (\lambda - 1) c^2 g_3(X_3, X_3). \tag{4.3}$$

Assume that the set $\{X_3, e_1, \ldots, e_{n_3 - 1}\}$ is a local orthogonal base for $\mathfrak{X}(M_3)$. The Riemannian curvature can be calculated as follows

$$\begin{aligned} R^3(X_3, e_j, X_3, e_j) &= g_3\left(R^3(X_3, e_j)X_3, e_j\right) \\ &= g_3\left(\nabla^3_{X_3}\nabla^3_{e_j}X_3 - \nabla^3_{e_j}\nabla^3_{X_3}X_3 - \nabla^3_{[X_3, e_j]}X_3, e_j\right) \\ &= 0, \end{aligned}$$

which leads to $\mathrm{Ric}^3(X_3, X_3) = 0$. From the equation (4.3) we get $\lambda = 1$. Therefore, M and M_3 are Ricci flat from the equations (4.1) and (4.2), respectively. When $Y = Y_2$ and $Z = Z_2$, we have

$$\begin{aligned} 0 = \mathrm{Ric}(Y, Z) &= \mathrm{Ric}(Y_2, Z_2) \\ &= \mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h}\overline{\mathrm{Hess}}h(Y_2, Z_2) \\ &= \mathrm{Ric}^2(Y_2, Z_2) \end{aligned}$$

since f and h are constant. Therefore M_2 is Ricci flat. The same argument works for M_1. It is obvious that $M_i (1 \leq i \leq 3)$ and M are Ricci soliton with $\lambda = 1$. When we choose the potential functions for M and $M_i (1 \leq i \leq 3)$,

$$\varphi = \frac{1}{2}g(X, X), \qquad \varphi_1 = \frac{1}{2}g_1(X_1, X_1),$$

$$\varphi_2 = \frac{1}{2}g_2(X_2, X_2), \qquad \varphi_3 = \frac{1}{2}g_3(X_3, X_3),$$

then each one of them is a gradient Ricci soliton.

Proof of (ii): For $Y \in \mathfrak{X}(M)$, we get

$$\sum_{i=1}^{3} Y_i = Y = \nabla_Y X = \sum_{j,k=1}^{3} \nabla_{Y_j} X_k.$$

After some calculation, we have the system following system

$$\begin{aligned} Y_1 &= \nabla^1_{Y_1}X_1 - f g_2(X_2, Y_2)\nabla^1 f - h g_3(X_3, Y_3)(\nabla h)^T, \\ Y_2 &= \nabla^2_{Y_2}X_2 + \frac{Y_1(f)}{f}X_2 + \frac{X_1(f)}{f}Y_2 - h g_3(X_3, Y_3)(\nabla h)^\perp, \\ Y_3 &= \nabla^3_{Y_3}X_3 + \frac{Y_1(h)}{h}X_3 + \frac{Y_2(h)}{h}X_3 + \frac{X_1(h)}{h}Y_3 + \frac{X_2(h)}{h}Y_3. \end{aligned}$$

Now suppose that the warping functions f and h are constant. Then, it is clear that each X_i is concurrent on M_i for $1 \leq i \leq 3$.

Conversely, assume that each nonzero X_i is concurrent for $1 \leq i \leq 3$, then the equations $Y_1 = \nabla^1_{Y_1}X_1$, $Y_2 = \nabla^2_{Y_2}X_2$ and $Y_3 = \nabla^3_{Y_3}X_3$ are satisfied.

Here choosing $Y = Y_1 + Y_2 + 0$ in the last equation of the above system, we have $(Y_1 + Y_2)(h) = 0$. Since $Y_1 + Y_2$ is an arbitrary, the function h must be constant. Therefore, we reach

$$f g_2(X_2, Y_2)\nabla^1 f = 0.$$

Here, Y_2 is arbitrary and X_2 is nonzero, hence f must be constant. □

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Received: 2023-03-21
Accepted: 2024-07-03