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CAYLEY-YOSIDA INCLUSION PROBLEM, COMPATIBLE GENERALIZED RESOLVENT EQUATION AND RELATED GAP FUNCTION WITH ERROR BOUND

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Abstract This paper introduces a novel inclusion problem, the Cayley-Yosida inclusion problem, which is constructed using generalized versions of the Cayley and Yosida operators. We also investigate the corresponding resolvent equation for this problem. To solve the Cayley-Yosida inclusion problem, we propose a fixed point formulation algorithm and prove the existence of a solution along with its convergence. Our results are demonstrated with and without the use of uniform and smoothness properties of Banach space. Additionally, we develop a gap function for the Cayley-Yosida inclusion problem and estimate the error bound for its solution. Overall, this paper presents a comprehensive analysis of the Cayley-Yosida inclusion problem and its resolvent equation along with the gap function for a given inclusion problem.

1 Introduction

Variational inequality and inclusion problems are fundamental concepts in mathematical optimization that deal with finding solutions to a set of interrelated equations. A variational inequality problem involves finding a solution to an equation or system of equations, where the solution must satisfy certain constraints or inequalities. In contrast, an inclusion problem, introduced by A. Hassouni and A. Moudafi [3], requires identifying a point in the intersection of two sets, which may represent feasible solutions to several criteria simultaneously. These problems find applications in many areas such as economics, engineering, biology, and operations research among others. Effective solutions require sophisticated mathematical techniques including convex analysis, operator theory, and fixed-point theory among others. Solving these mathematical challenges is very important for effective decision-making processes in various fields. Variational inequality is generalized in many environments, see examples [12, 14]. Similarly, the inclusion problem is generalized by many authors, see examples [1, 2, 4, 5, 7, 16].
In Hilbert space $\mathcal{H}$ and let $S : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is a set-valued map then the inclusion problem is to solve for $u \in \mathcal{H}$ such that $0 \in S(u)$; in general inclusion problems help to study the optimization problems, equilibrium problems, variational inequality problems, etc., see example [8]. Fang and Huang [9] introduced the resolvent operator in 2004. After this, resolvent operators and related operators were used in many areas such as partial differential equations and many areas mainly in convex analysis, see example [6].
Since the Cayley and Yosida operators are used to solve many problems in different areas such as computer programming, economics, engineering, and financial modeling, etc., we have constructed an inclusion problem that includes the generalized version of the Cayley and Yosida operator, which we call it Cayley-Yosida inclusion problem.
Presently, the gap function came to be very useful in the area of optimization theory, it converts

variational inequalities or inclusion problems into identical problems in optimization for studying the existence of solutions and many more, for example, [12].
Let us consider $\mathcal{H}$ to be the Hilbert space then for an identity operator I and a set-valued map E, we define the resolvent operator, $\mathcal{R}^{E}_{I,\theta} : \mathcal{H} \longrightarrow \mathcal{H}$ as, for all $u \in \mathcal{H}$,

$$\mathcal{R}^{E}_{I,\theta}(u) = (I + \theta E)^{-1}(u). \tag{1.1}$$

Define the Cayley operator, $\mathcal{C}^{E}_{I,\theta} : \mathcal{H} \longrightarrow \mathcal{H}$ as, for all $u \in \mathcal{H}$,

$$\mathcal{C}^{E}_{I,\theta}(u) = (2\mathcal{R}^{E}_{I,\theta} - I)(u). \tag{1.2}$$

Define the Yosida operator, $\mathcal{Y}^{E}_{I,\theta} : \mathcal{H} \longrightarrow \mathcal{H}$ as, for all $u \in \mathcal{H}$,

$$\mathcal{Y}^{E}_{I,\theta}(u) = \frac{1}{\theta}(I - \mathcal{R}^{E}_{I,\theta})(u). \tag{1.3}$$

where $\theta > 0$.

In this paper, we consider generalized forms of (1.1), (1.2) and (1.3).

2 Preliminaries

Throughout this paper, let us consider $\mathcal{B}$ to be a real Banach space with norm $||.||$ and $\mathcal{B}^*$ be a topological dual space of $\mathcal{B}$ with duality pairing $\langle ., . \rangle$ between $\mathcal{B}$ and $\mathcal{B}^*$. For $u \in \mathcal{B}$, define a normalized duality map $\mathcal{F} : \mathcal{B} \longrightarrow 2^{\mathcal{B}^*}$ as:

$$\mathcal{F}(u) = \{h \in \mathcal{B}^* : \langle u, h \rangle = ||u||.||h||, ||u|| = ||h||\}.$$

Also the function $\rho_{\mathcal{B}}$: $[0, \infty) \longrightarrow [0, \infty)$, which is modulus of smoothness of $\mathcal{B}$ given as:

$$\rho_{\mathcal{B}}(s) = \sup\{\frac{||u + w||}{2} + \frac{||u - w||}{2} - 1 : ||u|| \leq 1, ||w|| \leq s\},$$

and if $\lim_{s \to 0} \frac{\rho_{\mathcal{B}}(s)}{s} = 0$, then $\mathcal{B}$ is uniformly.

Definition 2.1. A map $X : \mathcal{B} \longrightarrow \mathcal{B}$ is called:

- Lipschitz continuos, if $\forall\ u, w \in \mathcal{B}$, $\exists\ a_X > 0$ such that,

$$||X(u) - X(w)|| \leq a_X||u - w||.$$

- accretive, if $\forall\ u, w \in \mathcal{B}$, $\exists\ f(u - w) \in \mathcal{F}(u - w)$ such that,

$$\langle X(u) - X(w), f(u - w) \rangle \geq 0.$$

- strongly accretive, if $\forall\ u, w \in \mathcal{B}$, $\exists\ f(u - w) \in \mathcal{F}(u - w)$ and $l_X > 0$ such that,

$$\langle X(u) - X(w), f(u - w) \rangle \geq l_X||u - w||^2.$$

Definition 2.2. [9] Let X be a map then $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ is called X-accretive if E is accretive and for $\theta > 0$, $(X + \theta E)(\mathcal{B}) = \mathcal{B}$.

Definition 2.3. [9] Let $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a single-valued map and $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be a set-valued map which is X-accretive then the generalized resolvent operator is defined for all $u \in \mathcal{B}$ as,

$$\mathcal{R}^{E}_{X,\theta}(u) = (X + \theta E)^{-1}(u). \tag{2.1}$$

Theorem 2.4. *[9] Let $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a strongly accretive map with $k_X > 0$ and $M : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be a set-valued X-accretive map then for all $u, w \in \mathcal{B}$,*

$$||\mathcal{R}^{E}_{X,\theta}(u) - \mathcal{R}^{E}_{X,\theta}(w)|| \leq \frac{1}{k_X}||u - w||.$$

that is, the generalized resolvent operator, $\mathcal{R}^{E}_{X,\theta}$ is Lipschitz continuous.

Lemma 2.5. *[15] Consider $\mathcal{B}$ be a Banach space with uniform and smoothness properties and $\mathcal{F} : \mathcal{B} \longrightarrow 2^{\mathcal{B}^*}$ then*

(i) $||u+w||^2 \leq ||u||^2 + 2\langle w, f(u+w)\rangle, \forall f(u+w) \in \mathcal{F}(u+w), \forall u, w \in \mathcal{B}$

(ii) $\langle u-w, f(u)-f(w)\rangle \leq 2d^2\rho_{\mathcal{B}}(\frac{4||u-w||}{d}), \forall f \in \mathcal{F}, \forall u, w \in \mathcal{B}$

where $d = \sqrt{(\frac{||u||^2+||w||^2}{2})}$.

Definition 2.6. The generalized Cayley operator, $\mathcal{C}^E_{X,\theta} : \mathcal{H} \longrightarrow \mathcal{H}$ as, for all $u \in \mathcal{B}$,

$$\mathcal{C}^E_{X,\theta}(u) = (2\mathcal{R}^E_{X,\theta} - X)(u). \tag{2.2}$$

The generalized Yosida operator, $\mathcal{Y}^E_{X,\theta} : \mathcal{H} \longrightarrow \mathcal{H}$ as, for all $u \in \mathcal{B}$,

$$\mathcal{Y}^E_{X,\theta}(u) = \frac{1}{\theta}(X - \mathcal{R}^E_{X,\theta})(u). \tag{2.3}$$

Now defining an operator $\mathcal{L}^E_{X,\theta}$ by using generalized Cayley and generalized Yosida operators as:

$$\mathcal{L}^E_{X,\theta}(u) = \alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u), \tag{2.4}$$

for all $u \in \mathcal{B}$ and where α and β are real constants.

Proposition 2.7. *Let $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a Lipschitz continuous with $k_X > 0$ then the operator $\mathcal{L}^E_{X,\theta}$ defined by (2.4) is a Lipschitz continuous if $\exists\, \gamma > 0$ such that, for all $u, w \in \mathcal{B}$,*

$$||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| \leq \gamma||u-w||.$$

Proof.

$$\begin{aligned}
||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| &= ||\alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u) - (\alpha\mathcal{C}^E_{X,\theta}(w) + \beta\mathcal{Y}^E_{X,\theta}(w))|| \\
&= ||\alpha(\mathcal{C}^E_{X,\theta}(u) - \mathcal{C}^E_{X,\theta}(w)) - \beta(\mathcal{Y}^E_{X,\theta}(u) - \mathcal{Y}^E_{X,\theta}(w))|| \\
&\leq |\alpha|||\mathcal{C}^E_{X,\theta}(u) - \mathcal{C}^E_{X,\theta}(w)|| + |\beta|||\mathcal{Y}^E_{X,\theta}(u) - \mathcal{Y}^E_{X,\theta}(w)||,
\end{aligned}$$

now using (2.2) and (2.3),

$$\begin{aligned}
&\leq |\alpha|||2\mathcal{R}^E_{X,\theta}(u) - X(u) - 2\mathcal{R}^E_{X,\theta}(w) + X(w)|| \\
&\qquad + |\beta|||\frac{1}{\theta}(X(u) - \mathcal{R}^E_{X,\theta}(u)) - \frac{1}{\theta}(X(w) - \mathcal{R}^E_{X,\theta}(w))||
\end{aligned}$$

$$\implies ||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| \leq (2|\alpha| + \frac{|\beta|}{\theta})||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(w)|| + (\frac{|\beta|}{\theta} + 1)||X(u) - X(w)||,$$

by theorem 2.4 and Lipschitz continuity of X,

$$||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| \leq (\frac{3|\alpha|\theta + (1+|\beta|)|\alpha|}{\theta k_X})||u-w||,$$

let $\gamma = (\frac{3|\alpha|\theta+(1+|\beta|)|\alpha|}{\theta k_X}) > 0$, we have the result,

$$||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| \leq \gamma||u-w||.$$

□

Let us consider $X : \mathcal{B} \longrightarrow \mathcal{B}$ and $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ to be single-valued and set-valued maps respectively and consider the operator defined by (2.4). The following problem is considered: To find $u \in \mathcal{B}$ such that,

$$0 \in \mathcal{L}^E_{X,\theta}(u) + E(u) = \alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u) + E(u). \tag{2.5}$$

We call the above problem (2.5) as Cayley-Yosida inclusion problem.

Now we will discuss some cases regarding problem (2.5) as follows:

- If $\alpha = 0$ and $\beta = 0$ then the problem (2.5) changes to the problem of finding $u \in \mathcal{B}$ such that,
$$0 \in E(u). \tag{2.6}$$
The above problem (2.6) is very common classical inclusion problem.
- If $\alpha = 1$ and $\beta = 0$ then the problem (2.5) changes to the problem of finding $u \in \mathcal{B}$ such that,
$$0 \in \mathcal{C}^E_{X,\theta}(u) + E(u). \tag{2.7}$$
The above problem (2.7) is a Cayley inclusion problem which is studied in uniformly smooth Banach space by Ahmad et al [11].
- If $\alpha = 0$ and $\beta = 1$ then the problem (2.5) changes to the problem of finding $u \in \mathcal{B}$ such that,
$$0 \in \mathcal{Y}^E_{X,\theta}(u) + E(u). \tag{2.8}$$
The above problem (2.8) is Yosida inclusion problem including generalized Yosida operator defined by (2.3) which is studied by Ahmad et al [10].

It is noted that maps X, E and all the operators defined by (1.1), (1.2), (1.3), (2.1), (2.2) and (2.3) are continuous.

3 Cayley-Yosida Inclusion Problem

In this section, we will prove the lemma about fixed point formulation and construct the algorithm for the solution of the Cayley-Yosida inclusion problem (2.5), which includes generalized Cayley and generalized Yosida operators. We will also establish the existence and convergence result for the Cayley-Yosida inclusion problem (2.5).

Lemma 3.1. *The Cayley-Yosida inclusion problem* (2.5) *has solution* $u \in \mathcal{B}$ *if and only if,*
$$u = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)) \tag{3.1}$$

Proof. Let $u \in \mathcal{B}$ be a solution of the Cayley-Yosida inclusion problem (2.5) then,
$$0 \in \alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u) + E(u)$$
for $\theta > 0$,
$$\begin{aligned}
&0 \in \theta\alpha\mathcal{C}^E_{X,\theta}(u) + \theta\beta\mathcal{Y}^E_{X,\theta}(u) + \theta E(u)\\
\iff\ & X(u) + \theta E(u) = X(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u) - \theta\beta\mathcal{Y}^E_{X,\theta}(u)\\
\iff\ & (X+\theta E)(u) = (X(u) - \mathcal{R}^E_{X,\theta}(u)) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u) - \theta\beta\mathcal{Y}^E_{X,\theta}(u)\\
\iff\ & u = (X+\theta E)^{-1}(\theta \times \frac{1}{\theta}(X(u) - \mathcal{R}^E_{X,\theta}(u)) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u) - \theta\beta\mathcal{Y}^E_{X,\theta}(u))\\
\iff\ & u = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)).
\end{aligned}$$
□

Now using lemma 3.1, we establish the following iterative algorithm for solution of the Cayley-Yosida inclusion problem (2.5).

Iterative Algorithm 3.1.

Consider $u_0 \in \mathcal{B}$ and for $\theta > 0$,
$$u_1 = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_0) + \mathcal{R}^E_{X,\theta}(u_0) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_0)) \in \mathcal{B}.$$
Similarly assume,
$$u_2 = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_1) + \mathcal{R}^E_{X,\theta}(u_1) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_1)) \in \mathcal{B}.$$

By continuing the above process, we will obtain the sequence as follows:

$$u_n = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1})) \in \mathcal{B}; n = 1, 2, 3..., .$$

By using above iterative algorithm, next we prove existence and convergence result for the Cayley-Yosida inclusion problem (2.5).

Theorem 3.2. *Let $\mathcal{B}$ be a Banach space with uniform and smoothness property such that, for some $K > 0$, the modulus of smoothness, $\rho_{\mathcal{B}}(s) \leq Ks^2$. Also let $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a Lipschitz continuous map and a set-valued map $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be a X-accretive. Assume $\mathcal{R}^E_{X,\theta}$ follows theorem 2.4. Consider $\mathcal{Y}^E_{X,\theta}$ is a strongly accretive and a Lipschitz continuous with constants $r_1 > 0$ and $r_2 > 0$ respectively, and $\mathcal{C}^E_{X,\theta}$ is a Lipschitz continuous with constant $r_3 > 0$. For some $\theta > 0$, if the condition:*

$$|\theta - \frac{k_X^2 - 1}{(1+|\alpha|r_3)}| < \frac{\sqrt{1 - 2(1-\beta)r_2 + 64K(\frac{(1-\beta)}{k_X})^2}}{(1+|\alpha|r_3)} \tag{3.2}$$

is satisfied then the Cayley-Yosida inclusion problem (2.5) (for $\beta < 0$) has at least a solution $u \in \mathcal{B}$ and the sequence obtained from iterative algorithm (3.1) strongly converges to u.

Proof. Using algorithm (3.1),

$$\begin{aligned}||u_{n+1} - u_n|| = ||&\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_n) + \mathcal{R}^E_{X,\theta}(u_n) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_n)) \\ &- \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}))||,\end{aligned}$$

using theorem 2.4,

$$\begin{aligned}&\leq \frac{1}{k_X}||\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_n) + \mathcal{R}^E_{X,\theta}(u_n) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_n) - (\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) \\ &\qquad + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}))||,\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{k_X}||\theta(1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1})) - \theta\alpha(\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1}))|| \\ &\qquad + \frac{1}{k_X}||\mathcal{R}^E_{X,\theta}(u_n) - \mathcal{R}^E_{X,\theta}(u_{n-1})||\end{aligned}$$

$$\begin{aligned}&\leq \frac{\theta}{k_X}||(u_n - u_{n-1}) - (1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}))|| \\ &\qquad + \frac{\theta}{k_X}||(u_n - u_{n-1}) - \alpha(\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1}))|| + \frac{1}{k_X^2}||u_n - u_{n-1}||.\end{aligned} \tag{3.3}$$

Now we have,

$$\begin{aligned}&||(u_n - u_{n-1}) - \alpha(\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1}))|| \\ &\qquad \leq ||(u_n - u_{n-1})|| + |\alpha|||\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1})|| \\ &\qquad \leq (1+|\alpha|r_3)||(u_n - u_{n-1})||.\end{aligned} \tag{3.4}$$

Also we have,

$$\begin{aligned}&||(u_n - u_{n-1}) - (1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}))||^2 \\ &\qquad \leq ||u_n - u_{n-1}||^2 - 2(1-\beta)\langle\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}), \\ &\qquad\qquad f((u_n - u_{n-1}) - (1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1})))\rangle,\end{aligned}$$

$$\leq ||u_n - u_{n-1}||^2 - 2(1-\beta)\langle \mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}), f(u_n - u_{n-1})\rangle$$
$$+ \langle (u_n - u_{n-1}) - 2(1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1})) - (u_n - u_{n-1}), f((u_n - u_{n-1})$$
$$- (1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}))) - f(u_n - u_{n-1}) - f((u_n - u_{n-1}))\rangle,$$

using lemma 2.5 and given condition $\rho_{\mathcal{B}}(s) \leq Ks^2$, we have

$$||(u_n - u_{n-1}) - (1-\beta)(\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1}))||$$
$$\leq \sqrt{(1 - 2(1-\beta)r_1 + 64K(\frac{1-\beta}{k_X})^2)}||u_n - u_{n-1}||. \quad (3.5)$$

Using (3.4) and (3.5) to evaluate (3.3), we get

$$||u_{n+1} - u_n|| \leq \mu ||u_n - u_{n-1}||,$$
$$\text{where} \quad \mu = \frac{k_X\theta(\sqrt{(1 - 2(1-\beta)r_1 + 64K(\frac{1-\beta}{k_X})^2)} + (1 + |\alpha| r_3)) + 1}{k_X^2}.$$

Using condition (3.2), we find that $\mu < 1$ and this implies, the sequence $\{u_n\}$ is Cauchy and converges to some $u \in \mathcal{B}$.
Hence the result follows by lemma 3.1 as map X, E and all the operators defined by (2.2), (2.3) and (2.4) are continuous. □

Now removing uniform and smoothness properties of $\mathcal{B}$.

Theorem 3.3. *Let $\mathcal{B}$ be a Banach space, $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a Lipschitz continuous and strongly accretive maps and also a set valued map $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be a X-accretive. Assume $\mathcal{R}^E_{X,\theta}$ follows theorem 2.4. For some $\theta > 0$, if the condition:*

$$|\theta - \frac{k_X^2 - 2}{(|\alpha| + k_X a_X)}| < \frac{|\beta|(1 + k_X a_X)}{(|\alpha| + k_X a_X)} \quad (3.6)$$

is satisfied then the Cayley-Yosida inclusion problem (2.5) has at least a solution $u \in \mathcal{B}$ and the sequence obtained from iterative algorithm (3.1) strongly converges to u.

Proof. Using algorithm (3.1),

$$||u_{n+1} - u_n|| = ||\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_n) + \mathcal{R}^E_{X,\theta}(u_n) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_n))$$
$$- \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}))||,$$

using theorem 2.4,

$$\leq \frac{1}{k_X}||\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_n) + \mathcal{R}^E_{X,\theta}(u_n) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_n) - (\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1})$$
$$+ \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}))||$$

$$\leq \frac{\theta(1+|\beta|)}{k_X}||\mathcal{Y}^E_{X,\theta}(u_n) - \mathcal{Y}^E_{X,\theta}(u_{n-1})|| + \frac{1}{k_X}||\mathcal{R}^E_{X,\theta}(u_n) - \mathcal{R}^E_{X,\theta}(u_{n-1})||$$
$$+ \frac{\theta|\alpha|}{k_X}||\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1})||$$

$$\leq \frac{\theta(1+|\beta|)}{k_X}(||X(u_n) - X(u_{n-1})|| + \frac{1}{\theta}||\mathcal{R}^E_{X,\theta}(u_n) - \mathcal{R}^E_{X,\theta}(u_{n-1})||) + \frac{1}{k_X}||\mathcal{R}^E_{X,\theta}(u_n)$$
$$- \mathcal{R}^E_{X,\theta}(u_{n-1})|| + \frac{\theta|\alpha|}{k_X}(2||\mathcal{R}^E_{X,\theta}(u_n) - \mathcal{R}^E_{X,\theta}(u_{n-1})|| + ||X(u_n) - X(u_{n-1})||)$$

$$= (\frac{2+|\beta|+\theta|\alpha|}{k_X})||\mathcal{R}^E_{X,\theta}(u_n) - \mathcal{R}^E_{X,\theta}(u_{n-1})|| + (\frac{\theta+|\alpha|+|\beta|}{k_X})||X(u_n) - X(u_{n-1})||,$$

using theorem 2.4 and Lipschitz continuity of X, we get

$$||u_{n+1} - u_n|| \leq \delta||u_n - u_{n-1}||, \text{ where } \delta = \frac{2+|\beta|+\theta|\alpha|}{k_X^2} + a_X\frac{\theta+|\alpha|+|\beta|}{k_X}.$$

Using condition (3.6), we find that $\delta < 1$ and this implies, the sequence $\{u_n\}$ is cauchy and converges to some $u \in \mathcal{B}$.
Hence the result follows by lemma 3.1 as map X, E and all the operators defined by (2.2), (2.3) and (2.4) are continuous. □

4 Generalized Resolvent Equation for Cayley-Yosida Inclusion Problem

Consider the problem, to find $u, t \in \mathcal{B}$, such that,

$$\alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u) + \theta^{-1}\mathcal{F}^E_{X,\theta}(t) = 0, \tag{4.1}$$

where $\mathcal{F}^E_{X,\theta}(t) = (I - X(\mathcal{R}^E_{X,\theta}))(t)$ and $X(\mathcal{R}^E_{X,\theta}(t)) = (X(\mathcal{F}^E_{X,\theta}))(t)$.

The above problem (4.1) is a generalized resolvent equation which we call it generalized resolvent equation for the Cayley-Yosida inclusion problem (2.5).

Proposition 4.1. *Let $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a one to one map and for $\theta > 0$,*

$$u = \mathcal{R}^E_{X,\theta}(t), \tag{4.2}$$

$$t = \theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u), \tag{4.3}$$

then the generalized resolvent equation (4.1) has a solution $u, t \in \mathcal{B}$ if and only if the Cayley-Yosida inclusion problem (2.5) has a solution $u \in \mathcal{B}$.

Proof. Let $u, t \in \mathcal{B}$ be the solution of (4.1), then,

$$\begin{aligned}
&\theta\alpha\mathcal{C}^E_{X,\theta}(u) + \theta\beta\mathcal{Y}^E_{X,\theta}(u) = -\mathcal{F}^E_{X,\theta}(t)\\
&\Rightarrow \theta\alpha\mathcal{C}^E_{X,\theta}(u) + \theta\beta\mathcal{Y}^E_{X,\theta}(u) = X(\mathcal{R}^E_{X,\theta}(t)) - t\\
&\Rightarrow \theta\alpha\mathcal{C}^E_{X,\theta}(u) + \theta\beta\mathcal{Y}^E_{X,\theta}(u) = X(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)))\\
&\qquad -(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))\\
&\Rightarrow \theta\mathcal{Y}^E_{X,\theta}(u) = X(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))) - \mathcal{R}^E_{X,\theta}(u)\\
&\Rightarrow X(u) - \mathcal{R}^E_{X,\theta}(u) = X(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))) - \mathcal{R}^E_{X,\theta}(u)\\
&\Rightarrow X(u) = X(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))),
\end{aligned}$$

since $X : \mathcal{B} \longrightarrow \mathcal{B}$ is one to one map then,

$$u = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)).$$

this implies, $u \in \mathcal{B}$ is solution of the Cayley-Yosida inclusion problem (2.5), follows from lemma 3.1.

Conversely,
let $u \in \mathcal{B}$ be a solution of the Cayley-Yosida inclusion problem (2.5), then,

$$u = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)).$$

since $t = \theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)$ and hence putting $u = \mathcal{R}^E_{X,\theta}(t)$ in t , we have,

$$\Rightarrow t = \theta(1-\beta)\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) + \mathcal{R}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) - \theta\alpha\mathcal{C}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t))$$
$$\Rightarrow t - \theta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) - \mathcal{R}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) = -\theta\beta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) - \theta\alpha\mathcal{C}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t))$$
$$\Rightarrow (I - \theta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}))(t) - \mathcal{R}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) = -\theta\beta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) - \theta\alpha\mathcal{C}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t))$$
$$\Rightarrow (I - (X - \mathcal{R}^E_{X,\theta}))(\mathcal{R}^E_{X,\theta})(t) - \mathcal{R}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) = -\theta\beta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) - \theta\alpha\mathcal{C}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t))$$
$$\Rightarrow (I - X(\mathcal{R}^E_{X,\theta}))(t) = -\theta(\beta\mathcal{Y}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)) + \alpha\mathcal{C}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(t)))$$
$$\Rightarrow \alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u) + \theta^{-1}\mathcal{F}^E_{X,\theta}(t) = 0,$$

hence the result follows. □

By proposition (4.1), the generalized resolvent equation (4.1) can be written as follows:

$$t = X(u) - \alpha\mathcal{C}^E_{X,\theta}(u) - \beta\mathcal{Y}^E_{X,\theta}(u) + (1-\theta^{-1})\mathcal{F}^E_{X,\theta}(t), \tag{4.4}$$

Also, the equation (4.1) can be written as:

$$u = u - \Delta(t - \theta(\alpha\mathcal{C}^E_{X,\theta}(u) + \beta\mathcal{Y}^E_{X,\theta}(u))), \tag{4.5}$$

where Δ is a positive step size.
Next we will construct the iterative algorithm using (4.2) and (4.3).

Iterative Algorithm 4.1.

Let $u_0, t_0 \in \mathcal{B}$ then,

$$u_0 = \mathcal{R}^E_{X,\theta}(t_0),$$

and

$$t_1 = \theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_0) + \mathcal{R}^E_{X,\theta}(u_0) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_0),$$

similarly, we can obtain the following iterative algorithm for the sequences $\{u_n\}$ and $\{t_n\}$,

$$u_{n-1} = \mathcal{R}^E_{X,\theta}(t_{n-1}), \tag{4.6}$$

and

$$t_n = \theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}); n =, 1, 2, 3, ..., . \tag{4.7}$$

Next, the iterative algorithm is constructed by (4.4) as follows:

Iterative Algorithm 4.2.

Let $u_0, t_0 \in \mathcal{B}$, we can find the following sequences $\{u_n\}$ and $\{t_n\}$:

$$u_{n-1} = \mathcal{R}^E_{X,\theta}(t_{n-1}), \tag{4.8}$$

and

$$t_n = X(u) - \alpha\mathcal{C}^E_{X,\theta}(u_{n-1}) - \beta\mathcal{Y}^E_{X,\theta}(u_{n-1}) + (1-\theta^{-1})\mathcal{F}^E_{X,\theta}(t_{n-1}); n =, 1, 2, 3, ..., . \tag{4.9}$$

Next we have constructed the iterative algorithm for (4.5) as follows:

Iterative Algorithm 4.3.

Let $u_0, t_0 \in \mathcal{B}$, we can obtain following sequences $\{u_n\}$ and $\{t_n\}$:

$$u_n = u_{n-1} - \Delta(t_{n-1}) - \theta(\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}) + \beta\mathcal{Y}^E_{X,\theta}(u_{n-1})); n = 1, 2, 3, ..., . \tag{4.10}$$

Theorem 4.2. *Let $\mathcal{B}$ be a Banach space, $X : \mathcal{B} \longrightarrow \mathcal{B}$ be a Lipschitz continuous map and $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be a X-accretive set-valued map. Assume $\mathcal{R}^E_{X,\theta}$, resolvent operator follows theorem 2.4. For some $\theta > 0$, if condition (3.6) given in theorem 3.3, that is,*

$$|\theta + \frac{|\beta|(1 + k_X a_X)}{(|\alpha| + k_X a_X)}| < \frac{k_X^2 + k_X a_X|\alpha| + 2}{(|\alpha| + k_X a_X)}$$

is satisfied then (4.1) has at least a solution $u \in \mathcal{B}$ and the sequences $\{u_n\}$ and $\{t_n\}$ obtained from iterative algorithm (4.1) strongly converges to u and t respectively.

Proof. Using iterative algorithm (4.1),

$$||t_{n+1} - t_n|| \leq ||\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_n) + \mathcal{R}^E_{X,\theta}(u_n) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1}) - (\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1})||$$

using same process as used in theorem 3.3,

$$||t_{n+1} - t_n|| \leq \delta||u_n - u_{n-1}||,$$

where

$$\delta = \frac{2 + |\beta| + \theta|\alpha|}{k_X^2} + a_X\frac{\theta + |\alpha| + |\beta|}{k_X}.$$

hence from (4.2), it is clear that,

$$||u_n - u_{n-1}|| \leq \frac{1}{k_X}||t_n - t_{n-1}||$$

Using condition (3.6), we find that $\delta < 1$ and this implies, the sequences $\{u_n\}$ and $\{t_n\}$ are Cauchy and converge to some $u, t \in \mathcal{B}$ respectively.
Hence the result follows by the proposition (4.1) as map X, E, all the operators defined by (2.2), (2.3) and (2.4) are continuous. □

4.1 Example

Example

Let a single-valued map, $X : \mathcal{B} \longrightarrow \mathcal{B}$ be given by, $X(u) = \frac{3}{2}u$ and a set-valued map, $E : \mathcal{B} \longrightarrow 2^{\mathcal{B}}$ be given by, $E(u) = \{\frac{1}{5}u\}$.

Now first we will show that X is a Lipschitz continuous.

$$||X(u) - X(w)|| = ||\frac{3}{2}u - \frac{3}{2}w|| = \frac{3}{2}||u - w|| \leq 2||u - w||,$$

that is, X satisfied the Lipschitz continuity with $a_X = 2$.

Now the generalized resolvent operator $\mathcal{R}^E_{X,\theta}$, for $\theta = 1$,

$$\mathcal{R}^E_{X,\theta}(u) = (X + \theta E)^{-1}(u) = \frac{10}{17}u,$$

hence,

$$||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(w)|| = ||\frac{10}{17}u - \frac{10}{17}w|| = \frac{10}{17}||u - w|| \leq \frac{2}{3}||u - w||,$$

implies that $\mathcal{R}^E_{X,\theta}$ is a Lipschitz continuous with $k_X = \frac{3}{2}$.

Now the generalized Cayley operator $\mathcal{C}^E_{X,\theta}$ and the generalized Yosida operator $\mathcal{Y}^E_{X,\theta}$,

$$\mathcal{C}^E_{X,\theta}(u) = (2\mathcal{R}^E_{X,\theta} - X)(u) = -\frac{11}{34}u,$$

and

$$\mathcal{Y}^E_{X,\theta}(u) = \frac{1}{\theta}(X - \mathcal{R}^E_{X,\theta})(u) = \frac{31}{34}u,$$

so that for $\alpha = 1$ and $\beta = 1$, the operator defined by (2.4), $\mathcal{L}^E_{X,\theta}(u) = \frac{10}{17}u$ and hence,

$$||\mathcal{L}^E_{X,\theta}(u) - \mathcal{L}^E_{X,\theta}(w)|| = ||\frac{10}{17}x - \frac{10}{17}y|| = \frac{10}{17}||u - w|| \leq \frac{12}{17}||u - w||,$$

that is $\mathcal{L}^E_{X,\theta}$ is a Lipschitz continuous with $\gamma = \frac{12}{17}$.

By using above considered argument, the condition (3.6) given in theorem 3.3 is satisfied.

Hence the sequence $\{u_n\}$ is computed by the above argument as:

$$u_n = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u_{n-1}) + \mathcal{R}^E_{X,\theta}(u_{n-1}) - \theta\alpha\mathcal{C}^E_{X,\theta}(u_{n-1})),$$
$$\implies u_n = \mathcal{R}^E_{X,\theta}(\frac{9}{34}u_{n-1}),$$
$$\implies u_n = \frac{45}{289}u_{n-1}.$$

The above sequence converges to 0 as $n \to \infty$.

Moreover from above argument and the iterative algorithm (4.1), we have,

$$u_{n-1} = \frac{10}{17}t_{n-1}$$

and

$$t_n = \frac{45}{289}t_{n-1}.$$

5 Gap Function and Error Bound

The theory of gap function for the Cayley inclusion problem (2.6) and the Yosida inclusion problem (2.8), and related error bound is not studied yet. So due to this fact, in this section, we discussed the theory of gap function for the Cayley-Yosida inclusion problem (2.7) and then obtained error bound for the solution with the help of gap function for Cayley-Yosida inclusion problem (2.5).

First, we define the gap function for the Cayley-Yosida inclusion problem (2.5).

Definition 5.1. A function $\eta : \mathcal{B} \longrightarrow \mathbb{R}$, is known to be gap (merit) function for (2.5) if,

(i) $\eta(u) \geq 0, \forall u \in \mathcal{B}$,

(ii) $\eta(u^*) = 0$ if and only if u^* solves (2.5).

For $u \in \mathcal{B}$, let us define residual function, $\mathbb{G} : \mathcal{B} \longrightarrow \mathbb{R}$ by,

$$\mathbb{G}(u) = u - \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)) \tag{5.1}$$

Theorem 5.2. *Let $\mathbb{G} : \mathcal{B} \longrightarrow \mathbb{R}$ be a function given by (5.1), then $||\mathbb{G}(u)||$ is a gap function for (2.5) if and only if u solves (2.5).*

Proof. Clearly for all $u \in \mathcal{B}$, $||\mathbb{G}(u)|| \geq 0$ and rest is obvious by lemma 3.1. □

Theorem 5.3. *Let $\mathbb{G} : \mathcal{B} \longrightarrow \mathbb{R}$ be a function given by (5.1) and u^* be the solution of (2.5). Also consider the condition given by (3.6) is satisfied then, for all $u \in \mathcal{B}$ and for $\Lambda, \lambda > 0$,*

$$\Lambda||\mathbb{G}(u)|| \leq ||u - u^*|| \leq \lambda||\mathbb{G}(u)||, \tag{5.2}$$

Proof. Since u^* solves (2.5), then,

$$u^* = \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*))$$

for $u \in \mathcal{B}$,

$$||u - u^*|| = ||u - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*)))||$$

$$\implies ||u-u^*|| \leq ||u - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u)))|| + ||(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))) - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*)))||$$

using theorem 2.4,

$$\implies ||u-u^*|| \leq ||\mathbb{G}(u)|| + \frac{1}{k_X}||\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u) - (\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*))||$$

$$\implies ||u-u^*|| \leq ||\mathbb{G}(u)|| + \frac{\theta(1+|\beta|)}{k_X}||\mathcal{Y}^E_{X,\theta}(u) - \mathcal{Y}^E_{X,\theta}(u^*)|| + \frac{1}{k_X}||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(u^*)|| + \frac{\theta|\alpha|}{k_X}||\mathcal{C}^E_{X,\theta}(u_n) - \mathcal{C}^E_{X,\theta}(u_{n-1})||$$

$$\implies ||u-u^*|| \leq ||\mathbb{G}(u)|| + \frac{\theta(1+|\beta|)}{k_X}(||X(u) - X(u^*)|| + \frac{1}{\theta}||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(u^*)||) + \frac{1}{k_X}||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(u^*)|| + \frac{\theta|\alpha|}{k_X}(2||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(u^*)|| + ||X(u) - X(u^*)||)$$

$$\implies ||u-u^*|| \leq ||\mathbb{G}(u)|| + (\frac{2+|\beta|+\theta|\alpha|}{k_X})||\mathcal{R}^E_{X,\theta}(u) - \mathcal{R}^E_{X,\theta}(u^*)|| + (\frac{\theta+|\alpha|+|\beta|}{k_X})||X(u) - X(u^*)||,$$

using theorem 2.4 and Lipschitz continuity of X,

$$||u-u^*|| \leq ||\mathbb{G}(u)|| + (\frac{2+|\beta|+\theta|\alpha|}{k_X^2} + a_X\frac{\theta+|\alpha|+|\beta|}{k_X})||u-u^*||,$$

put,

$$\lambda = \frac{1}{1 - (\frac{2+|\beta|+\theta|\alpha|}{k_X^2} + a_X\frac{\theta+|\alpha|+|\beta|}{k_X})}$$

we have,

$$||u-u^*|| \leq \lambda||\mathbb{G}(u)||,$$

hence by condition (3.6), we have $\lambda > 0$.
Since for $u \in \mathcal{B}$,

$$||\mathbb{G}(u)|| = ||u - \mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))||$$

$$\implies ||\mathbb{G}(u)|| = ||u - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))) - (u^* - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*))))||$$

$$\implies ||\mathbb{G}(u)|| \leq ||u-u^*|| + +||(\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u))) - (\mathcal{R}^E_{X,\theta}(\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*)))||$$

using theorem 2.4,

$$\implies ||\mathbb{G}(u)|| \leq ||u-u^*|| + + \frac{1}{k_X}||\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\alpha\mathcal{C}^E_{X,\theta}(u) - (\theta(1-\beta)\mathcal{Y}^E_{X,\theta}(u^*) + \mathcal{R}^E_{X,\theta}(u^*) - \theta\alpha\mathcal{C}^E_{X,\theta}(u^*))||$$

similarly from above process, we have,

$$||\mathbb{G}(u)|| \leq ||u-u^*|| + (\frac{2+|\beta|+\theta|\alpha|}{k_X^2} + a_X\frac{\theta+|\alpha|+|\beta|}{k_X})||u-u^*||,$$

put,

$$\Lambda = \frac{1}{1+(\frac{2+|\beta|+\theta|\alpha|}{k_X^2} + a_X\frac{\theta+|\alpha|+|\beta|}{k_X})} > 0$$

hence,

$$||u-u^*|| \geq \Lambda||\mathbb{G}(u)||.$$

□

Corollary 5.4. *If $\beta = 0$ and $\alpha = 1$ with all assumptions considered in the theorems 5.2 and 5.3 then the residual function $||\mathbb{G}_1(u)|| = ||u - \mathcal{R}^E_{X,\theta}(\theta\mathcal{Y}^E_{X,\theta}(u) + \mathcal{R}^E_{X,\theta}(u) - \theta\mathcal{C}^E_{X,\theta}(u))||$ is a gap function for the Cayley inclusion problem (2.6) and error bound,*

$$\Lambda_1||\mathbb{G}_1(u)|| \leq ||u-u^*|| \leq \lambda_1||\mathbb{G}_1(u)||,$$

where $\Lambda_1, \lambda_1 > 0$.

Corollary 5.5. *If $\beta = 1$ and $\alpha = 0$ with all assumptions considered in the theorems 5.2 and 5.3 then the residual function $||\mathbb{G}_2(u)|| = ||u - \mathcal{R}^E_{X,\theta}(\mathcal{R}^E_{X,\theta}(u))||$ is a gap function for the Yosida inclusion problem (2.7) and error bound,*

$$\Lambda_2||\mathbb{G}_2(u)|| \leq ||u-u^*|| \leq \lambda_2||\mathbb{G}_2(u)||,$$

where $\Lambda_2, \lambda_2 > 0$.

The above corollaries 5.4 and 5.5 can be proved by similar process as we proved theorems 5.2 and 5.3, which are not discussed yet.

6 Conclusion and Remarks

The purpose of this study is to delve into the theory of the Cayley-Yosida inclusion problem and its related gap function. We have developed an algorithm to solve the Cayley-Yosida inclusion problem and its corresponding resolvent equation. Additionally, we have explored the existence of theorems for the Cayley-Yosida inclusion problem and its related resolvent equation. As the gap function is a crucial component of optimization theory, we have dedicated the final section of our work to constructing gap function for the Cayley-Yosida inclusion problem, utilizing various values of α and β. Our findings provide valuable insights into this complex problem and offer practical solutions for optimization challenges.

References

[1] Q.B. Zhang, X.P. Ding and C.Z. Chang, *Resolvent operator technique for generalized implicit variational-like inclusion in Banach spaces*, J. Math. Anal. Appl., **361**, 283–292, (2010).

[2] C. Zhang and Y. Wang *Proximal algorithm for solving monotone variational inclusions*. Optimization, **67(8)**, 1197–1209, (2018).

[3] A. Hassouni and A. Moudafi, *A perturbed algorithm for variational inclusions*. J. Math. Anal. Appl., **183(3)**, 706–712, (1994).

[4] R. Ahmad and Q.H. Ansari, *An iterative algorithm for generalized nonlinear variational inclusions*, Appl. Math. Lett., **13(5)**, 23–26, (2002).

[5] A. Moudafi, *A duality algorithm for solving general variational inclusions*, Adv. Model Optim.,**13(2)**, 213–220, (2011).

[6] H.W. Cao, *Yosida approximation equations technique for system of generalized set-valued variational inclusions*, J. Inequal. Appl., 455, (2013). https://doi.org/10.1186/1029-242X-2013-455

[7] X. Zhao, D.R. Sahu and C.F. Wen, *Iterative methods for system of variational inclusions involving accretive operators and applications*. Fixed Point Theory, **19(2)**, 801–822, (2018).

[8] R. Ahmad, Q.H. Ansari and S.S. Irfan, *On generalized mixed co-quasi-variational inequalities with noncomapct valued mappings*, Bull. Austral. Math. Soc., **70**, 7–15, (2004).

[9] Y.P. Fang and N.J. Huang, *H-accretive operator and resolvent operator technique for variational inclusions in Banach spaces*, Appl. Math. Lett., **17(6)**, 647–653, (2004).

[10] R. Ahmad, M. Ishtiyak, M. Rahaman and I. Ahmad, *Graph convergence and generalized Yosida approximation with an application*, Math. Sci., **11**, 155-163, (2017).

[11] R. Ahmad, I. Ali, M. Rahaman, M. Ishtiyak and J.C. Yao, *Cayley inclusion problem with its corresponding generalized resolvent equation problem in uniformly smooth Banach spaces*, Appl. Anal., 1354-1368, (2022). https://doi.org/10.1080/00036811.2020.1781822

[12] Z. Khan and A.F. Ansari, *Gap function with error bound for random generalized variational-like inequality problems*, J. Sci. Arts, **53**, 917-930 (2020).

[13] S. S. Chang and Y. G. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets Syst., **32**, 359–367, (1989).

[14] Q.H. Ansari, M. Islam and J.C. Yao, *Nonsmooth variational inequalities on Hadamard manifolds*, Appl. Anal., **99(2)**, 340-358, (2018).

[15] W.V. Petershyn, *A characterization of strictly convexity of Banach spaces and other uses of duality mappings*, J. Funct. Anal., **6**, 282–291, (1970).

[16] S. Shafi, *A new convergence theorem for a system of variational-like inclusions and fixed point problems*, Palest. J. Math.**12(1)**, 955–967, (2023).

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