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ON GENERALIZED LEFT DERIVATIONS OF 3-PRIME NEAR-RINGS

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Abstract *Let $\mathcal{N}$ be a 3-prime near-ring with center $Z(\mathcal{N})$ and U a nonzero Lie ideal of $\mathcal{N}$. The aim of this paper is to prove some theorems showing that $\mathcal{N}$ must be commutative if it admits a nonzero generalized left derivation F with associated a left derivation D satisfying any one of the following properties: (i) $F(U) \subseteq Z(\mathcal{N})$, (ii) $F(U^2) = \{0\}$, (iii) $F(u) = u$ for all $u \in U$, and (iv) $D(U^2) \subseteq Z(\mathcal{N})$. We also give some examples to show that the hypotheses made in our results are not superfluous.*

1 Introduction

A right (resp. left) near-ring $\mathcal{N}$ is a triple $(\mathcal{N}, +, .)$ with two binary operations " + " and "." such that (i) $(\mathcal{N}, +)$ is a group (not necessarily abelian), (ii) $(\mathcal{N}, .)$ is a semigroup, (iii) $(x + y).z = x.z + y.z$ (resp. $x.(y + z) = x.y + x.z$) for all $x, y, z \in \mathcal{N}$. We denote by $Z(\mathcal{N})$ the multiplicative center of $\mathcal{N}$, and usually $\mathcal{N}$ will be 3-prime, if, for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A right (resp. left) near-ring $\mathcal{N}$ is a zero-symmetric if $x.0 = 0$ (resp. $0.x = 0$) for all $x \in \mathcal{N}$, (recall that right distributive yields $0.x = 0$ and left distributive gives $x.0 = 0$). For any pair of elements $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ stands for the Lie product. Recall that $\mathcal{N}$ is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. The Lie ideal U of $\mathcal{N}$ is an additive subgroup which has the property $[u, x] \in U$ for all $u \in U, x \in \mathcal{N}$. A Lie ideal U of $\mathcal{N}$ is said to be a square closed Lie ideal if $u^2 \in U$ for all $u \in U$. According to [12], an additive mapping $D : \mathcal{R} \to \mathcal{R}$ is a left derivation (resp. a Jordan left derivation) if $D(xy) = xD(y) + yD(x)$ (resp. $D(x^2) = 2xD(x)$) for all $x, y \in \mathcal{R}$. Obviously, every left derivation is a Jordan left derivation, but the converse is not generally true (see [17], Example 1.1.). Recently, M. Ashraf et al. [1] proved that the converse statement is true if the underlying ring is prime and 2-torsion free. According to [14], an additive mapping $G : \mathcal{N} \to \mathcal{N}$ is said to be a left generalized derivation of $\mathcal{N}$ if there exists a derivation $d : \mathcal{N} \to \mathcal{N}$ such that $G(xy) = xG(y) + d(x)y$ holds for all $x, y \in \mathcal{N}$. Inspired by the definition of generalized left derivation, Ashraf and Shakir [2] introduced the concepts of generalized left derivation and generalized Jordan left derivation on rings $\mathcal{R}$ as follows: an additive mapping $G : \mathcal{R} \to \mathcal{R}$ is called a generalized left derivation (respectively, generalized Jordan left derivation) if there exists a Jordan left derivation $\delta : \mathcal{R} \to \mathcal{R}$ such that $G(xy) = xG(y) + y\delta(y)$ (respectively, $G(x^2) = xG(x) + x\delta(x)$) holds for all $x, y \in \mathcal{R}$. In [16], S. Y. Kang and I. S. Chang introduced the concepts of generalized left derivation in algebra as follows: an additive mapping $G : \mathcal{A} \to \mathcal{A}$ is called a generalized left derivation if there exists a left derivation $d : \mathcal{A} \to \mathcal{A}$ such that $G(xy) = xG(y) + yd(x)$ for all $x, y \in \mathcal{N}$. Motivated by the concept of generalized left derivations in algebra (see, [16]), we introduce the concept of generalized left derivations in near rings with a similar manner: an additive mapping $F : \mathcal{N} \to \mathcal{N}$ is called a generalized left derivation if there exists a left derivation $D : \mathcal{N} \to \mathcal{N}$ such that $F(xy) = xF(y) + yD(x)$ for all $x, y \in \mathcal{N}$. It is obvious to see that every left derivation on a near-ring $\mathcal{N}$ is a generalized left derivation, but the opposite is not true in general. The following example justifies this:

Example 1.1. Let $\mathcal{R}$ be a near-ring. Define the set $\mathcal{N}$ and the maps $D, F : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathcal{R} \right\}$$

$$F \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \ D \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$$

It is easy to verify that F is a generalized left derivation of the near-ring $\mathcal{N}$ associated with the derivation D, but F is not a left derivation of $\mathcal{N}$.

In [6], Bergen showed that if U is a nonzero Lie ideal of a 2-torsion free prime ring $\mathcal{R}$ and d is a nonzero derivation of $\mathcal{R}$ such that $d(U) \subseteq Z(\mathcal{R})$, then $U \subseteq Z(\mathcal{R})$. In [7] and [9], the authors used Lie ideals and derivations to make a number of important discoveries, including the commutativity of addition in near-rings. In [15] Öznur Gölbaşi and K. Kaya has proved that if $\mathcal{R}$ is a prime ring of characteristic different from 2 which admits a nonzero Lie ideal U and if f is a nonzero generalized derivation associated with d of $\mathcal{R}$. Then we have the following results: (i) If $a \in \mathcal{R}$ and $[a, f(U)] = 0$ then $a \in Z(\mathcal{R})$ or $d(a) = 0$ for all $U \subseteq Z(\mathcal{R})$; (ii) If $f^2(U) = \{0\}$ then $U \subseteq Z(\mathcal{R})$, (iii) If $u^2 \in U$ for all $u \in U$ and f acts as a homomorphism or antihomomorphism on U then either $d = 0$ or $U \subseteq Z(\mathcal{R})$. It is my purpose to extend some comparable results to near-rings with generalized left derivation.

2 Some preliminaries

We start with the following lemmas they are essential for developing the proof of our results.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) [4, Lemma 1.2 (iii)] *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

(ii) [5, Lemma 3 (ii)] *If $Z(\mathcal{N})$ contains a nonzero element z of $\mathcal{N}$ which $z + z \in Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.*

(iii) [7, Lemma 3] *If $U \subseteq Z(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.*

Lemma 2.2 ([13], Theorem 3.1). *Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero left derivation d, then the following properties hold true:*

(i) If there exists a nonzero element a such that $d(a) = 0$, then $a \in Z(\mathcal{N})$.

(ii) $(\mathcal{N}, +)$ is abelian, if and only if $\mathcal{N}$ is a commutative ring.

(iii) [13, Lemma 3.2 (i)] *$d(U^2) = \{0\}$ if and only if $\mathcal{N}$ is a commutative ring.*

Lemma 2.3. *Let $\mathcal{N}$ be a 3-prime right near-ring. If $\mathcal{N}$ admits a nonzero generalized left derivation F associated with a left derivation D such that $F(a) = 0$, then*

$$a(xD(y) + yD(x)) = xaD(y) + yaD(x) \text{ for all } x, y \in \mathcal{N}.$$

Proof. Using the definition of F, we have

$$\begin{aligned} F(xya) &= xF(ya) + yaD(x) \\ &= x(yF(a) + aD(x)) + yaD(x) \\ &= xaD(x) + yaD(x) \text{ for all } x, y \in \mathcal{N}, \end{aligned}$$

and

$$\begin{aligned} F(xya) &= xyF(a) + aD(xy) \\ &= a(xD(y) + yD(x)) \text{ for all } x, y \in \mathcal{N}. \end{aligned}$$

By combining the last two expressions, we obviously have

$$a(xD(y) + yD(x)) = xaD(y) + yaD(x) \text{ for all } x, y \in \mathcal{N}.$$

□

3 Identities in 3-prime right near-rings with Lie ideals

Let $\mathcal{N}$ be a 2-torsion free 3-prime right near-ring. In [3], H. E. Bell proved that if $\mathcal{N}$ admits a non-zero generalized derivation f such that $f(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring. Our goal in the following result is to prove the same result by replacing the generalized derivation by a generalized left derivation without using the 2-torsion-free condition of $\mathcal{N}$.

Theorem 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation F associated with a left derivation D, then the following assertions are equivalent:*

(i) $F(U) \subseteq Z(\mathcal{N})$;

(ii) $\mathcal{N}$ is a commutative ring.

Proof. Clearly, (ii) implies (i).
$(i) \Rightarrow (ii)$ If $Z(\mathcal{N}) = \{0\}$, then

$$F(U) = \{0\}. \tag{3.1}$$

By (3.1), we can write $F([u, v]) = 0$ for all $u, v \in U$, which implies that $F(uv) = F(vu)$ for all $u, v \in U$. Using the definition of F together with (3.1), we get

$$uD(v) = vD(u) \text{ for all } u, v \in U. \tag{3.2}$$

Invoking (3.1), then $F[u, nu] = 0$ for all $u \in U, n \in \mathcal{N}$. Since $[u, nu] = [u, n]u$, we get $F([u, n]u) = 0$ for all $u \in U, n \in \mathcal{N}$, which means $[u, n]F(u) + ud([u, n]) = 0$ for all $u \in U, n \in \mathcal{N}$. By (3.1), we get $ud([u, n]) = 0$ for all $u \in U, n \in \mathcal{N}$, and from (3.2), we arrive at $[u, n]D(u) = 0$ for all $u \in U, n \in \mathcal{N}$, which gives

$$unD(u) = nuD(u) \text{ for all } u \in U, n \in \mathcal{N}. \tag{3.3}$$

Taking nm in place of n in (3.3) and using it again, we may write

$$unmD(u) = nmuD(u) \text{ for all } u \in U, m, n \in \mathcal{N}.$$

Which leads to

$$unmD(u) = numD(u) \text{ for all } u \in U, m, n \in \mathcal{N},$$

and therefore,

$$[u, n]mD(u) = 0 \text{ for all } u \in U, m \in \mathcal{N}.$$

So, $[u, n]\mathcal{N}D(u) = \{0\}$ for all $u \in U, n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$ and Lemma 2.2(i), we conclude that $U \subseteq Z(\mathcal{N})$. Using Lemma 2.1($iii$) and Lemma 2.2($ii$), we deduce that $\mathcal{N}$ is a commutative ring.
Now, suppose that $F(U) \subseteq Z(\mathcal{N})$ and $Z(\mathcal{N}) \neq \{0\}$, then there exists $z \in U \backslash \{0\}$ such that $F(z) \in Z(\mathcal{N})$ and $F(z) + F(z) = F(2z) \in Z(\mathcal{N})$, which implies that $(\mathcal{N}, +)$ is abelian by Lemma 2.1 (ii). Using Lemma 2.2(ii), we conclude that $\mathcal{N}$ is a commutative ring.

□

Corollary 3.2. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation F associated with a left derivation D, then the following assertions are equivalent:*

(i) $D(\mathcal{N}) \subseteq Z(\mathcal{N})$;

(ii) $D(U) \subseteq Z(\mathcal{N})$;

(iii) $F(\mathcal{N}) \subseteq Z(\mathcal{N})$;

(iv) *$\mathcal{N}$ is a commutative ring.*

Theorem 3.3. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation F associated with a left derivation D such that $F(U^2) = \{0\}$, then $\mathcal{N}$ is a commutative ring.*

Proof. Assume that

$$F(U^2) = \{0\}. \tag{3.4}$$

Invoking (3.4), then $F(v[u, nu]) = 0$ for all $u, v \in U, n \in \mathcal{N}$, using the definition of F and the fact that $[u, nu] = [u, n]u$, we get $vF([u, n]u) + [u, nu]D(v) = 0$ for all $u, v \in U, n \in \mathcal{N}$. Using (3.4), then the latter equation becomes $[u, n]uD(v) = 0$ for all $u, v \in U, n \in \mathcal{N}$, which gives

$$unuD(v) = nu^2D(v) \text{ for all } u \in U, n \in \mathcal{N}. \tag{3.5}$$

Taking nm in place of n in (3.5) and using it, we can write

$$unmD(v) = nmu^2D(v) \text{ for all } u \in U, m, n \in \mathcal{N}.$$

Which leads to

$$unmuD(v) = numuD(v) \text{ for all } u \in U, m, n \in \mathcal{N}.$$

And therefore,

$$[u, n]muD(v) = 0 \text{ for all } u, v \in U, n \in \mathcal{N}.$$

So $[u, n]\mathcal{N}uD(v) = \{0\}$ for all $u, v \in U, n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we may write

$$[u, n] = 0 \text{ or } uD(v) = 0 \text{ for all } u, v \in U, n \in \mathcal{N}. \tag{3.6}$$

Suppose that there exists $u_0 \in U$ such that

$$u_0D(v) = 0 \text{ for all } v \in U. \tag{3.7}$$

Using (3.4), then $F(vu_0) = 0$ for all $v \in U$, by the definition of F and (3.7), we get

$$vF(u_0) = 0 \text{ for all } v \in U. \tag{3.8}$$

Replacing v by $[u, n]$ in (3.8) and using it again, we obtain $unF(u_0) = 0$ for all $u \in U, n \in \mathcal{N}$, which gives $U\mathcal{N}F(u_0) = \{0\}$. Since $\mathcal{N}$ is 3-prime and $U \neq \{0\}$, we obtain

$$F(u_0) = 0 \tag{3.9}$$

Using (3.4) and (3.9) together with the definition of F, we get $u_0D(u_0) = 0$, then $D(u_0^2) = 0$, so $u_0^2 \in Z(\mathcal{N})$ by Lemma 2.2(i). Furthermore, $D(u_0^3) = 0$ then $u_0^3 \in Z(\mathcal{N})$ by Lemma 2.2(i). By lemma 2.1(i) we get $u_0 \in Z(\mathcal{N})$ or $u_0^2 = 0$.
If $u_0^2 = 0$, then $D(u_0^2n) = 0$ for all $n \in \mathcal{N}$, which implies $u_oD(u_0n) + u_0nD(u_0) = 0$ for all $n \in \mathcal{N}$. Using the lemma 2.3 and the fact that $u_0D(u_0) = 0$, we get $u_0\mathcal{N}D(u_0) = \{0\}$. By the 3-primeness of $\mathcal{N}$ together with the lemma 2.2(i), we deduce that $u_0 \in Z(\mathcal{N})$, then (3.6) becomes $U \subseteq Z(\mathcal{N})$. Using the Lemma 2.1($i$) and Lemma 2.2($ii$), we conclude that $\mathcal{N}$ is a commutative ring. □

Corollary 3.4. *Let $\mathcal{N}$ be a 3-prime near-ring, U be a nonzero Lie ideal of $\mathcal{N}$ and F is a generalized left derivation associated with a left derivation D. Then*

(i) If $D(U^2) = \{0\}$, then $\mathcal{N}$ is a commutative ring.

(ii) If $D(\mathcal{N}^2) = \{0\}$, then $\mathcal{N}$ is a commutative ring.

(iii) If $F(\mathcal{N}^2) = \{0\}$, then $\mathcal{N}$ is a commutative ring.

The following example proves that the 3-primeness of $\mathcal{N}$ cannot be omitted in the Theorem 3.1.

Example 3.5. Let $\mathcal{R}$ be a right near-ring which is not abelian. Define $\mathcal{N}$, U, d and F by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathcal{R} \right\}, \ U = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{pmatrix} \mid p, 0 \in \mathcal{R} \right\},$$

$$D \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \text{ and } F \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{pmatrix}$$

Then $\mathcal{N}$ is a right near-ring which is not 3-prime, U is a nonzero Lie ideal of $\mathcal{N}$, D is a nonzero left derivation of $\mathcal{N}$ which is not a derivation and F is a nonzero generalized left derivation with associated left derivation D of $\mathcal{N}$ which is not a left derivation. It is easy to see that

(i) $F(U) \subseteq Z(\mathcal{N})$;

(ii) $F(U^2) = \{0\}$.

However, $\mathcal{N}$ is not a commutative ring.

Theorem 3.6. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero closed Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a generalized left derivation F associated with a left derivation D such that $F(u) = u$ for all $u \in U$, then $\mathcal{N}$ is a commutative ring.*

Proof. Suppose that $F(u) = u$ for all $u \in U$, then $F([u, nu]) = [u, nu]$ for all $u \in U, n \in \mathcal{N}$. Using the definition of F with $[u, nu] = [u, n]u$, we get $uD([u, n]) = 0$ for all $u \in U, n \in \mathcal{N}$. Replace n with nu in the above equation and use it, we arrive at $u[u, n]D(u) = 0$ for all $u \in U, n \in \mathcal{N}$, which implies

$$u(unD(u) - nuD(u)) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{3.10}$$

Since $u^2 \in U$, by hypothesis given, we have $F(u^2) = u^2$ for all $u \in U$, and by the defintion of F, we obtain $uD(u) = 0$ for all $u \in U$, and therefore (3.10) becomes $u^2nD(u) = 0$ for all $u \in U, n \in \mathcal{N}$, which gives $u^2\mathcal{N}D(u) = \{0\}$ for all $u \in U$. By the 3-primeness of $\mathcal{N}$, we deduce that

$$u^2 = 0 \text{ or } D(u) = 0 \text{ for all } u \in U. \tag{3.11}$$

By Lemma 2.2(i), (3.11) becomes

$$u^2 = 0 \text{ or } u \in Z(\mathcal{N}) \text{ for all } u \in U. \tag{3.12}$$

Suppose that there exists an element $u_0 \in U$ such that $u_0^2 = 0$, then $D(u_0^2 n) = 0$ for all $n \in \mathcal{N}$, which implies that

$$u_0 D(u_0 n) + u_0 n D(u_0) = 0 \text{ for all } n \in \mathcal{N}. \tag{3.13}$$

On the other hand $F([u_0, nu_0]) = [u_0, nu_0]$ for all $n \in \mathcal{N}$, it follows that $F(u_0 n u_0) = u_0 n u_0$ for all $n \in \mathcal{N}$. Using the definition of F and $F(u_0) = u_0$, and simplifying we get $u_0 D(u_0 n) = 0$ for all $n \in \mathcal{N}$, then (3.13) becomes $u_0 n D(u_0) = 0$ for all $n \in \mathcal{N}$, thus $u_0 \mathcal{N} d(u_0) = \{0\}$ for all $u \in U$. By the 3-primeness of $\mathcal{N}$ and Lemma 2.2(i), we deduce $u_0 \in Z(\mathcal{N})$, and therefore (3.12) becomes $u \in Z(\mathcal{N})$ for all $u \in U$, which implies that $U \subseteq Z(\mathcal{N})$. Using Lemma 2.1$(iii)$ and Lemma 2.2(ii), we conclude that $\mathcal{N}$ is a commutative ring. ☐

Corollary 3.7. *Let $\mathcal{N}$ be a 3-prime near-ring, U be a nonzero closed Lie ideal of $\mathcal{N}$ and F is a generalized left derivation associated with a left derivation D of $\mathcal{N}$. Then*

(i) If $F(x) = x$ for all $x \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.

(ii) If $D(u) = u$ for all $u \in U$, then $\mathcal{N}$ is a commutative ring.

(iii) If $D(x) = x$ for all $x \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.

The following example proves that the 3-primeness of $\mathcal{N}$ cannot be omitted in the Theorem 3.6.

Example 3.8. Let $\mathcal{R}$ be a 2-torsion left near-ring which is not abelian. Define $\mathcal{N}$, U, D and F by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, 0 \in \mathcal{R} \right\}, \ U = \left\{ \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & q & 0 \end{pmatrix} \mid p, q, 0 \in \mathcal{R} \right\},$$

$$D \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}.$$

Then $\mathcal{N}$ is a left near-ring which is not 3-prime, U is a nonzero closed Lie ideal of $\mathcal{N}$, D is a nonzero left derivation of $\mathcal{N}$ which is not a derivation and F is a nonzero generalized left derivation with associated left derivation d of $\mathcal{N}$ which is not a left derivation. It is also easy to see that $F(u) = u$ for all $u \in U$. However, $\mathcal{N}$ is not a commutative ring.

4 Some results involving left derivations

In [5], H. E. Bell and G. Mason proved the following results: (i) If $\mathcal{N}$ is a 3-prime 2-torsion-free near-ring which admits a nonzero derivation D for which $D(\mathcal{N}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring; (ii) If $\mathcal{N}$ is 3-prime and 2-torsion-free and D is a derivation such that $D^2 = 0$, then $D = 0$. In the present section, our goal is to extend the above study to the setting of left derivations.

Theorem 4.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left derivation D, then the following assertions are equivalent:*

(i) $D(U^2) \subseteq Z(\mathcal{N})$;

(ii) $\mathcal{N}$ is a commutative ring.

Proof. It is easy to see that $(ii) \Rightarrow (i)$.
$(i) \Rightarrow (ii)$ If $Z(\mathcal{N}) = \{0\}$, then $D(U^2) = \{0\}$ and, by Lemma 2.2 (iii) we find that $\mathcal{N}$ is a commutative ring.
Now, suppose that $D(U^2) \subseteq Z(\mathcal{N})$ and $Z(\mathcal{N}) \neq \{0\}$, then $D(u^2) \in Z(\mathcal{N})$ for all $u \in U$, which gives $2uD(u) \in Z(\mathcal{N})$ for all $u \in U$. Substituting $[u, nu]$ for u in the last equation and using the fact that $[u, nu] = [u, n]u$, we obtain $2[u, nu]D([u, n]u) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, By Lemma 2.1 (i), we get

$$2[u, nu] \in Z(\mathcal{N}) \text{ or } D([u, nu]) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{4.1}$$

In view of Lemma 2.2 (i), (4.1) becomes

$$2[u, nu] \in Z(\mathcal{N}) \text{ for all } u \in U, n \in \mathcal{N}. \tag{4.2}$$

Replacing n by nu in (4.2), we obtain $2[u, nu]u \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$. Invoking Lemma 2.1 (i) together with the 2-torsion freeness of $\mathcal{N}$, we get

$$u \in Z(\mathcal{N}) \text{ or } [u, nu] = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{4.3}$$

Which implies that $[u, nu] = 0$ for all $u \in U, n \in \mathcal{N}$, then $unu = nu^2$ for all $u \in U, n \in \mathcal{N}$. Substituting nm for n in last equation and using it, we obtain $[u, n]\mathcal{N}u = \{0\}$ for all $u \in U, n \in \mathcal{N}$ and by 3-primeness of $\mathcal{N}$, we deduce $u \in Z(\mathcal{N})$, then (4.3) becomes $U \subseteq Z(\mathcal{N})$. The Lemma 2.1$(iii)$ and Lemma 2.2(ii) assure that $\mathcal{N}$ is a commutative ring. □

Corollary 4.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left derivation D, then the following assertions are equivalent:*

(i) $D(\mathcal{N}^2) \subseteq Z(\mathcal{N})$;

(ii) *$\mathcal{N}$ is a commutative ring .*

Theorem 4.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left derivation D, then the following assertions are equivalent:*

(i) $D^2(U) = \{0\}$;

(ii) $D^2(U^2) = \{0\}$;

(iii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is obvious that (iii) implies (i) and (ii) .
$(i) \Rightarrow (iii)$ Suppose that $D^2(U) = \{0\}$. By Lemma 2.2 (i), we get $D(U) \subseteq Z(\mathcal{N})$ and using Theorem 3.1, we conclude that $\mathcal{N}$ is a commutative ring.
$(ii) \Rightarrow (iii)$ Assume that $D^2(U^2) = \{0\}$. Using Lemma 2.2 (i), we obtain $D(U^2) \subseteq Z(\mathcal{N})$, and by Theorem 4.1, we find that $\mathcal{N}$ is a commutative ring.

□

Corollary 4.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left derivation D such that $D^2 = 0$, then $D = 0$.*

Proof. Suppose $D^2 = 0$, using the theorem 3.3, then $\mathcal{N}$ is a commutative ring. So $0 = D^2(x^2)y = 2D(x)D(x)y$ for all $x, y \in \mathcal{N}$, and the 2-torsion freeness of $\mathcal{N}$ forces that $D(x)\mathcal{N}D(x) = \{0\}$ for all $x \in \mathcal{N}$. So $D = 0$ by the 3-primeness of $\mathcal{N}$. □

The following example proves that the 3-primeness of $\mathcal{N}$ in Theorems 4.3 cannot be omitted.

Example 4.5. Let $\mathcal{R}$ be a 2-torsion free right near-ring which is not abelian. Define $\mathcal{N}$, J and D by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \mid r, s, t, 0 \in \mathcal{R} \right\}, U = \left\{ \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid p, 0 \in \mathcal{R} \right\},$$

$D \begin{pmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & o & 0 \end{pmatrix}$. Then, $\mathcal{N}$ is a right near-ring which is not 3-prime, U is a nonzero Lie ideal of $\mathcal{N}$, and D is a nonzero left derivation of $\mathcal{N}$ which is not a derivation. We can easily see that

(i) $D(U^2) \subseteq Z(\mathcal{N})$;

(ii) $D^2(U) = \{0\}$;

(iii) $D^2(U^2) = \{0\}$.

But $\mathcal{N}$ is not a commutative ring.

5 Results in right near-rings involving Lie ideals and right centralizers

The notion of generalized left derivations with $D = 0$ includes the notion of right centralizers (multipliers). An additive mapping $T : \mathcal{N} \to \mathcal{N}$ is a right centralizer (multiplier) if $T(xy) = xT(y)$ for all $x, y \in \mathcal{N}$. Our goal in this section is to establish similar results in [[8], Theorems 3.1, 3.11 and 4.1]. Furthermore, we investigate the structure of a 3-prime right near-ring $\mathcal{N}$ admitting a nonzero right centralizer T which satisfies certain differential identities on Lie ideals.

Theorem 5.1. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero right multiplier T such that $T([u, n]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, then $(\mathcal{N}, +)$ is abelian.*

Proof. Suppose that $Z(\mathcal{N}) = \{0\}$, then $T([u, n]) = 0$ for all $u \in U, n \in \mathcal{N}$. It follows that $T(un) = T(nu)$ for all $u \in U, n \in \mathcal{N}$, using the definition of T, then $uT(n) = nT(u)$ for all $u \in U, n \in \mathcal{N}$. Replacing u by $[u, m]$ in the last equation and using our assumption, then $[u, m]T(n) = 0$ for all $u \in U, n, m \in \mathcal{N}$. Now substituting xy instead of n and using the definition of T, we find $[u, m]xT(y) = 0$ for all $u \in U, m, x, y \in \mathcal{N}$, which gives $[u, m]\mathcal{N}T(y) = \{0\}$ for all $u \in U, m, y \in \mathcal{N}$. Since $T \neq 0$, we deduce from the 3-primeness of $\mathcal{N}$ that $U \subseteq Z(\mathcal{N})$, and Lemma 2.1 (i) assure that $(\mathcal{N}, +)$ is abelian.
Now assume that $Z(\mathcal{N}) \neq \{0\}$ and $T([u, n]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, then $T([[u, n], n[u, n]]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, using the definition of T and the fact that $[[u, n], n[u, n]] = [[u, n], n][u, n]$ for all $u \in U, n \in \mathcal{N}$, we

obtain $[[u,n],n]T([u,n]) \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$, using Lemma 2.1 (i), then $[[u,n],n] \in Z(\mathcal{N})$ or $T([u,n]) = 0$ for all $u \in U, n \in \mathcal{N}$. If there exists $u_0 \in U \setminus \{0\}$ such that $T([u_0,n]) = 0$ for all $n \in \mathcal{N}$, using the same previous demonstrations with the necessary modifications, we can easily find $u_0 \in Z(\mathcal{N})$, which gives $[[u_0,n],n] \in Z(\mathcal{N})$ for all $n \in \mathcal{N}$, then the latter expression becomes $[[u,n],n] \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$. Substituting $n[u,n]$ instead of n, we obtain $[[u,n],n][u,n] \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$. By Lemma 2.1 (i), we can see that $[u,n] \in Z(\mathcal{N})$ for all $u \in U, n \in \mathcal{N}$. Now substituting nu instead of n in the above equation, by Lemma 2.1 (i) we can obtain $U \subseteq Z(\mathcal{N})$ which assure that $(\mathcal{N},+)$ is abelian by Lemma 2.1 (i). □

Theorem 5.2. *Let $\mathcal{N}$ be a 3-prime near-ring and U a nonzero Lie ideal of $\mathcal{N}$. There is no nonzero right multiplier T satisfying any one of the following assertions:*

(i) $T(U) = \{0\}$;

(ii) $T(U^2) = \{0\}$.

Proof. (i) Suppose that $T(U) = \{0\}$, then $T([u,n]) = 0$ for all $u \in U, n \in \mathcal{N}$. Which implies $T(un) = T(nu)$ for all $u \in U, n \in \mathcal{N}$, using the definition of T, then $uT(n) = nT(u)$ for all $u \in U, n \in \mathcal{N}$. Using our hypothesis, then $uT(n) = 0$ for all $u \in U, n \in \mathcal{N}$. Replacing n by nm, we can easily arrive at $U\mathcal{N}T(m) = \{0\}$ for all $m \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime, we find that $U = \{0\}$ or $T = 0$, a contradiction.

(ii) Suppose that $T(U^2) = \{0\}$, then $T(u^2) = 0$ for all $u \in U$, it follows that

$$uT(u) = 0 \text{ for all } u \in U. \tag{5.1}$$

Using our hypothesis, then $T([u,nu]) = 0$ for all $u \in U, n \in \mathcal{N}$, thus $T(unu) = T(nu^2)$ for all $u \in U, n \in \mathcal{N}$, using the definition of T and (5.1), we find that $unT(u) = 0$ for all $u \in U, n \in \mathcal{N}$, which gives $u\mathcal{N}T(u) = \{0\}$ for all $u \in U$. By the 3-primeness of $\mathcal{N}$, we deduce that $T(u) = 0$ for all $u \in U$, which leads to a contradiction by (i). □

6 Left near-ring involving Lie ideals and generalized left derivations

The present section is motivated by [9, Lemma 3 (iii)] and [5, Theorem 2]. Our goal is to extend these results to 3-prime near rings admitting a non-zero left derivation.

Theorem 6.1. *Let $\mathcal{N}$ be a 3-prime near-ring, D be a left derivation of $\mathcal{N}$ and U be a nonzero Lie ideal of $\mathcal{N}$. Then, we have the following results:*

(i) If $aD(x) = 0$ for all $x \in \mathcal{N}$, then $a = 0$;

(ii) If $D([u,n]) = [u,n]$ for all $u \in U, n \in \mathcal{N}$, then $(\mathcal{N},+)$ is abelian.

Proof. (i) Suppose that $aD(x) = 0$ for all $x \in \mathcal{N}$. Replacing x by xy, we obtain $axD(y) + ayD(x) = 0$ for all $x, y \in \mathcal{N}$. Taking ya instead of y in the above equation, we get $axD(ya) = 0$ for all $x, y \in \mathcal{N}$, which gives $ax(yD(a) + aD(y)) = 0$ for all $x, y \in \mathcal{N}$, using our hypothesis, we find that $axyD(a) = 0$ for all $x, y \in \mathcal{N}$, it follows that $ax\mathcal{N}D(a) = \{0\}$ for all $x \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we deduce that $ax = 0$ or $D(a) = 0$ for all $x \in \mathcal{N}$, which implies that $axa = 0$ or $D(a) = 0$ for all $x \in \mathcal{N}$ and the 3-primeness hypothesis yields $a = 0$ or $D(a) = 0$.

If $D(a) = 0$, then

$$D(axy) = aD(xy) + xyD(a) = 0 \text{ for all } x, y \in \mathcal{N}.$$

On the other hand

$$\begin{aligned} D(axy) &= axD(y) + yD(xa) \\ &= axD(y) + y(xD(a) + aD(x)) \\ &= axD(y) \text{ for all } x, y \in \mathcal{N}. \end{aligned}$$

Comparing the two last expressions, we obtain $axD(y) = 0$ for all $x, y \in \mathcal{N}$, which implies that $a\mathcal{N}D(y) = \{0\}$ for all $y \in \mathcal{N}$ and by the 3-primeness of $\mathcal{N}$, we conclude that $a = 0$.

(ii) Assume that

$$D([u,n]) = [u,n] \text{ for all } u \in U, n \in \mathcal{N}. \tag{6.1}$$

Substituting un for n in (6.1) and using it together with the definition of D, one can easily see that

$$[u,n]D(u) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{6.2}$$

Applying D to both sides of the equation (6.1) and invoking it, we find that

$$[u,n]D^2(u) + D(u)D([u,n]) = 0 \text{ for all } u \in U, n \in \mathcal{N}. \tag{6.3}$$

Taking $[v,m]$ instead of u in (6.3) and using (6.2), it is obvious to see that

$$[[v,m],n][v,m] = 0 \text{ for all } v \in U, n, m \in \mathcal{N}. \tag{6.4}$$

Applying D to both sides of the equation (6.4) and invoking it, we find that

$$[v,m][[v,m],n] = 0 \text{ for all } v \in U, n, m \in \mathcal{N}. \tag{6.5}$$

Which implies that

$$[v,m]^2 n = [v,m]n[v,m] \text{ for all } v \in U, n, m \in \mathcal{N}$$

Taking nt in place of n in above equation and using it, we may write

$$[v,m]^2 nt = [v,m]nt[v,m] \text{ for all } v \in U, n, m \in \mathcal{N}.$$

Which gives

$$[v,m]n[v,m]t = [v,m]nt[v,m] \text{ for all } v \in U, n, m, t \in \mathcal{N}.$$

And therefore,

$$[v,m]n[[v,m],t] = 0 \text{ for all } v \in U, n, m, t \in \mathcal{N}.$$

So, $[v,m]\mathcal{N}[[v,m],t] = \{0\}$ for all , $v \in U, m, t \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we can write

$$[v,m] \in Z(\mathcal{N}) \text{ for all } v \in U, m \in \mathcal{N}. \tag{6.6}$$

Putting vm instead of v in (6.6) one can easily find $v[v,m] \in Z(\mathcal{N})$ for all , $v \in U, m \in \mathcal{N}$ and by Lemma 2.1(i), we deduce that $v \in Z(\mathcal{N})$ or $[v,m] = 0$ for all , $v \in U, m \in \mathcal{N}$, which leads to $U \subseteq Z(\mathcal{N})$ and by lemma 2.1(iii), we conclude that $(\mathcal{N},+)$ is abelian. □

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