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A numerical approach of the space-time-fractional telegraph equations with variable coefficients

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Abstract *In this paper, a numerical approach is proposed for solving the one-dimensional space-time-fractional telegraph equations with variable coefficients and Robin's boundary conditions. The fractional derivatives are described in the conformable sense. Based on the Legendre collocation method, our problem is reduced to a linear system of second order differential equations and the Generalized-α method is applied to solve this system. With five examples, we present a comparative study between our algorithm and some numerical methods available in the literature. This algorithm gives an excellent approximation with a small number of collocation points.*

1 Introduction

Fractional calculus generalized the classical calculus to an arbitrary (non-integer) order. The history of this theory was begun from a letter written by L'Hôspital to Leibniz in 1695 *asking him if* $n = \frac{1}{2}$, *what does it mean* $\frac{d^n f}{dx^n}$. *Leibniz then responded saying "An apparent paradox", different explanations of the fractional derivative are presented. There are almost* 25 *definitions of the fractional derivative. The most famous of them are Caputo and R-L derivatives. Both these definitions include integral in their definitions. Few properties of these fractional-order derivatives are similar to the classical order derivatives. However, there are a few complications, see [30].*

- ☞ *These definitions are non-local, which makes them unsuitable for investigating properties related to local scaling or fractional differentiability.*
- ☞ *Riemann Liouville's derivative does not fulfill* $\mathcal{D}^{\alpha}(1) = 0$.
- ☞ *For Caputo's derivative, we have to assume that the function is differentiable. Otherwise, we cannot apply this definition.*

All fractional derivatives are deficient in some mathematical properties like product rule, chain rule, and quotient rule. Therefore, the solution of differential equations is not easy to obtain using these definitions.

A new type of fractional derivative was introduced by Khalil et al., [20] and developed by Abdeljawad, [1] called "conformable fractional derivative". This definition is different from other fractional derivatives and similar to the classical definition of the derivative. It depends on the limit definition of the derivative of a function. So this definition seems to be a natural extension of the ordinary derivative. Other fractional derivatives do not have geometrical interpretation but conformable derivative has [19]. This theory has attracted many researchers to work within and so many new concepts are introduced in conformable fractional calculus, see [5, 10, 29, 28, 34]. Recently, the authors in [6] introduced a fuzzy conformable derivative of order Ψ *and extended* *by Younus et al. in [31, 32, 33]. In [13] the authors introduced a new class of mixed fractional differential equations involving the conformable and Caputo derivatives with integral boundary conditions. The exact solutions of conformable time-fractional modified nonlinear Schrödinger equation is obtained by Direct algebraic method and Sine-Gordon expansion method [7].*

Modelling of real-life problems with conformable derivatives was done in [11] and [22] where the authors modelled the dynamic cobweb and the gray system, respectively.

In the last three decades, communication systems have come to play a vital role in many of the world's problems. A typical engineering problem involves the transmission of a signal from one point to another. In order to optimize the transmission media it is needed to determine signal losses. To evaluate these losses, it is necessary to formulate some kind of equations which can calculate these losses efficiently.

In this paper, we consider the fractional telegraph equation with variable coefficients:

$$\begin{aligned}\mathcal{D}_t^{(1+\alpha)}u(x,t)+2a(x,t)\,\mathcal{D}_t^{(\alpha)}u(x,t)+b^2(x,t)\,u(x,t)=\omega(x,t)\,\mathcal{D}_x^{(\beta)}(x,t)\\ +f(x,t)\,,\ (x,t)\in\Omega_T,\\ 0<\alpha\le 1,\ 1<\beta\le 2\end{aligned} \tag{1.1}$$

where $\Omega_T := \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$ *with* $T > 0$ *are given,* $\mathcal{D}_t^{(\alpha)}$ *and* $\mathcal{D}_x^{(\beta)}$ *represent the left-conformable fractional derivatives of order* α *and* β *with respect to* t *and* x*, respectively.* $u(x,t)$ *can be voltage or current through the wire at position* x *and time* t*,* $f(x,t)$ *is the source term and* a*,* b*,* ω *are given functions such that* $a \ge b > 0$ *and* $\omega > 0$*.*

For $\beta = \alpha+1 = 2$ *and* a, b, ω *are constants, equation* (1.1) *is the classical telegraph equation introduced by Oliver Heaviside [17]. This equation is a second-order linear hyperbolic equation and it models several phenomena in many different fields such as signal analysis [18], wave propagation [27], random walk theory [8].*

We consider the following initial conditions:

$$u(x,0)=\varphi(x)\,,\quad u_t(x,0)=\psi(x)\,,\quad 0\le x\le 1, \tag{1.2}$$

and Robin's boundary conditions:

$$\begin{cases}\lambda_1 u(0,t)+\mu_1 u_x(0,t)=g(t)\,,\ \lambda_1\mu_1\le 0,\ |\lambda_1|+|\mu_1|>0\\ \lambda_2 u(1,t)+\mu_2 u_x(1,t)=h(t)\,,\ \lambda_2\mu_2\ge 0,\ |\lambda_2|+|\mu_2|>0\end{cases},\quad 0\le t\le T, \tag{1.3}$$

where φ, ψ, g, h *are given functions.*

Special cases of the initial-boundary value problem (1.1)-(1.3) *have been studied previously and summarized as follows:*

- ✓ *When* $\beta = 1+\alpha = 2$*,* $\lambda_2 = -\lambda_1 = 1$ *and* $\mu_1 = \mu_2 = 0$*, the problem* (1.1)-(1.3) *has been studied by different numerical methods such as the Crank-Nicolson scheme and the Haar wavelets [25], semi-analytical method [21] and the shifted Jacobi collocation method [16].*
- ✓ *When* a*,* b*,* ω *are constants,* $\lambda_2 = -\lambda_1 = 1$*,* $\mu_1 = \mu_2 = 0$ *and fractional derivatives in time and space in the sense of Caputo, many authors have contributed to the development of numerical methods to solve the problem* (1.1)-(1.3) *such as cubic B-spline collocation method [23], a mesh-free approach to the collocation method [9] and Bernstein polynomials operational matrices [15].*
- ✓ *When* a*,* b*,* ω *are constants, and* $\beta = 2$*, an analytical method has been presented to solve the problem* (1.1)-(1.3) *based the Fourier's method, [4].*
- ✓ *When* a*,* b*,* ω *are constants,* $\lambda_2 = -\lambda_1 = 1$ *and* $\mu_1 = \mu_2 = 0$*, the problem* (1.1)-(1.3) *has been studied by an efficient algorithm based on Chebyshev polynomials of the fourth kind and Newmark's method, [3].*

Our objective in this paper is to propose a numerical approach to solve the problem (1.1)-(1.3)*. This approach based on shifted Legendre polynomials of the first kind and the Generalized-*α *method. This study is a generalization of work published in [3].*

The rest of this paper is structured as follows: Section 2 deals with some description of conformable fractional derivative and its properties. In Section 3, deals with some properties of *shifted Legendre polynomials of the first kind. Section 4 is devoted to evaluation of the conformable fractional derivative using shifted Legendre polynomials of the first kind. In section 5, we gave three lemmas to prove our result on uniform convergence. In section 6, a Legendre collocation method was proposed to reduce the problem (1.1)-(1.3) to a linear system of second order differential equations. In Section 7, the Generalized-α method has been proposed to solve this system. With five examples, we presented a comparative study between this algorithm and some numerical methods available in the literature in Section 8.*

2 Preliminaries on conformable fractional derivative

In this section, we start by recalling some concepts about conformable fractional calculus.

Definition 2.1 ([1]). Let $\varphi : [a, +\infty[\to \mathbb{R}$ is a given function and $\alpha \in]0, 1]$. Then, the left-conformable fractional derivative of order α is defined by:

$$\mathcal{D}_t^{(\alpha)}(\varphi)(t) := \lim_{\varepsilon \to 0} \frac{\varphi\left(t + \varepsilon (t-a)^{1-\alpha}\right) - \varphi(t)}{\varepsilon}. \tag{2.1}$$

If $\mathcal{D}_t^{(\alpha)}(\varphi)(t)$ exists on $]a, +\infty[$, then $\mathcal{D}_t^{(\alpha)}(\varphi)(a) = \lim_{t \to a^+} \mathcal{D}_t^{(\alpha)}(\varphi)(t)$. If $a = 0$, the definition (2.1) is introduced by Khalil et al. [20]. In this case, we say that φ is α-differentiable.

Some important properties of the conformable fractional derivative are as follows:

Proposition 2.2 ([1, 19]). *Let $f, g : [0, +\infty[\to \mathbb{R}$ and $0 < \alpha \leq 1$. We have the following properties:*

$$\text{If } f \text{ is } \alpha\text{-differentiable, then } f \text{ is continuous.} \tag{2.2}$$

$$\mathcal{D}^{(\alpha)}(af + bg) = a\mathcal{D}^{(\alpha)}(f) + b\mathcal{D}^{(\alpha)}(g), \text{ for all } a, b \in \mathbb{R}. \tag{2.3}$$

$$\mathcal{D}^{(\alpha)}(C) = 0 \text{ where } C \text{ is a constant}, \tag{2.4}$$

$$\mathcal{D}^{(\alpha)} t^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k-n)} t^{k-\alpha} & \text{If } k \in \mathbb{N} \text{ and } k > \alpha, \\ 0 & \text{If } k \in \mathbb{N} \text{ and } k < \alpha, \end{cases}, \tag{2.5}$$

where $\Gamma(\cdot)$ is Euler's Gamma function and $n < \alpha \leq n+1$.

3 Some properties of Legendre polynomials

Definition 3.1 ([14]). The Legendre polynomial of the first kind is a polynomial of degree n in x defined on $[-1, 1]$ by the Rodrigues formula:

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}\left[\left(x^2 - 1\right)^n\right], \tag{3.1}$$

where x is a real or complex variable.

To use the Legendre polynomials defined by (3.1) on the interval [0, 1], we define the so-called shifted Legendre polynomials of the first kind by making the following change of variable:

$$z = 2x - 1 \text{ or } x = \frac{1}{2}(z+1).$$

In this case, the shifted Legendre polynomials $P_n^*(x)$ of order n in x are defined on $[0, 1]$ by:

$$P_n^*(x) = P_n(z) = P_n(2x - 1). \tag{3.2}$$

With (3.2) and Proposition 1.6 in [2, Page 23], the shifted Legendre polynomials of the first kind $P_n^*(x)$ verify the following recurrence formula:

$$\begin{cases} P_0^*(x) = 1, \\ P_1^*(x) = 2x - 1, \\ P_{n+1}^*(x) = \frac{(2n+1)(2x-1)}{n+1} P_n^*(x) - \frac{n}{n+1} P_{n-1}^*(x), \text{ for all } n \in \mathbb{N}^*. \end{cases} \tag{3.3}$$

Using (3.2) *and* (3.1)*, we obtain the explicit form of the shifted Legendre polynomials of the first kind* $P_n^*(x)$ *of degree* n *at* x *given by:*

$$P_n^*(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{\Gamma(n+k+1)}{\Gamma(n-k+1)\left(\Gamma(k+1)\right)^2} x^k, \tag{3.4}$$

where $\Gamma(\cdot)$ *is Euler Gamma function. On the other hand we have:*

$$\begin{cases} P_n^*(0) = (-1)^n, & (P_n^*(0))' = (-1)^{n-1} n(n+1), \\ P_n^*(1) = 1, & (P_n^*(1))' = n(n+1). \end{cases} \tag{3.5}$$

According to (3.2) *and Lemme 1.3 in [2, Page 23], the polynomials* $P_n^*(x)$ *are orthogonal on the interval* $[0,1]$*, i.e.*

$$\left\langle P_i^*(x), P_j^*(x)\right\rangle = \int_0^1 P_i^*(x) P_j^*(x)\, dx = \begin{cases} \frac{1}{2i+1}, & \text{if} \quad i=j, \\ 0, & \text{if} \quad i \neq j. \end{cases} \tag{3.6}$$

A function $\Phi(x)$*, square integrable in* $[0,1]$*, may be expressed in terms of the shifted Legendre polynomials as*

$$\Phi(x) = \sum_{i=0}^{+\infty} c_i P_i^*(x), \tag{3.7}$$

where the coefficients c_i *are given by*

$$c_i = (2i+1)\int_0^1 \Phi(x) P_i^*(x)\, dx, \quad \text{for all } i \in \mathbb{N}. \tag{3.8}$$

In practice, only the first $(m+1)$*-terms shifted Legendre polynomials are considered. Then we have:*

$$\Phi_m(x) = \sum_{i=0}^{m} c_i P_i^*(x). \tag{3.9}$$

4 Evaluation of the conformable fractional derivative

In this section, we present the evaluation of the conformable fractional derivative for the function $\Phi_m(x)$ *defined by* (3.9)*. This evaluation is given by the following theorem:*

Theorem 4.1. *Let* $\Phi_m(x)$ *be an approximate function in terms of shifted Legendre polynomials of the first kind defined by* (3.9)*. For all* $\xi > 0$*, we have:*

$$\mathcal{D}_x^{(\xi)} \Phi_m(x) = \sum_{i=n+1}^{m} \sum_{k=n+1}^{i} c_i \mathcal{N}_{i,k}^n x^{k-\xi}, \quad n < \xi \leq n+1, \tag{4.1}$$

where $\mathcal{N}_{i,k}^n$ *is given by*

$$\mathcal{N}_{i,k}^n = \frac{(-1)^{i+k}\Gamma(i+k+1)}{\Gamma(i-k+1)\Gamma(k+1)\Gamma(k-n)} \tag{4.2}$$

Proof. Using (3.9) and (2.3), we get:

$$\mathcal{D}_x^{(\xi)} \Phi_m(x) = \sum_{i=0}^{m} c_i \mathcal{D}_x^{(\alpha)} P_i^*(x), \quad \text{for all } \xi > 0. \tag{4.3}$$

Moreover, from (3.4) and (2.5), we have:

$$\mathcal{D}_x^{(\xi)} P_i^*(x) = \sum_{k=n+1}^{i} (-1)^{i+k} \frac{\Gamma(i+k+1)}{\Gamma(i-k+1)\Gamma(k+1)\Gamma(k-n)} x^{k-\xi}, \quad n < \xi \leq n+1. \tag{4.4}$$

By substituting (4.4) in (4.3), we obtain:

$$\mathcal{D}_x^{(\xi)}\Phi_m(x) = \sum_{i=n+1}^{m}\sum_{k=n+1}^{i}(-1)^{i+k}c_i\mathcal{N}_{i,k}^{n}x^{k-\xi}. \tag{4.5}$$

where $\mathcal{N}_{i,k}^{n}$ is defined by (4.2). □

Example 4.2. Consider $\Phi(x) = x^3$ and $m = 3$, $\xi = 2.5$ and $n = 2$. Using (2.5), we obtain:

$$\mathcal{D}^{(2.5)}\Phi(x) = 6\sqrt{x}.$$

Now, using Theorem 4.1 we obtain:

$$\mathcal{D}^{(2.5)}\Phi(x) = \sum_{i=3}^{3}\sum_{k=3}^{3}c_i\mathcal{M}_{i,k}^{2}x^{k-2.5} = c_3\mathcal{N}_{3,3}^{2}x^{0.5}$$

From (3.5), (3.6), (3.8) and (4.2), we obtain :

$$c_3 = \frac{1}{20},\ \mathcal{N}_{3,3}^{2} = 120.$$

Then, $\mathcal{D}_x^{(2.5)}x^3 = 6\sqrt{x}$.

5 Convergence analysis

Before proving our main theorem on convergence, we need the following three lemmas:

Lemma 5.1. *We have*

$$|P_n^*(x)| < \sqrt{\frac{\pi}{8n(x-x^2)}},\ \text{for all } x \in\ ]0,1[\ \text{and } n \in \mathbb{N}^*. \tag{5.1}$$

Proof. The first Laplace integral for $P_n(z)$ with $z \in [-1,1]$, see [26, page 171], is given by:

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi}\left[z + \left(z^2-1\right)^{1/2}\cos(\theta)\right]^n d\theta \tag{5.2}$$

We put $z = 2x-1$ in (5.2), we obtain

$$\begin{aligned} P_n^*(x) &= \frac{1}{\pi}\int_0^{\pi}\left[2x-1+\left(4x^2-4x\right)^{1/2}\cos(\theta)\right]^n d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\left[2x-1+2i\left(x-x^2\right)^{1/2}\cos(\theta)\right]^n d\theta. \end{aligned} \tag{5.3}$$

For all $x \in\ ]0,1[$, we have

$$\begin{aligned} \left|2x-1+2i\left(x-x^2\right)^{1/2}\cos(\theta)\right| &= \sqrt{(2x-1)^2+4\left(x-x^2\right)\cos^2(\theta)} \\ &= \sqrt{1-4\left(x-x^2\right)\sin^2(\theta)}. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain

$$|P_n^*(x)| \le \frac{1}{\pi}\int_0^{\pi}\left[1-4\left(x-x^2\right)\sin^2(\theta)\right]^{n/2} d\theta \le \frac{2}{\pi}\int_0^{\pi/2}\left[1-4\left(x-x^2\right)\sin^2(\theta)\right]^{n/2} d\theta.$$

We have

$$\begin{cases} \forall\theta \in\ ]0,\pi/2[\,,\ \sin(\theta) > \frac{2\theta}{\pi}, \\ \forall y > 0,\ 1-y < e^{-y}. \end{cases}$$

So we have

$$1-4(x-x^2)\sin^2(\theta)<1-\frac{16\theta^2(x-x^2)}{\pi^2}<e^{\left(-\frac{16\theta^2(x-x^2)}{\pi^2}\right)}.$$

Therefore, we get:

$$|P_n^*(x)|<\frac{2}{\pi}\int_0^{\frac{\pi}{2}}e^{\left(\frac{-8n\theta^2(x-x^2)}{\pi^2}\right)}d\theta<\frac{2}{\pi}\int_0^{+\infty}e^{\left(\frac{-8n\theta^2(x-x^2)}{\pi^2}\right)}d\theta.$$

With the change of variable $s=\frac{2\theta}{\pi}\sqrt{2n\left(x-x^2\right)}$, we get:

$$|P_n^*(x)|<\frac{2}{\pi}\frac{\pi/2}{\sqrt{2n\left(x-x^2\right)}}\int_0^{+\infty}e^{-s^2}ds=\sqrt{\frac{\pi}{8n\left(x-x^2\right)}},$$

where $\int_0^{+\infty}e^{-s^2}ds=\sqrt{\pi}/2.$ □

Lemma 5.2. *If* $\Phi\in L^2\left(0,1\right)$*, then*

$$\lim_{n\to+\infty}\sqrt{2n+1}\int_0^1\Phi\left(x\right)P_n^*\left(x\right)dx=0. \tag{5.5}$$

Proof. Let be the partial sum given by:

$$\Phi_m\left(x\right)=\sum_{n=0}^{m}c_nP_n^*\left(x\right). \tag{5.6}$$

Consider the following positive quantity:

$$\int_0^1\left[\Phi\left(x\right)-\Phi_m\left(x\right)\right]^2dx=\int_0^1\Phi^2\left(x\right)dx-2\int_0^1\Phi\left(x\right)\Phi_m\left(x\right)dx+\int_0^1\Phi_m^2\left(x\right)dx\geq 0. \tag{5.7}$$

From (5.6), (3.8) and (3.6), we obtain

$$\int_0^1\Phi\left(x\right)\Phi_m\left(x\right)dx=\sum_{i=0}^{m}c_n\int_0^1\Phi\left(x\right)P_n^*\left(x\right)dx=\sum_{n=0}^{m}\frac{c_n^2}{2n+1} \tag{5.8}$$

and

$$\begin{aligned}\int_0^1\Phi_m^2\left(x\right)dx&=\sum_{n=0}^{m}\sum_{k=0}^{m}c_nc_k\int_0^1P_n^*\left(x\right)P_k^*\left(x\right)dx\\&=\sum_{n=0}^{m}c_n^2\int_0^1\left[P_n^*\left(x\right)\right]^2dx\\&=\sum_{n=0}^{m}\frac{c_n^2}{2n+1}.\end{aligned} \tag{5.9}$$

Replacing (5.8) and (5.9) in (5.7), we obtain:

$$\sum_{n=0}^{m}\frac{c_n^2}{2n+1}\leq\int_0^1\Phi^2\left(x\right)dx. \tag{5.10}$$

Passing to limit $m\to+\infty$, the inequality (5.10) becomes:

$$\sum_{n=0}^{+\infty}\frac{c_n^2}{2n+1}\leq\int_0^1\Phi^2\left(x\right)dx. \tag{5.11}$$

Since $\Phi \in L^2(0,1)$, then the series defined in (5.11) is convergent. Therefore, we have:

$$\lim_{n\to+\infty} \frac{c_n^2}{2n+1} = 0. \tag{5.12}$$

Using (3.8) and (5.12), we obtain (5.5). □

Lemma 5.3. *The shifted Legendre polynomials of the first kind satisfy the following identity:*

$$\sum_{n=0}^{m} (2n+1) P_n^*(x) P_n^*(y) = \frac{m+1}{2(y-x)} \left[P_{m+1}^*(y) P_m^*(x) - P_m^*(y) P_{m+1}^*(x)\right]. \tag{5.13}$$

Proof. By multiplying the recurrence relation (3.3) by $P_n^*(y)$, we get:

$$(2n+1)(2x-1) P_n^*(y) P_n^*(x) = (n+1) P_n^*(y) P_{n+1}^*(x) + n P_n^*(y) P_{n-1}^*(x).$$

We change the roles between x and y in this expression and by the difference, we get:

$$\begin{aligned} 2(2n+1)(y-x) P_n^*(y) P_n^*(x) &= (n+1)\left[P_{n+1}^*(y) P_n^*(x) - P_n^*(y) P_{n+1}^*(x)\right] \\ &\quad - n\left[P_n^*(y) P_{n-1}^*(x) - P_{n-1}^*(y) P_n^*(x)\right]. \end{aligned} \tag{5.14}$$

Summing both sides of (5.14) with $n = 0, 1, 2, ..., m$, we get:

$$\begin{aligned} 2(y-x)\sum_{n=0}^{m}[(2n+1)P_n^*(y)P_n^*(x)] &= \sum_{n=0}^{m}[(n+1)(P_{n+1}^*(y)P_n^*(x) - P_n^*(y)P_{n+1}^*(x)) \\ &\quad -n(P_n^*(y)P_{n-1}^*(x) - P_{n-1}^*(y)P_n^*(x))] \\ &= (m+1)\left[P_{m+1}^*(y)P_m^*(x) - P_m^*(y)P_{m+1}^*(x)\right]. \end{aligned} \tag{5.15}$$

Dividing (5.15) by $2(y-x)$, we get (5.13). □

Remark 5.4. Integrating (5.13) over $[0,1]$ gives

$$\sum_{n=0}^{m} (2n+1) P_n^*(x) \int_0^1 P_n^*(y)\,dy = \frac{m+1}{2} \int_0^1 \frac{P_{m+1}^*(y) P_m^*(x) - P_m^*(y) P_{m+1}^*(x)}{y-x} dy,$$

and from (3.8), we deduce

$$(m+1) \int_0^1 \frac{P_{m+1}^*(y) P_m^*(x) - P_m^*(y) P_{m+1}^*(x)}{y-x} dy = 2. \tag{5.16}$$

For convergence, we have the following theorem:

Theorem 5.5. *Let* $\Phi(x) = \sum_{n=0}^{+\infty} c_n P_n^*(x)$ *is a differentiable function on* $]0,1[$. *Then, for all* $0 < \varepsilon \le 1/2$, *the sequence of partial sums defined by:*

$$S_m(x) = \sum_{n=0}^{m} c_n P_n^*(x)$$

converges uniformly to $\Phi(x)$ *on* $[\varepsilon, 1-\varepsilon]$.

Proof. Let $x \in [0,1]$. We have

$$S_m(x) = \sum_{n=0}^{m} c_n P_n^*(x) = \sum_{n=0}^{m} \left[(2n+1)\int_0^1 \Phi(y) P_n^*(y)\,dy\right] P_n^*(x).$$

By reversing the order of summation and integration and using Lemma 5.3, we obtain:

$$
\begin{aligned}
S_m(x) &= \int_0^1 \Phi(y) \sum_{n=0}^{m} (2n+1) P_n^*(y) P_n^*(x) dy \\
&= \frac{m+1}{2} \int_0^1 \frac{\Phi(y) \left[P_{m+1}^*(y)P_m^*(x) - P_m^*(y)P_{m+1}^*(x)\right] dy}{y-x} \\
&= \frac{m+1}{2} \Phi(x) \int_0^1 \frac{\left[P_{m+1}^*(y)P_m^*(x) - P_m^*(y)P_{m+1}^*(x)\right] dy}{y-x} \\
&+ \frac{m+1}{2} \int_0^1 \frac{\Phi(y)-\Phi(x)}{y-x} \left[P_{m+1}^*(y)P_m^*(x) - P_m^*(y)P_{m+1}^*(x)\right] dy.
\end{aligned}
$$

We introduce the function $\hbar$ defined by:

$$
\hbar(y) = \begin{cases} \frac{\Phi(y)-\Phi(x)}{y-x}, & \text{si } y \neq x, \\ \Phi'(x), & \text{si } y = x. \end{cases}
$$

The function $\hbar$ is continuous on $[0,1]$ and using (5.16) we get:

$$
S_m(x) = \Phi(x) + \frac{m+1}{2} P_m^*(x) \int_0^1 \hbar(y) P_{m+1}^*(y) dy - \frac{m+1}{2} P_{m+1}^*(x) \int_0^1 \hbar(y) P_m^*(y) dy. \tag{5.17}
$$

Let $b_m = \sqrt{2m+1} \int_0^1 \hbar(y) P_n^*(y)\, dy$. Equation (5.17) becomes in the following form:

$$
|\Phi(x) - S_m(x)| = \left| \sum_{n=m+1}^{+\infty} c_n P_n^*(x) \right| \leq \frac{(m+1)|b_m| \left|P_{m+1}^*(x)\right|}{2\sqrt{2m+1}} + \frac{(m+1)|b_{m+1}| \left|P_m^*(x)\right|}{2\sqrt{2m+3}}. \tag{5.18}
$$

Using Lemma 5.1, we get:

$$
\frac{(m+1)|P_m^*(x)|}{2\sqrt{2m+3}} < \left[\frac{(m+1)^2}{2m^2+3m} \times \frac{\pi}{32(x-x^2)}\right]^{1/2} < \sqrt{\frac{\pi}{32(x-x^2)}} \leq M_\varepsilon \tag{5.19}
$$

and

$$
\frac{(m+1)\left|P_{m+1}^*(x)\right|}{2\sqrt{2m+1}} < \left[\frac{m+1}{2m+1} \times \frac{\pi}{32(x-x^2)}\right]^{1/2} < \sqrt{\frac{\pi}{32(x-x^2)}} \leq M_\varepsilon, \tag{5.20}
$$

where $M_\varepsilon = \sup\limits_{x\in[\varepsilon,1-\varepsilon]} \sqrt{\frac{\pi}{32(x-x^2)}}$.

From (5.18)-(5.20), we obtain:

$$
\left| \sum_{n=m+1}^{+\infty} c_n P_n^*(x) \right| \leq M_\varepsilon (|b_m| + |b_{m+1}|) \tag{5.21}
$$

According to Lemma 5.2, we get $\lim\limits_{m\to+\infty} b_m = \lim\limits_{m\to+\infty} b_{m+1} = 0$. Therefore, we have

$$
\left| \sum_{n=m+1}^{+\infty} c_n P_n^*(x) \right| \xrightarrow[m\to+\infty]{} 0,
$$

then $(S_m(x))$ converges uniformly to $\Phi(x)$ over $[\varepsilon, 1-\varepsilon]$ with $0 < \varepsilon \leq 1/2$. □

6 Legendre collocation method

In this section, we apply Legendre collocation method to the problem (1.1)-(1.3). Let $u_m(x,t)$ be the approximation of $u(x,t)$ given in the following form:

$$u_m(x,t) = \sum_{i=0}^{m} c_i(t) P_i^*(x). \tag{6.1}$$

By applying Theorem (4.1) and (6.1) to the equation (1.1), we obtain:

$$\sum_{i=0}^{m} t^{1-\alpha} c_i''(t) P_i^*(x) + 2a(x,t) \sum_{i=0}^{m} t^{1-\alpha} c_i'(t) P_i^*(x) + b^2(x,t) \sum_{i=0}^{m} c_i(t) P_i^*(x) = \omega(x,t) \sum_{i=2}^{m} \sum_{k=2}^{i} c_i(t) \mathcal{N}_{i,k}^1 x^{k-\xi} + f(x,t). \tag{6.2}$$

We put in (6.2) $x = x_p$, $p = 1, \ldots, m-1$ the roots of the shifted Legendre polynomial $P_n^*(x)$, we have:

$$\sum_{i=0}^{m} \left[t^{1-\alpha} c_i''(t) P_i^*(x_p) + 2a(x_p,t) t^{1-\alpha} c_i'(t) P_i^*(x_p) + c_i(t) R_i(x_p)\right] = f(x_p,t), \tag{6.3}$$

where

$$\begin{cases} S_0(x_p) = S_1(x_p) = 0, \\ S_i(x_p) = \sum_{k=2}^{i} \mathcal{N}_{i,k}^1 x_p^{k-\xi}, \quad i = 2,3,...,m, \\ R_i(x_p) = b^2(x_p,t) P_i^*(x_p) - \omega(x_p,t) S_i(x_p), \quad i = 0,1,2,...,m. \end{cases} \tag{6.4}$$

From (6.1), (1.3) and (3.5), we obtain:

$$\begin{cases} \sum_{i=0}^{m} \left[(-1)^i \lambda_1 + (-1)^{i-1} i(i+1) \mu_1\right] c_i(t) = g(t), \\ \sum_{i=0}^{m} \left[\lambda_2 + i(i+1)\mu_2\right] c_i(t) = h(t). \end{cases} \tag{6.5}$$

We introduce the vectors $X(t)$, $\dot{X}(t)$, $\ddot{X}(t)$ and $F(t)$ defined by:

$$\begin{cases} X(t) = (c_0(t), c_1(t), ..., c_m(t))^T, \\ \dot{X}(t) = (c_0'(t), c_1'(t), ..., c_m'(t))^T, \\ \ddot{X}(t) = (c_0''(t), c_1''(t), ..., c_m''(t))^T, \\ F(t) = (f(x_1,t), f(x_2,t), ..., f(x_{m-1},t), g(t), h(t))^T. \end{cases} \tag{6.6}$$

By replacing (6.1) in the initial conditions (1.2), and using (3.8), we obtain:

$$\begin{cases} c_i(0) = (2i+1) \int_0^1 \varphi(x) P_i^*(x)\,dx. \\ c_i'(0) = (2i+1) \int_0^1 \psi(x) P_i^*(x)\,dx. \end{cases}, \quad i = 0,1,\ldots,m. \tag{6.7}$$

By combining equations (6.3), (6.5) and (6.6), we find the following matrix form:

$$\begin{cases} M(t)\ddot{X}(t) + C(t)\dot{X}(t) + K(t)X(t) = F(t), \\ X(0) = (c_0(0), c_1(0), ..., c_m(0))^T, \\ \dot{X}(0) = (c_0'(0), c_1'(0), ..., c_m'(t))^T, \end{cases} \tag{6.8}$$

where $M(t)$ is the mass matrix given by:

$$M(t)=t^{1-\alpha}\begin{pmatrix} P_0^*(x_1) & P_1^*(x_1) & \cdots & P_m^*(x_1)\\ P_0^*(x_2) & P_1^*(x_2) & \cdots & P_m^*(x_2)\\ \vdots & \ddots & \vdots & \vdots\\ P_0^*(x_{m-1}) & P_1^*(x_{m-1}) & \cdots & P_m^*(x_{m-1})\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$C(t)$ is the damping matrix given by:

$$C(t)=2t^{1-\alpha}\begin{pmatrix} a(x_1,t)P_0^*(x_1) & a(x_1,t)P_1^*(x_1) & \cdots & a(x_1,t)P_m^*(x_1)\\ a(x_2,t)P_0^*(x_2) & a(x_2,t)P_1^*(x_2) & \cdots & a(x_2,t)P_m^*(x_2)\\ \vdots & \ddots & \vdots & \vdots\\ a(x_{m-1},t)P_0^*(x_{m-1}) & a(x_{m-1},t)P_1^*(x_{m-1}) & \cdots & a(x_{m-1},t)P_m^*(x_{m-1})\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $K(t)$ is the stiffness matrix given by:

$$K(t)=\begin{pmatrix} R_0(x_1) & R_1(x_1) & \cdots & R_m(x_1)\\ R_0(x_2) & R_1(x_2) & \cdots & R_m(x_2)\\ \vdots & \ddots & \vdots & \vdots\\ R_0(x_{m-1}) & R_1(x_{m-1}) & \cdots & R_m(x_{m-1})\\ \lambda_1 & -\lambda_1+2\mu_1 & \cdots & (-1)^m\lambda_1+(-1)^{m-1}m(m+1)\mu_1\\ \lambda_2 & \lambda_2+2\mu_2 & \cdots & \lambda_2+m(m+1)\mu_2 \end{pmatrix}.$$

To solve the system (6.8), we use the Generalized-α method, [12].

7 The Generalized-α method

Let $0=t_0<t_1<\ldots<t_{N_t}=T$ a uniform partition of the interval $[0,T]$ and $\tau=t_{j+1}-t_j$ is the time step. We define $t_j=\tau j$ where $j=0,1,\ldots,N_t$. We introduce the following notations:

$$c_i(t_j)=c_i^j,\ g(t_j)=g_j,\ h(t_j)=h_j,\ F_j=(f(x_1,t_j),f(x_2,t_j),\ldots,f(x_{m-1},t_j),g_j,h_j)^T,$$

and

$$M^j=M(t_j),\ C^j=C(t_j),\ K^j=K(t_j),\ j=0,1,\ldots,N_t.$$

In 1993, Chung and Hulbert proposed a method called the Generalized-α method, see [12]. The basic form of this method is given by:

$$d_{j+1}=d_j+\tau v_j+\tau^2\left[\left(\frac{1}{2}-\theta\right)A_j+\theta A_{j+1}\right], \tag{7.1}$$

$$v_{j+1}=v_j+\tau\left[(1-\gamma)A_j+\gamma A_{j+1}\right], \tag{7.2}$$

$$M^{j+1}A_{j+1-\alpha_m}+C^{j+1}v_{j+1-\alpha_f}+K^{j+1}d_{j+1-\alpha_f}=F_{j+1-\alpha_f}, \tag{7.3}$$

$$d_0=X(0),\ v_0=\dot{X}(0),\ A_0=\ddot{X}(0), \tag{7.4}$$

where

$$d_{j+1-\alpha_f}=(1-\alpha_f)d_{j+1}+\alpha_f d_j, \tag{7.5}$$

$$v_{j+1-\alpha_f}=(1-\alpha_f)v_{j+1}+\alpha_f v_j, \tag{7.6}$$

$$A_{j+1-\alpha_m}=(1-\alpha_m)A_{j+1}+\alpha_m A_j, \tag{7.7}$$

$$F_{j+1-\alpha_f}=F_{(1-\alpha_f)t_{j+1}+\alpha_f t_j}, \tag{7.8}$$

in which d_j , v_j *and* A_j *are respectively the displacement, velocity and acceleration vectors at time* t_j *with* $j = 0, 1, \ldots, N_t$.

According to (7.1)*, we have:*

$$A_{j+1} = \frac{1}{\tau^2\theta}\left(d_{j+1} - d_j\right) - \frac{v_j}{\tau\theta} - \left(\frac{1}{2\theta} - 1\right) A_j. \tag{7.9}$$

By replacing (7.9) *in* (7.2)*, we get:*

$$v_{j+1} = \frac{\gamma}{\tau\theta}\left(d_{j+1} - d_j\right) + \left(1 - \frac{\gamma}{\theta}\right) v_j + \tau\left(1 - \frac{\gamma}{2\theta}\right) A_j. \tag{7.10}$$

By replacing (7.9) *and* (7.10) *in* (7.3) *and with* (7.5)-(7.8)*, we obtain the following linear system:*

$$D^{j+1} d_{j+1} = B_{j+1}, \tag{7.11}$$

where

$$D^{j+1} = \frac{1-\alpha_m}{\tau^2\theta} M^{j+1} + \frac{\gamma\left(1-\alpha_f\right)}{\tau\theta} C^{j+1} + \left(1-\alpha_f\right) K^{j+1}, \tag{7.12}$$

$$\begin{aligned} B^{j+1} = {} & F_{j+1-\alpha_f} + M^{j+1}\left[\frac{1-\alpha_m}{\tau^2\theta} d_j + \frac{1-\alpha_m}{\tau\theta} v_j + \left(\frac{1-\alpha_m}{\tau\theta} - 1\right) A_j\right] \\ & + C^{j+1}\left[\frac{\gamma\left(1-\alpha_m\right)}{\tau\theta} d_j + \left(\frac{\gamma\left(1-\alpha_m\right)}{\theta} - 1\right) v_j + \tau\left(1-\alpha_f\right)\left(\frac{\gamma}{2\theta} - 1\right) A_j\right] - \alpha_f K^{j+1} d_j. \end{aligned} \tag{7.13}$$

The complete algorithm using the Generalized-α *method is below.*

Algorithm 7.1 (The Generalized-α method)

1: **Initializations:**

(i) Give the coefficients $a(x,t)$, $b(x,t)$ and $\omega(x,t)$.

(ii) Give fractional orders $0 < \alpha \leq 1$ and $1 < \beta \leq 2$.

(iii) Give the source term $f(x,t)$, the initial data $\varphi(x)$ and $\psi(x)$.

(iv) Give Robin's conditions $g(t)$, $h(t)$, μ_1 , μ_2, λ_1 and λ_2.

(v) Give the shifted Legendre polynomials $P_0^*(x), P_1^*(x), ..., P_m^*(x)$.

(vi) Give $x_1, x_2, ..., x_{m-1}$ the roots of shifted Legendre polynomial $P_{m-1}^*(x)$.

(vii) Give the time step $\tau = T/N_t$ and the step of space $h = 1/(N_x + 1)$.

(viii) Calculate the following parameters:

$$\rho_\infty \in [0,1], \alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \theta = \frac{1}{4}(1 - \alpha_m + \alpha_f)^2, \gamma = \frac{1}{2} - \alpha_m + \alpha_f.$$

(ix) Give the constants:

$$a_0 = \frac{1-\alpha_m}{\beta\tau^2}, a_1 = \frac{\gamma(1-\alpha_f)}{\beta\tau}, a_2 = 1 - \alpha_f, a_3 = \frac{1-\alpha_m}{\beta\tau}, a_4 = \frac{1-\alpha_m}{2\beta} - 1,$$

$$a_5 = \frac{\gamma(1-\alpha_f)}{\beta} - 1, a_6 = \tau(1-\alpha_f)\left(\frac{\gamma}{2\beta} - 1\right), a_7 = \frac{\gamma}{\beta\tau}, a_8 = 1 - \frac{\gamma}{\beta},$$

$$a_9 = \tau\left(1 - \frac{\gamma}{2\beta}\right), a_{10} = \frac{1}{\beta\tau^2}, a_{11} = \frac{1}{\beta\tau}, a_{12} = \frac{1}{2\beta} - 1.$$

(x) Calculate vectors d_0, v_0 and A_0.

2: **For each time step:**

(i) Calculate the mass matrix M^{j+1}, the damping matrix C^{j+1}, the stiffness matrix K^{j+1} and the source term $F_{j+1-\alpha_f}$.

(ii) Calculate the matrix $D^{j+1} = a_0 M^{j+1} + a_1 C^{j+1} + a_2 k^{j+1}$ and the second member

$$B^{j+1} = F_{j+1-\alpha_f} + M(a_0 d_j + a_3 v_j + a_4 a_j) + C(a_1 d_j + a_5 v_j + a_6 a_j) - \alpha_f K d_j.$$

(iii) Solve the linear system $A^{j+1} d_{j+1} = B^{j+1}$.

(iv) Calculate velocity and acceleration vectors:

$$v_{j+1} = a_7(d_{j+1} - d_j) + a_8 v_j + a_9 A_j \text{ and } A_{j+1} = a_{10}(d_{j+1} - d_j) - a_{11} v_j - a_{12} A_j.$$

(v) Ask $j = j + 1$ and go to step 2:1.

8 Comparative study

In this section, we present five numerical examples for the problem (1.1)-(1.3)*. Let L_2 and L_∞ be the errors defined by:*

$$L_2 = \|u_e - u_m\|_2 = \sqrt{h \sum_{i=0}^{N} |u_e(x_i,t) - u_m(x_i,t)|^2},$$

$$L_\infty = \|u_e - u_m\|_\infty = \max_{0 \leq i \leq N} |u_e(x_i,t) - u_m(x_i,t)|,$$

where u_e is the exact solution and u_m is the numerical solution.

We compare the numerical solutions obtained by Algorithm 7.1 with solutions obtained by numerical methods available in the literature.

All numerical calculations were performed by MATLAB R2014b in Windows 10 (64-bit) operating system with Intel(R) Core(TM) i7-2670QM, 2.20 GHz processor and 8 GB of memory.

Example 8.1 ([3]). We consider the following data:

$$\begin{cases} a(x,t) = \frac{1}{2}, \;\; b(x,t) = \omega(x,t) = 1, \;\; \beta = 2, \\ f(x,t) = \left[\left(x - x^2 \right) \left[\alpha + \alpha^2 - (1+\alpha)\, t + t^{1+\alpha} \right] + 2t^{1+\alpha} \right] e^{-t}, \\ \varphi(x) = 0, \;\; \psi(x) = 0, \;\; g(t) = h(t) = 0, \\ \mu_1 = \mu_2 = 0, \;\; \lambda_1 = -1, \;\; \lambda_2 = 1. \end{cases}$$

For these data, the problem (1.1)-(1.3) is given by:

$$\begin{cases} \mathcal{D}_t^{(1+\alpha)} u(x,t) + \mathcal{D}_t^{(\alpha)} u(x,t) + u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \\ u(x,0) = u_t(x,0) = 0, \;\; 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, \;\; 0 < t \le T. \end{cases}$$

The exact solution to this problem is $u_e = \left(x - x^2 \right) t^{1+\alpha} e^{-t}$. This example has been studied by different numerical methods in [3], and in [21] for $\alpha = 1$.

Table 1 compares the L_2 and L_∞ errors obtained by our algorithm and with the numerical methods studied in [21] and [3]. In Table 2, we compare the L_2 and L_∞ errors at different values of T and α with $\tau = 0.0001$ and $h = 0.01$ with the results obtained by the method studied in [3]. With this comparative study, we can conclude that the numerical solutions obtained by our algorithm are very good. In Figure 1, we present the curves of the exact and numerical solutions for $\alpha = 1$ at $T = 1, 2, 3, 4$. The curves of the exact and numerical solutions for $T = 2$ at $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ are given by Figure 2.

Table 1: L_2 and L_∞ errors with $\tau = 0,001$, $h = 0,01$ and $\alpha = 1$.

	Our algorithm		Abdelkebir[3]		Lin et al.[21]	
T	L_2	L_∞	L_2	L_∞	L_2	L_∞
1.0	3.32E-09	5.45E-09	8.15E-07	1.46E-06	2.67E-08	5.17E-09
2.0	9.60E-09	1.25E-08	1.30E-05	1.83E-05	8.78E-08	1.27E-08
3.0	5.86E-10	1.13E-09	4.75E-06	7.12E-06	3.66E-09	1.07E-09
4.0	1.72E-09	2.60E-09	9.09E-06	1.26E-05	1.47E-08	2.62E-09
5.0	1.67E-10	2.45E-10	4.43E-06	6.03E-06	1.79E-09	3.52E-10

Table 2: L_2 and L_∞ errors with $\tau = 0,0001$, $h = 0,01$ and $T = 2$.

	Our algorithm		Abdelkebir[3]	
α	L_2	L_∞	L_2	L_∞
0.2	1.81E-03	2.57E-03	2.08E-03	2.93E-03
0.4	3.09E-04	4.32E-04	4.10E-04	5.77E-04
0.6	3.96E-05	5.54E-05	6.24E-05	8.91E-05
0.8	3.50E-06	4.97E-06	6.73E-06	9.72E-06
1.0	3.06E-10	4.14E-10	1.25E-06	1.75E-06

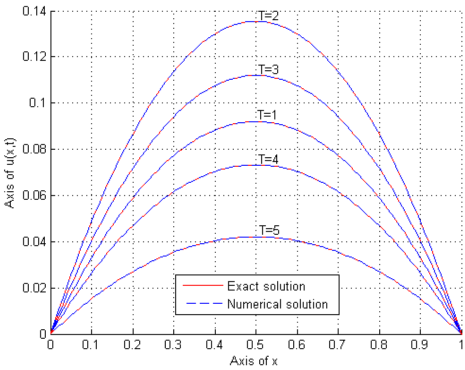


Figure 1: Exact and numerical solutions for $\alpha = 1$ at $T = 1, 2, 3, 4, 5$.

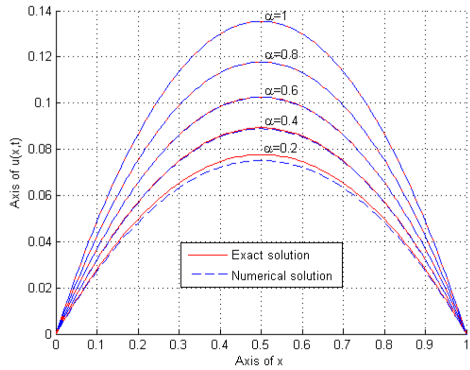


Figure 2: Exact and numerical solutions for $T = 2$ at $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$.

Example 8.2 ([3]). We consider the following data:

$$\begin{cases} a(x,t) = 6, \ \ b(x,t) = 2, \ \ \omega(x,t) = 1, \ \ \alpha = 1, \\ f(x,t) = \left[3\cos(t) - 12\sin(t) + (\beta-1)^2 x^{2-\beta}\cos(t)\right]\sin((\beta-1)x), \\ \varphi(x) = \sin((\beta-1)x), \ \psi(x) = 0, \\ g(t) = 0, \ \ h(t) = \cos(t)\sin(\beta-1), \\ \mu_1 = \mu_2 = 0, \ \lambda_1 = -1, \ \lambda_2 = 1. \end{cases}$$

For these data, the problem (1.1)-(1.3) is given by:

$$\begin{cases} u_{tt}(x,t) + 12u_t(x,t) + 4u(x,t) = \mathcal{D}_x^{(\beta)} u(x,t) + f(x,t), \\ u(x,0) = \sin((\beta-1)x), \ u_t(x,0) = 0, \ 0 \le x \le 1, \\ u(0,t) = 0, \ u(1,t) = \cos(t)\sin(\beta-1), \ 0 < t \le T. \end{cases}$$

The exact solution to this problem is $u_e(x,t) = \cos(t)\sin((\beta-1)x)$. This example has been studied by different numerical methods in [3], and in [24] for the case $\beta = 2$.

In Table 3, we compare the L_2 and L_∞ errors of our algorithm with results obtained by the numerical methods studied in [3, 24] for $\beta = 2$ at $T = 0.2, 0.4, 0.6, 0.8, 1.0$. With this comparative study, we conclude that our algorithm is very efficient. Figure 3 and Figure 4 represent the exact and numerical solutions for different values of T and β.

Table 3: L_2 and L_∞ errors with $\tau = 0.0001$, $h = 0.01$ and $\beta = 2$.

	Our algorithm		Abdelkebir [3]		Nazir et al. [24]	
T	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	4.93E-08	8.07E-08	5.07E-07	1.03E-06	2.96E-06	4.63E-06
0.4	4.27E-08	7.35E-08	3.48E-07	6.12E-07	6.77E-06	1.01E-05
0.6	3.97E-08	6.90E-08	9.23E-07	1.70E-06	9.81E-06	1.42E-05
0.8	3.40E-08	5.95E-08	1.69E-06	2.69E-06	1.20E-05	1.71E-05
1.0	2.67E-08	4.73E-08	2.01E-06	2.91E-06	1.34E-05	1.90E-05

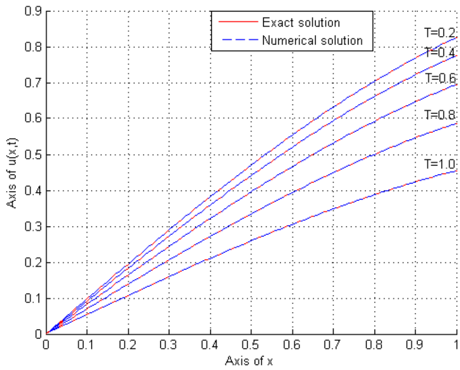


Figure 3: Exact and numerical solutions for $\beta = 2$ at $T = 0.2, 0.4, 0.6, 0.8, 1.0$.

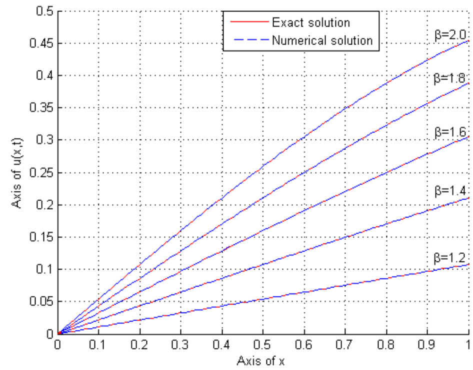


Figure 4: Exact and numerical solutions for $T = 1$ at $\beta = 1.2, 1.4, 1.6, 1.8, 2.0$.

Example 8.3 (Regular variable coefficients). In this example we have the following data:

$$\begin{cases} a\left(x,t\right) = e^{x+t}, \ \ b\left(x,t\right) = \sin\left(x+t\right), \ \ \omega\left(x,t\right) = 1 + x^2, \ \ \beta = \alpha + 1, \\ f(x,t) = \left(4t^{1-\alpha} - 4t^{1-\alpha}e^{x+t} + \sin^2\left(x+t\right) - \alpha^2 x^{1-\alpha}\left(1+x^2\right)\right) e^{-2t}\sin\left(\alpha x\right), \\ \varphi(x) = \sinh\left(\alpha x\right), \ \psi(x) = -2\sinh\left(\alpha x\right), \\ g(t) = 0, \ \ h(t) = e^{-2t}\sinh\left(\alpha\right), \\ \mu_1 = \mu_2 = 0, \ \lambda_1 = -1, \ \lambda_2 = 1. \end{cases}$$

For these data, the problem (1.1)-(1.3) is given by:

$$\begin{cases} \mathcal{D}_t^{(1+\alpha)} u\left(x,t\right) + 2e^{x+t}\mathcal{D}_t^{(\alpha)} u\left(x,t\right) + \sin^2\left(x+t\right) u\left(x,t\right) = \left(1+x^2\right)\mathcal{D}_x^{(1+\alpha)} u\left(x,t\right) + f(x,t), \\ u\left(x,0\right) = \sinh\left(\alpha x\right), \ u_t\left(x,0\right) = -2\sinh\left(\alpha x\right), \ 0 \le x \le 1, \\ u\left(0,t\right) = 0, \ u\left(1,t\right) = e^{-2t}\sinh\left(\alpha\right), \ 0 < t \le T. \end{cases}$$

The exact solution to this problem is $u_e\left(x,t\right) = e^{-2t}\sinh\left(\alpha x\right)$. For $\alpha = 1$, this example has been studied by different numerical methods in [21] and [25].

In Table 4, we compare the L_∞ error of our algorithm with the numerical methods studied in [21] and [25] for $T = 1, 2, 3, 4, 5$ with $\tau = 0.0005$, $h = 0.01$ and $\alpha = 1$. Thanks to this comparative study, we conclude that our algorithm is more efficient and reliable. The L_2 and L_∞ errors for different values of T and α are presented in Table 5. The space-time graph of the numerical solution for $T = 1$ and $\alpha = 1$ is given by Figure 5. Finally, Figure 6 represents the graphs of the numerical solution for $T = 1$ and $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

Table 4: L_∞ error for $T = 1, 2, 3, 4, 5$ with $\tau = 0,0005$, $h = 0.01$ and $\alpha = 1$.

	Our algorithm	Pandit et al. [25]	Lin et al.[21]
T	L_∞	L_∞	L_∞
1.0	4.7539E-08	8.1073E-03	1.2776E-04
2.0	2.2249E-08	1.1022E-03	1.9609E-05
3.0	1.8700E-08	1.4738E-04	3.9246E-06
4.0	1.7072E-08	2.0202E-05	8.8376E-07
5.0	1.6388E-08	2.8410E-06	5.2078E-08

Table 5: L_2 and L_∞ errors for different values of T and α with $\tau = 0.0005$, $h = 0.01$.

	T=1		T=2		T=3		T=4	
α	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	2.53E-09	3.70E-09	2.91E-09	4.32E-09	2.96E-09	4.33E-09	2.91E-09	4.26E-09
0.4	6.66E-09	9.57E-09	6.34E-09	9.13E-09	6.21E-09	8.88E-09	6.05E-09	8.64E-09
0.6	1.25E-08	1.82E-08	9.97E-09	1.41E-08	9.38E-09	1.32E-08	9.00E-09	1.27E-08
0.8	1.99E-08	2.89E-08	1.34E-08	1.86E-08	1.20E-08	1.68E-08	1.12E-08	1.58E-08
1.0	2.98E-08	4.75E-08	1.62E-08	2.22E-08	1.34E-08	1.87E-08	1.21E-08	1.71E-08

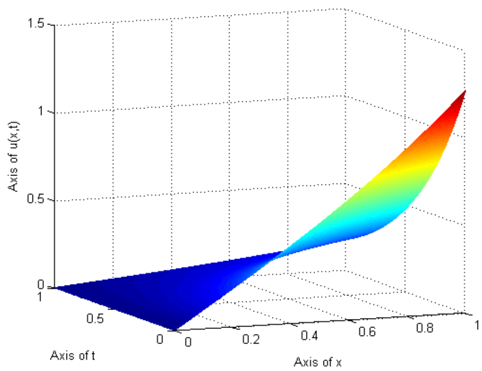


Figure 5: space-time graph of the numerical solution for $T = 1$ and $\alpha = 1$.

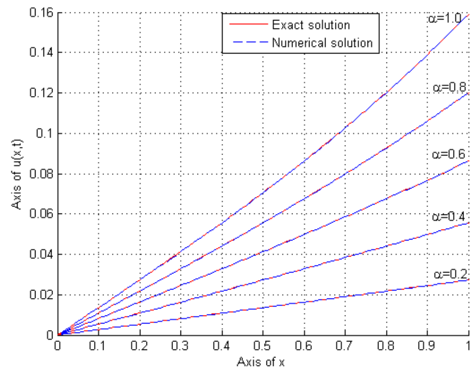


Figure 6: Graph of the numerical solution for $T = 1$ and $\alpha = 0.2, 0.4, 0.6, 0.8, 1$.

Example 8.4 (Singular coefficients). In this example we have the following data:

$$\begin{cases} a(x,t) = \frac{1}{x^2}, \quad b(x,t) = \frac{1}{x}, \quad \omega(x,t) = 1 + x^2, \quad \beta = \alpha + 1, \\ f(x,t) = \left(4t^{1-\alpha} + \frac{1-4t^{1-\alpha}}{x^2} - \alpha^2 x^{1-\alpha}\left(1 + x^2\right)\right) e^{-2t} \sinh(\alpha x), \\ \varphi(x) = \sinh(\alpha x), \quad \psi(x) = -2\sinh(\alpha x), \\ g(t) = 0, \quad h(t) = e^{-2t}\sinh(\alpha), \\ \mu_1 = \mu_2 = 0, \quad \lambda_1 = -1, \quad \lambda_2 = 1. \end{cases}$$

For these data, the problem (1.1)-(1.3) is given by:

$$\begin{cases} \mathcal{D}_t^{(1+\alpha)} u(x,t) + \frac{2}{x^2}\mathcal{D}_t^{(\alpha)} u(x,t) + \frac{1}{x^2} u(x,t) = \left(1 + x^2\right) \mathcal{D}_x^{(1+\alpha)} u(x,t) + f(x,t), \\ u(x,0) = \sinh(\alpha x), \quad u_t(x,0) = -2\sinh(\alpha x), \quad 0 \le x \le 1, \\ u(0,t) = 0, \quad u(1,t) = e^{-2t}\sinh(\alpha), \quad 0 < t \le T. \end{cases}$$

The exact solution to this problem is $u_e(x,t) = e^{-2t}\sinh(\alpha x)$. In Table 6, we compare the error L_∞ of our method with $\alpha = 1$ and $T = 1, 2, 3, 4, 5$, as well as the error L_∞ given in [21] and [25].

Example 8.5. In this example we have the following data:

$$\begin{cases} a(x,t) = x^2 + t^2, \quad b(x,t) = 2xt, \quad \omega(x,t) = 1 + x^2 t^2, \quad \beta = \alpha + 1, \\ f(x,t) = \left[(\alpha+2)(\alpha+1)t + 2(\alpha+2)t^2 + 4x^2 t^{\alpha+4}\right]\left(x^{\alpha+2} - \cos(\pi x)\right) \\ -\left(1 + x^2 t^2\right) x^{1-\alpha} t^{\alpha+2}\left[(\alpha+2)(\alpha+1)x^\alpha + \pi^2\cos(\pi x)\right], \\ \varphi(x) = \psi(x) = 0, \\ g(t) = -\lambda_1 t^{2+\alpha}, \quad h(t) = \left[(2+\alpha)\mu_2 + 2\lambda_2\right] t^{2+\alpha}. \end{cases}$$

Table 6: L_∞ error for $T = 1, 2, 3, 4, 5$ with $\tau = 0.0001$, $h = 0.01$ and $\alpha = 1$.

	Our algorithm	Pandit et al. [25]	Lin et al.[21]
T	L_∞	L_∞	L_∞
1.0	2.1983E-08	4.9526E-03	7.3025E-05
2.0	4.0365E-09	7.7597E-03	4.3352E-06
3.0	1.0183E-09	1.3503E-04	3.5757E-07
4.0	1.5180E-10	2.0606E-05	3.4843E-08
5.0	7.5489E-11	2.9512E-06	9.3298E-10

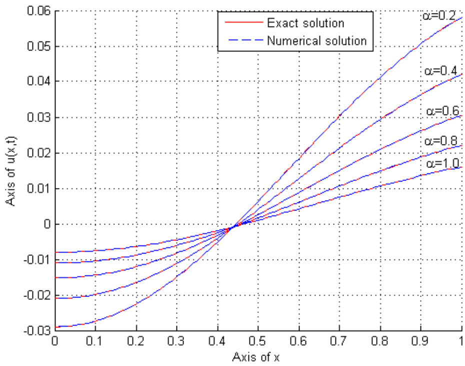
For these data, the problem (1.1)-(1.3) is given by:

$$\begin{cases} \mathcal{D}_t^{(1+\alpha)}u(x,t) + 2\left(x^2+t^2\right)\mathcal{D}_t^{(\alpha)}u(x,t) + 4x^2t^2u(x,t) = \left(1+x^2t^2\right)\mathcal{D}_x^{(1+\alpha)}u(x,t) + f(x,t), \\ u(x,0) = 0,\ u_t(x,0) = 0,\ 0 \le x \le 1, \\ \lambda_1 u(0,t) + \mu_1 u_x(0,t) = -\lambda_1 t^{2+\alpha}, \\ \lambda_2 u(1,t) + \mu_2 u_x(1,t) = \left[(2+\alpha)\mu_2 + 2\lambda_2\right]t^{2+\alpha},\ 0 < t \le T. \end{cases}$$

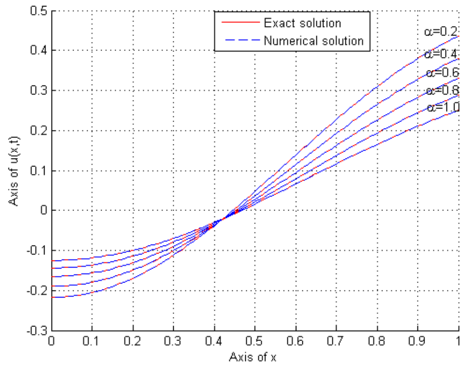
The exact solution to this problem is $u_e(x,t) = t^{2+\alpha}\left(x^{2+\alpha} - \cos(\pi x)\right)$. Table 7 contains errors L_2 and L_∞ for different values of T and α. The graphs of the exact and numerical solutions are given by Figure 7.

Table 7: L_2 and L_∞ errors for different values of T and α with $\tau = 0.0001$, $h = 0.01$.

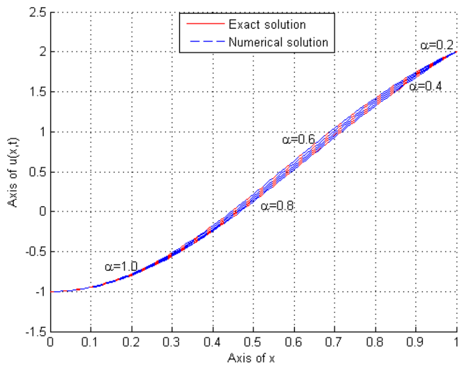
	T=0.2		T=0.5		T=1.0		T=2.0	
α	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	3.35E-06	6.61E-06	2.59E-05	5.45E-05	1.38E-04	3.71E-04	6.05E-04	1.60E-03
0.4	1.80E-06	3.54E-06	2.18E-05	4.94E-05	1.40E-04	3.86E-04	7.44E-04	2.01E-03
0.6	1.39E-06	2.61E-06	1.94E-05	3.37E-05	1.34E-04	2.51E-04	8.36E-04	1.58E-03
0.8	1.20E-06	2.11E-06	1.87E-05	3.24E-05	1.39E-04	2.38E-04	9.75E-04	1.74E-03
1.0	9.99E-07	1.69E-06	1.79E-05	3.06E-05	1.51E-04	2.57E-04	1.22E-03	2.09E-03



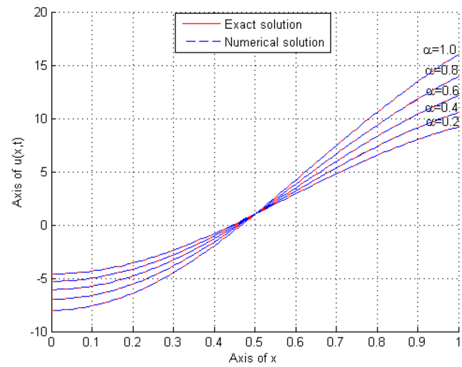
(a) $T = 0.2$



(b) $T = 0.5$



(c) $T = 1$



(d) $T = 2$

Figure 7: Graphs of exact and numerical solutions for different values of T and α with $h = 0.01$ and $\tau = 0.0001$.

9 Conclusion

We have proposed a numerical approach based on the Legendre collocation method and the Generalized-α method to compute the numerical solution of the conformable fractional telegraph equations with variable coefficients and Robin's boundary conditions. A comparative study between our algorithm and the numerical methods available in the literature. We obtain an excellent approximation with a small number of collocation points.

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