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RESULTS FOR NONLINEAR IMPULSIVE HYBRID DIFFERENTIAL EQUATION WITH LINEARr AND NONLINEAR PERTURBATION

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Abstract *In this manuscript, we prove the existence and uniqueness of solutions for impulsive nonlinear hybrid fractional differential equations. This encompasses scenarios involving both linear and nonlinear perturbations. Our methodology is rooted in the nonlinear alternative of Leray-Schauder type, coupled with the application of Banach's fixed-point theorem. Moreover, we offer an illustrative example to demonstrate the practical applicability of our findings.*

1 Introduction

Fractional calculus explores the theory and applications of integrals and derivatives of non-integer orders. This field of mathematical analysis, extensively investigated in recent years, has proven to be a potent tool for mathematically modeling various engineering and scientific phenomena. The widespread appeal of this subject can be attributed to the nonlocal nature of fractional-order operators.

The use of fractional-order operators is particularly valuable in describing the hereditary properties of numerous materials and processes, as evidenced by a shift in focus from classical integer-order models to fractional-order models in the relevant literature. Notably, this characteristic has found applications in applied and biomedical sciences and engineering, as highlighted in books such as [4, 12].

Recent contributions to the field can be explored in works like [19, 17, 32, 33] and their associated references. The examination of coupled systems of fractional-order differential equations holds significance, especially in biosciences, and interested readers can delve into papers like [18, 25] and their referenced works for detailed insights and examples.

Researchers have also delved into the study of hybrid fractional differential equations, a class involving the fractional derivative of an unknown function hybridized with a dependent nonlinearity. Noteworthy results on hybrid differential equations can be found in a series of papers, including [16, 13]. For more details about hybrid differential equations, we refer to [6, 8, 20, 21, 5].

The authors of [25], S. Melliani, A. El Allaoui, and L. S. Chadli, examined a boundary value problem involving nonlinear hybrid differential equations with both linear and nonlinear perturbations.

$$\begin{cases} \frac{d}{dt}\Big(\vartheta(\hat{\varkappa})\eta(\hat{\varkappa},\vartheta(\hat{\varkappa}))-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\xi(\hat{\varkappa},\vartheta(\hat{\varkappa})), \hat{\varkappa}\in I=[0,a], a>0, \\ \\ \vartheta(0)\eta(0,\vartheta(0))+\hat{\nu}\vartheta(a)\eta(a,\vartheta(a))=\vartheta(0)\chi(0,\vartheta(0))+\hat{\nu}\chi(a,\vartheta(a))+\beta, \end{cases} \tag{1.1}$$

where $\eta \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ *and* $\chi, \xi \in C(I \times \mathbb{R}, \mathbb{R})$ *are given functions and* $\hat{\nu}, \beta \in \mathbb{R}$ *such that* $\hat{\nu} \neq -1$.
Motivated by the good effect of model (1.1), *we consider the following problem of impulsive hybrid fractional differential equation:*

$$\begin{cases} D^{\hat{\nu}}\Big(\dfrac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))\Big) = \xi(\hat{\varkappa}, \vartheta(\hat{\varkappa})), \hat{\varkappa} \in \hat{\mathfrak{J}} = [0,1], \hat{\varkappa} \neq \hat{\varkappa}_i, i = 1, 2, \ldots, n, 0 < \alpha < 1, \\ \vartheta(\hat{\varkappa}_i^+) = \vartheta(\hat{\varkappa}_i^-) + I_i(\vartheta(\hat{\varkappa}_i^-)), \hat{\varkappa}_i \in (0,1), i = 1, 2, \ldots, n, \\ \dfrac{\vartheta(0)}{\varpi(0, \vartheta(0))} - \chi(0, 0)) = \phi(\vartheta), \end{cases} \tag{1.2}$$

where $D^{\hat{\nu}}$ *denote the Caputo fractional derivative of order* $\hat{\nu}$. *The functions* $\varpi \in \mathcal{C}(\hat{\mathfrak{J}} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ *and* ξ, χ *belong to* $\mathcal{C}(\hat{\mathfrak{J}} \times \mathbb{R}, \mathbb{R})$, *while* $\phi : \mathcal{C}(\hat{\mathfrak{J}}, \mathbb{R}) \longrightarrow \mathbb{R}$ *is continuous, defined as* $\phi(\vartheta) = \sum_{i=1}^{n} \lambda_i \vartheta(\xi_i)$. *Here,* $\xi_i \in (0,1)$ *for* $i = 1, 2, ..., n$, *and* $I_i : \mathbb{R} \longrightarrow \mathbb{R}$. *The notation* $\vartheta(\hat{\varkappa}_i^+)$ *and* $\vartheta(\hat{\varkappa}_i^-)$ *represents the right and left limits of* $\vartheta(\hat{\varkappa})$ *at* $\hat{\varkappa} = \hat{\varkappa}_i$, *where* $\vartheta(\hat{\varkappa}_i^+) = \lim_{\epsilon \to 0^+} \vartheta(\hat{\varkappa}_i + \epsilon)$ *and* $\vartheta(\hat{\varkappa}_i^-) = \lim_{\epsilon \to 0^-} \vartheta(\hat{\varkappa}_i + \epsilon)$. *The function* ϕ *is continuous, defined over the space* $\mathcal{C}(\hat{\mathfrak{J}}, \mathbb{R})$.
By a solution of the peoblem (1.2) we mean a function $\vartheta \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ *such that*
(i) the function $\hat{\varkappa} \longrightarrow \dfrac{\vartheta}{\varpi(\hat{\varkappa}, \vartheta)}$ *is increasing in* $\mathbb{R}$, *and*
(ii) ϑ *satisfies the equations in (1.2).*

The paper is organized as follows. Section 2 provides a concise overview of fundamental concepts, fractional calculation laws, and introduces preliminary results. In Section 3, we examine the existence and uniqueness of solutions to the initial value problem (1.2), employing both the Banach contraction mapping principle (BCMP) and Leray-Schauder fixed point theorem. In Section 4, we present an example that serves to illustrate the findings of our study. Lastly, Section 5 contains concluding remarks and proposes potential avenues for future research.

2 Preliminaries

In this section, we offer a brief overview of the essential concepts and properties of fractional calculus theory. Furthermore, we present several preliminary findings that will be employed in our subsequent analysis. Throughout this paper denotes $\hat{\mathfrak{J}}_0 = [0, \hat{\varkappa}_1]$, $\hat{\mathfrak{J}}_1 = (\hat{\varkappa}_1, \hat{\varkappa}_2]$, $\ldots$, $\hat{\mathfrak{J}}_{n-1} = (\hat{\varkappa}_{n-1}, \hat{\varkappa}_n]$, $\hat{\mathfrak{J}}_n = (\hat{\varkappa}_n, 1], n \in \mathbb{N}, n > 1$.
For $\hat{\varkappa}_i \in (0,1)$ *such that* $\hat{\varkappa}_1 < \hat{\varkappa}_2 < \ldots < \hat{\varkappa}_n$, *we define the following spaces:*
$\hat{\mathfrak{J}}' = \hat{\mathfrak{J}} \setminus \{\hat{\varkappa}_1, \hat{\varkappa}_2, ..., \hat{\varkappa}_n\}$,

$$\hat{\mathfrak{X}} = \{\vartheta \in \mathcal{C}(\hat{\mathfrak{J}}, \mathbb{R}) : \vartheta \in C(\hat{\mathfrak{J}}') \text{ and left } \vartheta(\hat{\varkappa}_i^+) \text{ and right limit } \vartheta(\hat{\varkappa}_i^-)) \text{ exist and } \vartheta(\hat{\varkappa}_i^-) = \vartheta(\hat{\varkappa}_i), 1 \leq i \leq n\}.$$

Then, clearly $(\hat{\mathfrak{X}}, \|.\|)$ *is a Banach space under the norm* $\|\vartheta\| = \max_{\hat{\varkappa} \in [0,1]} |\vartheta(\hat{\varkappa})|$.

Definition 2.1. [3] The fractional integral of the function $\xi \in L^1([a,b], \mathbb{R}^+)$ of order $\hat{\nu} \in \mathbb{R}^+$ is defined by

$$I_a^{\hat{\nu}} \xi(\hat{\varkappa}) = \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s) ds,$$

where Γ is the gamma function.

Definition 2.2. [3] For a function ξ defined on the interval $[a,b]$, the Riemann-Liouville fractional-order derivative of ξ, is defined by

$$({}^R D_{a^+}^{\hat{\nu}} \xi)(\hat{\varkappa}) = \frac{1}{\Gamma(n - \hat{\nu})} \left(\frac{d}{dt}\right)^n \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{n-\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s) ds,$$

where $n = [\hat{\nu}] + 1$ and $[\hat{\nu}]$ denotes the integer part of $\hat{\nu}$.

Definition 2.3. [3] For a function ξ given on the interval $[a,b]$, the Caputo fractional-order derivative of ξ, is defined by

$$({}^cD^{\hat{\nu}}_{a^+}\xi)(\hat{\varkappa}) = \frac{1}{\Gamma(n-\hat{\nu})}\int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{n-\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi^{(n)}(s)ds,$$

where $n = [\hat{\nu}]+1$ and $[\hat{\nu}]$ denotes the integer part of $\hat{\nu}$.

In this section, we introduce the notation, definitions, and lemmas that will be utilized in our proofs later.

Lemma 2.4. *[1] Let $n \in \mathbb{N}$ and $n-1 < \hat{\nu} < n$. If η is a continuous function, then we have*

$$I^{\hat{\nu}}\ {}^cD^{\hat{\nu}}\eta(\hat{\varkappa}) = \eta(\hat{\varkappa}) + a_0 + a_1\hat{\varkappa} + a_2\hat{\varkappa}^2 + ... + a_{n-1}\hat{\varkappa}^{n-1}.$$

Lemma 2.5. *(Leray-Schauder alternative see [2]). Let $\hat{\mathfrak{F}} : \hat{\mathfrak{G}} \longrightarrow \hat{\mathfrak{G}}$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in $\hat{\mathfrak{G}}$ is compact). Let $\hat{\mathfrak{P}}(\hat{\mathfrak{F}}) = \{\vartheta \in \hat{\mathfrak{G}} : \vartheta = \lambda\hat{\mathfrak{F}}\vartheta \quad for \quad some \quad 0 < \lambda < 1\}$. Then either the set $\hat{\mathfrak{P}}(\hat{\mathfrak{F}})$ is unbounded or $\hat{\mathfrak{F}}$ has at least one fxed point.*

3 Main results

In this section, we will prove the existence of a mild solution for problem (1.2) .
To obtain the existence of a mild solution, we will need the following assumptions:

(H_1) *i) The functions η and χ are continuous and bounded, that is, there exist positive numbers $\nu_\varpi > 0$ and $\mu_\chi > 0$, such that*
$|\varpi(\hat{\varkappa},\vartheta)| \le \nu_\varpi$ *and* $|\chi(\hat{\varkappa},\vartheta)| \le \mu_\chi$ *for all* $(\hat{\varkappa},\vartheta) \in [0,1]\times\mathbb{R}$.
ii) There exist positive numbers $M_\eta > 0$ and $M_\chi > 0$, such that

$$|\eta(\hat{\varkappa},\vartheta) - \eta(\hat{\varkappa},\bar{\vartheta})| \le M_\eta|\vartheta-\bar{\vartheta}|,$$

and

$$|\chi(\hat{\varkappa},\vartheta) - \chi(\hat{\varkappa},\bar{\vartheta})| \le M_\chi|u-\bar{\vartheta}|.$$

for all $\vartheta,\bar{\vartheta} \in \mathbb{R}$ and $\hat{\varkappa} \in [0,1]$.

(H_2) *There exist positive number $M_\xi > 0$, such that*

$$|\xi(\hat{\varkappa},\vartheta) - \xi(\hat{\varkappa},\bar{\vartheta})| \le M_\xi|\vartheta-\bar{\vartheta}|,$$

for all $\vartheta,\bar{\vartheta} \in \mathbb{R}$ and $\hat{\varkappa} \in [0,1]$.

(H_3) *There exists constant $A > 0$, such that for all*

$$|I_i(\vartheta) - I_i(\bar{\vartheta})| \le A|\vartheta-\bar{\vartheta}|, \qquad i = 1,2,...,n, \forall\vartheta,\bar{\vartheta} \in \mathbb{R}.$$

(H_4) *There exist constant K_ϕ, such that*
$|\phi(\vartheta)| \le K_\phi\|\vartheta\|,$ *for all* $\vartheta \in C([0,1],\mathbb{R})$,

(H_5) *There exist constant $M_\phi, N_\vartheta > 0$, such that*
$|\phi(\vartheta)| \le M_\phi\|\vartheta\|,$ *for all* $\vartheta \in C([0,1],\mathbb{R})$,
$|I_i(\vartheta)| \le N_\vartheta\|\vartheta\|, \quad i = 1,2,...,n,$ *for all* $\vartheta \in \mathbb{R}$,

(H_6) *There exists constant $\rho > 0$, such that*
$|\phi(\vartheta)| \le \rho, \quad \forall\vartheta \in C([0,1],\mathbb{R}).$

(H_7) *There exist constants $\rho_0, \rho_1 > 0$, such that*
$|\xi(\hat{\varkappa},\vartheta)| \le \rho_0 + \rho_1\|\vartheta\|,$ *for all* $\vartheta \in \mathbb{X}$ *and* $\hat{\varkappa} \in [0,1]$.

For brevity, let us set

$$d = \sum_{i=1}^{n} \Big| \chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))} \Big|,$$

$$\pi = \nu_\varpi \Big(M_\chi + K_\phi + \frac{n}{\nu_\varpi}(M_\chi + A) + \frac{M_\xi}{\Gamma(\alpha+1)} \Big). \tag{3.1}$$

Lemma 3.1. *: Let $\hat{\nu} \in (0,1)$ and $\psi : [0,a] \longrightarrow \mathbb{R}$ be continuous. A function $\vartheta \in \mathcal{C}([0,a],\mathbb{R})$ is a solution to the fractional integral equation*

$$\vartheta(\hat{\varkappa}) = \vartheta_0 - \int_0^a \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\alpha)} \psi(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \psi(s)ds \Big),$$

if and only if ϑ is a solution to the following fractional Cauchy problem:

$$\begin{cases} D^{\hat{\nu}} \vartheta(\hat{\varkappa}) = \psi(\hat{\varkappa}), \hat{\varkappa} \in [0,a] \\ \vartheta(a) = \vartheta_0, \quad a > 0, \end{cases} \tag{3.2}$$

Lemma 3.2. *Let's assume that hypotheses (H_1) and (H_3) hold. Let $\hat{\nu} \in (0,1)$ and $\psi : \hat{\mathfrak{J}} \longrightarrow \mathbb{R}$ be continuous. A function ϑ is a solution to the fractional integral equation*

$$\vartheta(\hat{\varkappa}) = \varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) \Big(\phi(\vartheta) + \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \theta(\hat{\varkappa}) \sum_{i=1}^{n} \Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))} \Big)$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \psi(s)ds \Big), \hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}], \tag{3.3}$$

where

$$\theta(\hat{\varkappa}) = \begin{cases} 0, & \hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1], \\ 1, & \hat{\varkappa} \notin [\hat{\varkappa}_0, \hat{\varkappa}_1[, \end{cases}$$

if and only if ϑ is a solution of the following impulsive problem:

$$\begin{cases} D^{\hat{\nu}} \Big(\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) \Big) = \psi(\hat{\varkappa}), \hat{\varkappa} \in \mathfrak{J} = [0,1], \hat{\varkappa} \neq \hat{\varkappa}_i, i = 1,2,\dots,n, 0 < \hat{\nu} < 1, \\ \vartheta(\hat{\varkappa}_i^+) = \vartheta(\hat{\varkappa}_i^-) + I_i(\vartheta(\hat{\varkappa}_i^-)), \qquad \hat{\varkappa}_i \in (0,1), i = 1,2,\dots,n \\ \frac{\vartheta(0)}{\varpi(0, \vartheta(0))} - \chi(0, \vartheta(0)) = \phi(\vartheta), \end{cases} \tag{3.4}$$

Proof. Assume that ϑ satisfies (3.4). If $\hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1[$, then

$$D^{\hat{\nu}} \Big(\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) \Big) = \psi(\hat{\varkappa}), \hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1[, \tag{3.5}$$

$$\frac{\vartheta(0)}{\varpi(0, \vartheta(0))} - \chi(0, \vartheta(0)) = \phi(\vartheta), \tag{3.6}$$

Applying $I^{\hat{\nu}}$ on both sides of (3.5), we obtain

$$\begin{aligned} \frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) &= \frac{\vartheta(0)}{\varpi(0, \vartheta(0))} - \chi(0, \vartheta(0)) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \psi(s)ds \\ &= \phi(\vartheta) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \psi(s)ds, \end{aligned}$$

Then we get

$$\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} = \chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) + \phi(\vartheta) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds.$$

If $\hat{\varkappa} \in [\hat{\varkappa}_1, \hat{\varkappa}_2[$, then

$$D^{\hat{\nu}}\Big(\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big) = \xi(\hat{\varkappa}), \quad \hat{\varkappa} \in [\hat{\varkappa}_1, \hat{\varkappa}_2[, \tag{3.7}$$

$$\vartheta(\hat{\varkappa}_1^+) = \vartheta(\hat{\varkappa}_1^-) + I_1(\vartheta(\hat{\varkappa}_1^-)), \tag{3.8}$$

According to Lemma 3.1 and the continuity of $\hat{\varkappa} \longrightarrow \dfrac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}$, we have

$$\begin{aligned}\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) &= \frac{\vartheta(\hat{\varkappa}_1^+)}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))} - \chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)) - \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds \\ &+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds \\ &= \frac{(\vartheta(\hat{\varkappa}_1^-) + I_1(\vartheta(\hat{\varkappa}_1^-)))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))} - \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\alpha)}\psi(s)ds \\ &+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds.\end{aligned}$$

Since

$$\vartheta(\hat{\varkappa}_1^-) = \varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))\Big(\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)) + \phi(\vartheta) + \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds,$$

then we get

$$\begin{aligned}\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) &= \frac{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))\big[\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)) + \phi(\vartheta) + \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\big] + I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))} \\ &- \int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds \\ &= \phi(\vartheta) + \frac{I_1(u(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))} + \chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds,\end{aligned}$$

So, one has

$$\vartheta(\hat{\varkappa}) = \varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\phi(\vartheta) + \chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) + \frac{I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))} + \chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big).$$

If $\hat{\varkappa} \in [\hat{\varkappa}_2, \hat{\varkappa}_3[$, we have

$$\begin{aligned}\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) &= \frac{\vartheta(\hat{\varkappa}_2^+)}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))} - \chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2)) - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds \\ &+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds \\ &= \frac{(\vartheta(\hat{\varkappa}_2^-) + I_2(\vartheta(\hat{\varkappa}_2^-)))}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))} - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2-s)^{\hat{\nu}-1}}{\Gamma(\alpha)}\psi(s)ds \\ &+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds.\end{aligned}$$

and

$$\vartheta(\hat{\varkappa}_2^-) \quad = \varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\phi(\vartheta)+\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))+\frac{I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}+\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big),$$

Therefore, we obtain

$$\frac{\vartheta(\hat{\varkappa}}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))=$$

$$\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))\frac{[(\phi(\vartheta)+\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))+\frac{I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}+\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds]}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}$$

$$+\frac{I_2(\vartheta(\hat{\varkappa}_2^-))}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}-\int_0^{\hat{\varkappa}_2}\frac{(\hat{\varkappa}_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds,$$

$$=\phi(\vartheta)+\chi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))+\frac{I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}+\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))+\frac{I_2(\vartheta(\hat{\varkappa}_2^-))}{\varpi(\hat{\varkappa}_2,\vartheta(\hat{\varkappa}_2))}$$

$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds,$$

Consequently, we get

$$\vartheta(\hat{\varkappa})=\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\sum_{i=1}^{2}\Big(\chi(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))+\frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))}\Big)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big).$$

If $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}[(i=3,4,...,n)$, using the same method, one has

$$\vartheta(\hat{\varkappa})=\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))+\frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i,\vartheta(\hat{\varkappa}_i))}\Big)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big).$$

Conversely, assume that ϑ satisfies (3.3). If $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[$, we have

$$\vartheta(\hat{\varkappa}) \quad =\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\phi(\vartheta)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big). \tag{3.9}$$

Then, we divide by $\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\nu}}$ on both sides of (3.9), we get equation (3.5) . Again, substituting $\hat{\varkappa}=0$ in (3.9), we obtain $\dfrac{\vartheta(0)}{\varpi(0,\vartheta(0))}-\chi(0,\vartheta(0))=\phi(\vartheta)$. by (H_0), for $\hat{\varkappa}\in[\hat{\varkappa}_0,\hat{\varkappa}_1[$, the map $\hat{\varkappa}\longrightarrow\dfrac{\vartheta}{\varpi(\hat{\varkappa},\vartheta)}$ is injective in $\mathbb{R}$. Then we get (3.6).
Similarly, for $\hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[$, we get

$$\vartheta(\hat{\varkappa})=\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big(\phi(\vartheta)+\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))+\chi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1)+\frac{I_1(\vartheta(\hat{\varkappa}_1^-))}{\varpi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1))}$$
$$+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\psi(s)ds\Big). \tag{3.10}$$

Then, we divide by $\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\nu}}$ on both sides of (3.10), we get equation (3.11) . Again by (H_0), substituting $\hat{\varkappa}=\hat{\varkappa}_1$ in (3.9) and taking the limit of (3.10), then (3.10) minus (3.9) gives (3.12).
If $\hat{\varkappa}\in[\hat{\varkappa}_i,\hat{\varkappa}_{i+1}[(i=2,3,...,n)$, similarly we get

$$D^{\hat{\nu}}\Big(\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))}-\chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big)=\psi(\hat{\varkappa}),\hat{\varkappa}\in[\hat{\varkappa}_k,\hat{\varkappa}_{k+1}[\tag{3.11}$$

$$\vartheta(\hat{\varkappa}_1^+)=\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-)), \tag{3.12}$$

□

This completes the proof.

Lemma 3.3. *Let ξ be continuous, then $\vartheta \in \mathfrak{X}$ is a solution of (1.2) if and only if ϑ is the solution of the integral equations*

$$\vartheta(\hat{\varkappa}) = \varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa}) \sum_{i=1}^{n} \Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \psi(s) ds\Big), \hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}]$$

where

$$\theta(\hat{\varkappa}) = \begin{cases} 0, & \hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1], \\ 1, & \hat{\varkappa} \notin [\hat{\varkappa}_0, \hat{\varkappa}_1[, \end{cases}$$

We define an operator $\Theta : \mathfrak{X} \longrightarrow \mathfrak{X}$ by

$$\Theta(\vartheta)(\hat{\varkappa}) = \varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa}) \sum_{i=1}^{n} \Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s, \vartheta(s)) ds\Big). \tag{3.13}$$

First result

At this point, we are prepared to present our initial finding, focusing on the existence and uniqueness of solutions for the problem defined in (1.2). This result is established through the application of Banach's contraction mapping principle.

Theorem 3.4. *Assume that conditions $(H_1) - (H_7)$ holds . Then problem (1.2) has an unique solution provided that $\pi < 1$, π is the constant given in equation (3.1).*

Proof. Let us set $\sup_{\hat{\varkappa} \in J} \xi(\hat{\varkappa}, 0) = \kappa < \infty$, and defne a closed ball $\bar{B}$ as follows

$$\bar{B} = \{\vartheta \in \mathfrak{X} : \|\vartheta\| \leq r\},$$

where

$$r \geq \frac{\mu_\chi + |d| + \frac{\kappa}{\Gamma(\hat{\nu}+1)}}{\nu_\varpi - (M_\phi + \frac{M_\xi}{\Gamma(\hat{\nu}+1)})}. \tag{3.14}$$

We show that $\Theta\bar{B} \subset \bar{B}$. For $\vartheta \in \bar{B}$, we obtain

$$\begin{aligned} |\Theta(\vartheta)(\hat{\varkappa})| &\leq |\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))| \Big|\chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa}) \sum_{i=1}^{n} \Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big) \\ &\quad + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} \xi(s, \vartheta)(s)) ds\Big| \\ &\leq \nu_\varpi \Big(\mu_\chi + M_\phi\|\vartheta\| + |d| + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})} (|\xi(s, \vartheta(s))) - \xi(s, 0)| + |\xi(s, 0)|) ds \\ &\leq \nu_\varpi \Big(\mu_\chi + M_\phi\|\vartheta\| + |d| + \frac{M_\xi\|\vartheta\| + \kappa}{\Gamma(\alpha+1)}\Big) \\ &\leq \nu_\varpi \Big(\mu_\chi + M_\phi r + |d| + \frac{M_\xi\|\vartheta\| + \kappa}{\Gamma(\hat{\nu}+1)}\Big), \end{aligned}$$

Hence, we get

$$\|\Theta(\vartheta)\| \leq \nu_{\varpi}\Big(\mu_{\chi} + M_{\phi}r + |d| + \frac{M_{\xi}\|\vartheta\| + \kappa}{\Gamma(\hat{\nu}+1)}\Big). \tag{3.15}$$

From (3.15), it follows that $\|\Theta(\vartheta)\| \leq r$.
Next, for $(\vartheta, \bar{\vartheta}) \in \bar{B}^2$and for any $\hat{\varkappa} \in [0,1]$, we have

$$\begin{aligned}
|\Theta(\vartheta)(\hat{\varkappa}) - \Theta(\bar{\vartheta})(\hat{\varkappa})| = \Big|&\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa}, \vartheta(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa})\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big)\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s, \vartheta)(s))ds\Big)\\
&- \varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}))\Big(\chi(\hat{\varkappa}, \bar{\vartheta}(\hat{\varkappa})) + \phi(\vartheta) + \theta(\hat{\varkappa})\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \bar{\vartheta}(\hat{\varkappa}_i)) + \frac{I_i(\bar{\vartheta}(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \bar{\vartheta}(\hat{\varkappa}_i))}\Big)\\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\xi(s, \vartheta)(s))ds\Big)\Big|\\
&\leq \nu_{\varpi}\Big(M_{\chi}|\vartheta - \bar{\vartheta}| + K_{\phi}|\vartheta - \bar{\vartheta}| + \frac{n}{\nu_{\varpi}}(M_{\chi}|\vartheta - \bar{\vartheta}| + A|\vartheta - \bar{\vartheta}|)\\
&+ \frac{M_{\xi}}{\Gamma(\hat{\nu}+1)}|\vartheta - \bar{\vartheta}|\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\Theta(\vartheta) - \Theta(\bar{\vartheta})\| &\leq \nu_{\varpi}\Big(M_{\chi} + K_{\phi} + \frac{n}{\nu_{\varpi}}(M_{\chi} + A) + \frac{M_{\xi}}{\Gamma(\alpha+1)}\Big)(\|\vartheta - \bar{\vartheta}\|)\\
&= \pi\|\vartheta - \bar{\vartheta}\|.
\end{aligned} \tag{3.16}$$

From (3.16), we deduce that

$$\|\Theta(\vartheta) - \Theta(\bar{\vartheta})\| \leq \pi\|\vartheta - \bar{\vartheta}\|.$$

□

Because of the condition $\pi < 1$, we can assert that Θ acts as a contraction operator. Consequently, Banach's fixed-point theorem is applicable, guaranteeing that the operator Θ possesses a single fixed point. This unique fixed point serves as the sole solution to the Cauchy problem (1.2). This concludes the proof.

Second result

Our second result focuses on establishing the existence of solutions for the problem (1.2) using the Leray-Schauder alternative. For brevity, let us set

$$\Lambda_1 = \frac{\nu_{\varpi}}{\Gamma(\hat{\nu}+1)}, \tag{3.17}$$

$$\Lambda_0 = 1 - \Lambda_1\rho_1. \tag{3.18}$$

Theorem 3.5. *Assume that conditions $(H_1) - (H_2)$ and $(H_6) - (H_7)$ hold. Furthermore, it is assumed that $\Lambda_1\rho_1 < 1$, where Λ_1 isgiven by (3.17). Then the boundary value problem (1.2) has at least one solution.*

Proof. We will show that the operator $\Pi : \mathfrak{X} \longrightarrow \mathfrak{X}$ satisfes all the assumptions of Lemma 2.5.
Step 1: We will prove that the operator Π is completely continuous.
Clearly, it follows by the continuity of functions ϖ, χ, ξ that the operator Π is continuous.
Let $\mathfrak{S} \subset \mathfrak{X}$ be bounded. Then we can find positive constant Ω such that:

$|\xi(\hat{\varkappa}, \vartheta)| \leq \Omega, \quad \forall \vartheta \in \mathfrak{S}.$
Thus, for any $\vartheta \in \mathfrak{S}$, we can get

$$|\Pi(\vartheta)(\hat{\varkappa})| \leq \nu_{\varpi}\Big(\mu_{\chi} + \rho + |d| + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}\Omega ds\Big)$$
$$\leq \nu_{\varpi}\big(\mu_{\chi} + \rho + |d| + \frac{\Omega}{\Gamma(\alpha+1)}\big).$$

which yields

$$\|\Pi(\vartheta)\| \quad \leq \quad \nu_{\varpi}\big(\mu_{\chi} + \rho + |d| + \frac{\Omega}{\Gamma(\hat{\nu}+1)}\big). \tag{3.19}$$

From the inequalities (3.19), we deduce that the operator Π is uniformly bounded.
Setep 2: Now we show that the operator Π is equicontinuous.
We take $\tau_1, \tau_2 \in \hat{\mathfrak{J}}$ with $\tau_1 < \tau_2$ we obtain:

$$|\Pi(\vartheta(\tau_2)) - \Pi(\vartheta(\tau_1))|$$
$$\leq \Big|\varpi(\tau_2, \vartheta(\tau_2))\Big(\phi(\vartheta) + \chi(\tau_2, \vartheta(\tau_2)) + \theta(\tau_2)\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big)$$
$$+ \Omega\int_0^{\tau_2}\frac{(\tau_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}ds\Big)$$
$$- \varpi(\tau_1, \vartheta(\tau_1))\Big(\phi(\vartheta) + \chi(\tau_1, \vartheta(\tau_1)) + \theta(\tau_1)\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big)$$
$$+ \Omega\int_0^{\tau_1}\frac{(\tau_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}ds\Big)\Big|$$
$$\leq \nu_{\varpi}\Big(\Big|(\chi(\tau_2, \vartheta(\tau_2)) - \chi(\tau_1, \vartheta(\tau_1)) + (\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big)\Big|$$
$$+ \Omega\Big|\int_0^{\tau_2}\frac{(\tau_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}ds - \int_0^{\tau_1}\frac{(\tau_1-s)^{\hat{\nu}-1}}{\Gamma(\hat{\mu})}ds\Big|\Big)$$
$$\leq \nu_{\varpi}\Big(\Big|(\chi(\tau_2, \vartheta(\tau_2)) - \chi(\tau_1, \vartheta(\tau_1)) + (\theta(\tau_2) - \theta(\tau_1))\sum_{i=1}^{n}\Big(\chi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i)) + \frac{I_i(\vartheta(\hat{\varkappa}_i^-))}{\varpi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i))}\Big)\Big|$$
$$+ \Omega\Big|\int_0^{\tau_1}\frac{(\tau_1-s)^{\hat{\nu}-1} - (\tau_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}ds - \int_{\tau_2}^{\tau_2}\frac{(\tau_2-s)^{\hat{\nu}-1}}{\Gamma(\hat{\nu})}ds\Big|\Big).$$

Which tend to 0 independently of ϑ. This implies that the operator $\Pi(\vartheta)$ is equicontinuous. Thus, by the above fndings, the operator $\Pi(\vartheta)$ is completely continuous.
In the next step, it will be established that the set $P = \{\vartheta \in \mathfrak{X}/\vartheta = \lambda\Pi(\vartheta), 0 < \lambda < 1\}$ is bounded.
Let $\vartheta \in \mathcal{P}$. We have $\vartheta = \lambda\Pi(\vartheta)$. Thus, for any $\hat{\varkappa} \in [0,1]$, we can write

$$\vartheta(\hat{\varkappa}) = \lambda\Pi(\hat{\varkappa})(\hat{\varkappa}),$$

Hence, we get

$$\|\vartheta\| \leq \nu_{\varpi}\Big(\mu_{\chi} + \rho + |d| + \frac{1}{\Gamma(\hat{\nu}+1)}(\rho_0 + \rho_1\|\vartheta\|)\Big)$$
$$\leq \nu_{\varpi}(\mu_{\chi} + \rho + |d|) + \Lambda_1(\rho_0 + \rho_1\|\vartheta\|),$$

which, in view of (3.18), can be expressed as

$$\|\vartheta\| \quad \leq \quad \frac{\nu_{\varpi}\big(\mu_{\chi} + \rho + nC\big) + \Lambda_1\rho_0}{\Lambda_0}.$$

This demonstrates that the set $\mathcal{P}$ is bounded. As a result, all the conditions of Lemma 2.5 are satisfied. Therefore, the operator Π has at least one fixed point, which corresponds to a solution of problem (1.2). This completes the proof. □

4 Example

Consider the following impulsive hybrid fractional differential equation::

$$\begin{cases} D^{\frac{1}{2}}\Big(\frac{\vartheta(\hat{\varkappa})}{\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa}))} - \chi(\hat{\varkappa},\vartheta(\hat{\varkappa}))\Big) = \xi(\hat{\varkappa},\vartheta(\hat{\varkappa})), \hat{\varkappa}\in[0,1]\setminus\{\hat{\varkappa}_1\}, \\ \vartheta(\hat{\varkappa}_1^+) = \vartheta(\hat{\varkappa}_1^-) + (-2u(\hat{\varkappa}_1^-)), \quad \hat{\varkappa}_1 \neq 0,1, \\ \frac{\vartheta(0)}{\varpi(0,\vartheta(0)))} - \chi(0,\vartheta(0)) = \sum_{i=1}^{n} c_i\vartheta(\hat{\varkappa}_i), \end{cases} \tag{4.1}$$

Here, we have

$$\varpi(\hat{\varkappa},\vartheta(\hat{\varkappa})) = \frac{\arctan\hat{\varkappa}}{3}|\vartheta(\hat{\varkappa})| + 1,$$

$$\chi(\hat{\varkappa},\vartheta(\hat{\varkappa})) = \frac{1}{7} + \frac{1}{9}\vartheta(\hat{\varkappa}),$$

$$\xi(\hat{\varkappa},\vartheta(\hat{\varkappa})) = \frac{1}{4\hat{\varkappa}^2}(\vartheta(\hat{\varkappa}) + \sqrt{2}),$$

Note that

$$|\chi(\hat{\varkappa},\vartheta_1) - \chi(\hat{\varkappa},\vartheta_2)| \leq \frac{1}{9} \mid \vartheta_2 - \vartheta_1 \mid,$$
$$\hat{\varkappa}\in[0.1], \vartheta_1,\vartheta_2\in\mathbb{R}.$$

and

$$|\xi(\hat{\varkappa},\vartheta_1) - \xi(\hat{\varkappa},\vartheta_2)| \leq \frac{1}{4} \mid \vartheta_2 - \vartheta_1 \mid, \hat{\varkappa}\in[0.1], \vartheta_1,\vartheta_2\in\mathbb{R}.$$

$$\pi \;=\; \nu_{\varpi}\Big(M_\chi + K_\phi + M_\eta + nA + \frac{M_\xi}{\Gamma(\hat{\nu}+1)}\Big) = 0.12345678 < 1,$$

As all of the assumptions in Theorem 3.4 are satisfied, our results can be directly applied to the problem (4.1).

5 Conclusion

The main focus of this paper is to explore the existence of solutions for impulsive nonlinear hybrid fractional differential equations involving both linear and nonlinear perturbations. Our results not only improve upon existing findings in this research area but also provide a more generalized perspective. Furthermore, we anticipate that the theory we have developed can be extended to address broader problems related to impulsive fractional differential equations featuring both linear and nonlinear perturbations. The fixed-point theorems employed in our analysis can also be applied to investigate the existence of solutions for other types of impulsive fractional differential equations, including those involving alternative forms of fractional derivatives such as Hilfer's and Hadamard's derivatives. By contributing to the advancement of more comprehensive and efficient tools for studying these problems, we aim to enhance our understanding of the dynamics of complex systems and their behavior in impulsive conditions.

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