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The Restricted Action on the Higher Dimensional Bell Numbers and their Centralizer Algebras

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Abstract The tensor product partition algebra is the algebra of all transformations commuting with the action of the direct product of symmetric groups on tensor products of its permutation representation. In particular, we restrict the action of the direct product of the symmetric groups to the action of the direct product of the group of even permutations and consider the centralizer over this restricted action. In this instance, we describe a basis for the centralizer and compute the dimension formula for the centralizer using the index sets X_t^m and $Y_{t,s}^m$. Also, we demonstrate that the centralizer algebra is indeed the tensor product partition algebra under certain conditions.

1 Introduction

The partition algebras $P_k(x)$ have been defined independently in the work of Martin [8] and Jones [3]. Jones investigated the algebra $P_k(x)$ as a Potts model in statistical mechanics and Martin as a generalization of the Temperley-Lieb algebras. Jones considered the algebra $P_k(n)$ as a centralizer algebra of the symmetric group S_n acting on $V^{\otimes k}$, where $V = \mathbb{C}^n$ is the permutation module of S_n.

The G-vertex colored partition algebra $P_k(x, G)$ and extended G-vertex colored partition algebra $\widehat{P}_k(x, G)$ have been defined in [6] and [7] respectively, they have been realized as the centralizer algebras of the subgroups $S_n \times G$ and S_n of the group $G \wr S_n$ respectively, where $x = n \geq 2k$.

The tensor product partition algebra $P_k(x_1)\otimes P_k(x_2)\otimes \cdots \otimes P_k(x_m)$ has been recently studied by Kennedy and Jaish in [4]. The algebra was realized as the centralizer algebra of the direct product group $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ acting on the tensor space $W^{\otimes k}$, where $W = \mathbb{C}^{n_1 n_2 \cdots n_m}$ is the permutation module for the direct product group.

In [2], Bloss focused on the Jone's result and restricted the action of the symmetric group on n letters to the action of the group of even permutations, as acting on the tensor product of the permutation representation of S_n, for the algebra $P_k(n)$. Bloss exhibited that for $n \geq 2k + 2$, $End_{A_n}(V^{\otimes k}) \cong P_k(n)$.

In [5], A J Kennedy *et al.*(2023) restricted the action of $S_n \times G$ to $A_n \times G$ for the algebra $P_k(n, G)$ and described the basis for the centralizer algebra. Also, they computed the dimension of the centralizer and showed that for $n \geq 2k + 2$, $End_{A_n \times G}(W^{\otimes k}) \cong P_k(n, G)$, where $W = \mathbb{C}^{n|G|}$. In addition, they did the same for the extended G-vertex colored partition algebra $\widehat{P}_k(n, G)$.

In this paper, we restrict the action of the direct product of symmetric groups to the action of the direct product of alternating groups, and we describe the basis for the centralizer algebra. Let W denote the permutation representation of the direct product of symmetric groups. The direct product group $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$, acts diagonally on $W^{\otimes k}$ by the restriction. Clearly $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k}) \subseteq End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$. We compute the dimensions of the centralizer $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$ using the index sets $X_t^m = \Big\{x = (x_1, x_2, ..., x_t) \;\Big|\; x_1 < x_2 < \cdots < x_t$ and $1 \le x_i \le m, \forall\ 1 \le i \le t\Big\}$ and $Y_{t,s}^m = \Big\{y = (x_1, x_2, ..., x_t, y_1, y_2, ..., y_s) \;\Big|\; x_1 < x_2 < \cdots < x_t,\ y_1 < y_2 < \cdots < y_s$ and $1 \le x_i \ne y_j \le m$, $\forall\ 1 \le i \le t, 1 \le j \le s\Big\}$. Also, we demonstrate that for $n_1, n_2, ..., n_m \ge 2k+2$, the centralizer is isomorphic to the tensor product partition algebra $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$. Our method uses an explicit computation of the basis for $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$.

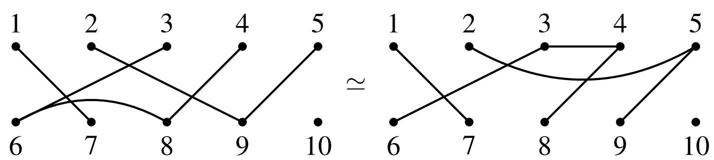
2 Preliminaries

In this section, we discuss some results needed for our main results.

2.1 The partition algebra $P_k(x)$

Let k be a positive integer. A set partition of the set $\{1, 2, ..., k, k + 1, k + 2, ..., 2k\}$ of $2k$ elements is of the form $\{B_1, B_2, ..., B_l\}$, where $B_1, B_2, ..., B_l$ are disjoint subsets with $\sqcup_{i=1}^l B_i = \{1, 2, ..., k, k + 1, k + 2, ..., 2k\}$.

A set partition can be represented as a simple graph of one above the other of two rows of k-vertices, called a k-partition diagram. Number the vertices $1, 2, ..., k$ in the upper row from left to right, and $k+1, k+2, ..., 2k$ in the lower row from left to right of a k-partition diagram. Draw an edge between any two vertices $i, j \in \{1, 2, ..., 2k\}$ if and only if $i, j \in B_s, 1 \le s \le l$. Two k-partition diagrams determine the same partitions of $2k$ vertices, then they are equivalent. The following is an example for equivalent 5-diagrams.



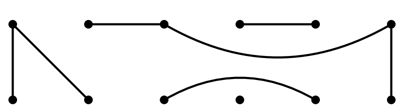
When we talk about the diagrams, we are actually speaking about the related equivalence classes of a k-partition diagram. Let F always represent a field with an arbitrary characteristic throughout the paper, and let x represent an element of the field F. The following is known as the product of two k-diagrams d and d':

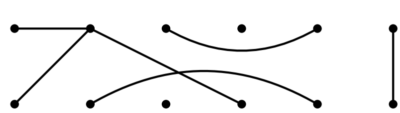
(i) Set d at the top and d' below it.

(ii) Join the vertex $(k+j)$ of d to the vertex j of d'. We now have a diagram with an upper line, middle line, and lower line of vertices. This diagram is named the attachment of d and d'. Let λ be the number of components in the middle line.

(iii) Construct a diagram d'' by deleting the vertices in the middle line but keeping the lower line and the upper line of vertices, and the connections between them. Every "component" in the middle line is replaced by the variable x. That is, $d'd = x^{\lambda}d''$.

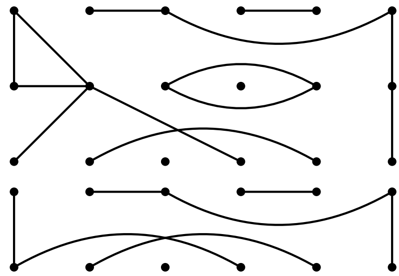
For example,

$d =$ 

$d' =$ 

$d'd =$

$d'd = x^2$



This product is associative and well defined up to equivalence. The F-span of all k-partition diagrams is the *partition algebra* $P_k(x)$, which is an associative algebra with identity. The dimension of $P_k(x)$ is the Bell number $B(2k)$, where

$$B(2k) = \sum_{l=1}^{l=2k} S(2k, l) \tag{2.1}$$

and where the number of equivalence relations with exactly l parts for a set of $2k$ elements is a Stirling number $S(2k, l)$.

The identity element is given by the diagram with each vertex in the upper row connected only to the vertex below it in the lower row. By convention, $P_0(x) = F$.

2.2 Schur - Weyl Duality of $P_k(x)$

Let $V = \mathbb{C}^n$, where V is the permutation module for S_n with standard basis $v_1, v_2, ..., v_n$. Then $\pi(v_i) = v_{\pi(i)}$, for $\pi \in S_n$ and $1 \le i \le n$. For every positive integer k, the tensor product space $V^{\otimes k}$ is a module for S_n with a standard basis given by $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$, where $1 \le i_j \le n$. The action of $\pi \in S_n$ on a basis vector is given by

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = v_{\pi(i_1)} \otimes v_{\pi(i_2)} \otimes \cdots \otimes v_{\pi(i_k)}. \tag{2.2}$$

For every diagram d and every integer sequence $i_1, i_2, ..., i_{2k}$ with $1 \le i_s \le n$, define

$$\psi(d)^{i_1, i_2, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} = \begin{cases} 1 & \text{if } i_r = i_s \text{ whenever vertices } s \text{ and } r \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Define the action of a diagram $d \in P_k(n)$ on $V^{\otimes k}$ by stating it on the standard basis as

$$d(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = \sum_{1 \le i_{k+1}, \ldots, i_{2k} \le n} \psi(d)^{i_1, i_2, \ldots, i_k}_{i_{k+1}, \ldots, i_{2k}} v_{i_{k+1}} \otimes v_{i_{k+2}} \otimes \cdots \otimes v_{i_{2k}}. \tag{2.4}$$

Theorem 2.1. [3] $\mathbb{C}[S_n]$ *and* $P_k(n)$ *generate full centralizers of each other in End*$(V^{\otimes k})$.
In particular, for $n \ge 2k$,
(a) $P_k(n) \cong \mathit{End}_{S_n}(V^{\otimes k})$,
(b) S_n *generates* $\mathit{End}_{P_k(n)}(V^{\otimes k})$.

Theorem 2.2. [2] $\mathbb{C}[A_n]$ *and* $P_k(n)$ *generate full centralizers of each other in End*$(V^{\otimes k})$.
In particular, for $n \ge 2k+2$,
(a) $P_k(n) \cong \mathit{End}_{A_n}(V^{\otimes k})$,
(b) A_n *generates* $\mathit{End}_{P_k(n)}(V^{\otimes k})$.

2.3 The tensor product partition algebra

The tensor product partition algebra $P_k(x_1) \otimes P_k(x_2) \otimes \cdots \otimes P_k(x_m)$ is the tensor product of m partition algebras $P_k(x_1), P_k(x_2), ..., P_k(x_m)$. The standard basis for this algebra is

$$\mathcal{A}_k := \{(d_1 \otimes d_2 \otimes \cdots \otimes d_m) \mid d_1, d_2, ..., d_m \text{ are } k\text{-partition diagrams}\}$$

with dimension $[B(2k)]^m$.

The multiplication of two tensor product partition diagrams as follows:

Let $(d'_1 \otimes d'_2 \otimes \cdots \otimes d'_m), (d''_1 \otimes d''_2 \otimes \cdots \otimes d''_m) \in \mathcal{A}_k$, then $(d''_1 \otimes d''_2 \otimes \cdots \otimes d''_m)(d'_1 \otimes d'_2 \otimes \cdots \otimes d'_m) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}(d_1 \otimes d_2 \otimes \cdots \otimes d_m)$, where $d''_1 d'_1 = x_1^{\lambda_1} d_1$ in $P_k(x_1)$, $d''_2 d'_2 = x_2^{\lambda_2} d_2$ in $P_k(x_2), ..., d''_m d'_m = x_m^{\lambda_m} d_m$ in $P_k(x_m)$. Thus, the product of any two elements in $\mathcal{A}_k$ is a scalar product of some element in $\mathcal{A}_k$. Hence, the F-span of all tensor product partition diagrams is defined to be the *tensor product partition algebra* with identity.

Let $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ be the direct product group of order $n_1! n_2! \cdots n_m!$ with elements of the form $(\pi_1, \pi_2, ..., \pi_m)$.

$$\text{Let} \qquad W = \text{Span}_{\mathbb{C}}\{v_{(i,j,...,s)} |\ 1 \leq i \leq n_1, 1 \leq j \leq n_2, ..., 1 \leq s \leq n_m\}. \tag{2.5}$$

The action of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ on W is as follows

$$(\pi_1, \pi_2, ..., \pi_m)(v_{(i,j,...,s)}) = v_{(\pi_1(i), \pi_2(j), ..., \pi_m(s))}. \tag{2.6}$$

Let $\mathbb{S} := [n_1] \times [n_2] \times \cdots \times [n_m]$ be an index set for the basis of W, where $[n_i] = \{1, 2, ..., n_i\}$ and let $I = ((i_1, j_1, ..., s_1), (i_2, j_2, ..., s_2), ..., (i_k, j_k, ..., s_k))$, $J = ((i_{k+1}, ..., s_{k+1}), (i_{k+2}, ..., s_{k+2}), ..., (i_{2k}, j_{2k}, ..., s_{2k}))$ in $\mathbb{S}^k$. Extend the action of the group $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ on the set $\mathbb{S}$ by $(\pi_1, \pi_2, ..., \pi_m)(i, j, ..., s) = (\pi_1(i), \pi_2(j), ..., \pi_m(s))$ to a component wise action on $\mathbb{S}^{2k}$ by $(\pi_1, \pi_2, ..., \pi_m)(I, J) = ((\pi_1, \pi_2, ..., \pi_m)(I), (\pi_1, \pi_2, ..., \pi_m)(J))$.

Diagonally extend the action of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ on W to an action of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ on $W^{\otimes k}$ as follows.

$$(\pi_1, \pi_2, ..., \pi_m)(v_{(i_1, j_1, ..., s_1)} \otimes \cdots \otimes v_{(i_k, j_k, ..., s_k)})$$

$$= v_{(\pi_1(i_1), ..., \pi_m(s_1))} \otimes v_{(\pi_1(i_2), ..., \pi_m(s_2))} \otimes \cdots \otimes v_{(\pi_1(i_k), ..., \pi_m(s_k))}.$$

We will write the above as $(\pi_1, \pi_2, ..., \pi_m)(v_I) = v_{(\pi_1, \pi_2, ..., \pi_m)(I)}$.

Let $A \in \text{End}(W^{\otimes k})$. Define $A(v_J) = \sum_I A_I^J(v_I)$, where $A_I^J \in \mathbb{C}$ is the $(I, J)^{th}$ entry of A $(I, J \in \mathbb{S}^k)$ and v_I is a basis element of $W^{\otimes k}$.

Lemma 2.3. [4] $A \in End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{(\pi_1, \pi_2, ..., \pi_m)(I)}^{(\pi_1, \pi_2, ..., \pi_m)(J)}, \forall\ (\pi_1, \pi_2, ..., \pi_m) \in S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$.

Lemma 2.4. [4] (a) $dim\ End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1, \\ i \in [m]}}^{n_i} \prod_{j=1}^{m} S(2k, l_j)$.

(b) *When* $n_1, n_2, ..., n_m \geq 2k$, $dim\ End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k}) = [B(2k)]^m$.

Theorem 2.5. [4] $\mathbb{C}[S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}]$ *and* $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)$ *generate full centralizers of each other in* $End(W^{\otimes k})$. *In particular, for* $n_1, n_2, ..., n_m \geq 2k$,

(a) $P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m) \cong End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k})$,

(b) $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ *generates* $End_{P_k(n_1) \otimes P_k(n_2) \otimes \cdots \otimes P_k(n_m)}(W^{\otimes k})$.

3 The Algebra $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$

Let $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$ denote the direct product group, where A_{n_i} is the group of even permutations on n_i symbols. Let W denote the permutation module for the direct product of symmetric groups $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$. The group $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$ acts diagonally on $W^{\otimes k}$ and this action may be restricted to the direct product of alternating groups $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$. Now, we describe a basis of the centralizer $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$ and compute the dimensions of the centralizer algebra in the case of $m = 2$ and $m = 3$. Clearly $End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k}) \subseteq End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$.

Lemma 3.1. $A \in End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k}) \Leftrightarrow A_I^J = A_{(\pi_1, \pi_2, ..., \pi_m)(I)}^{(\pi_1, \pi_2, ..., \pi_m)(J)}, \forall\ (\pi_1, \pi_2, ..., \pi_m) \in A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$.

Proof. We have $A \in End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$

$$\Leftrightarrow (\pi_1,\pi_2,...,\pi_m)A = A(\pi_1,\pi_2,...,\pi_m),\ \forall\ (\pi_1,\pi_2,...,\pi_m)\in A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$$
$$\Leftrightarrow (\pi_1,\pi_2,...,\pi_m)A(v_J) = A(\pi_1,\pi_2,...,\pi_m)(v_J),\ \forall\ v_J$$
$$\Leftrightarrow (\pi_1,\pi_2,...,\pi_m)\sum_I A_I^J(v_I) = A(v_{(\pi_1,\pi_2,...,\pi_m)(J)})$$
$$\Leftrightarrow \sum_I A_I^J(\pi_1,\pi_2,...,\pi_m)(v_I) = \sum_I A_I^{(\pi_1,\pi_2,...,\pi_m)(J)}(v_I)$$
$$\Leftrightarrow \sum_I A_I^J(v_{(\pi_1,\pi_2,...,\pi_m)(I)}) = \sum_I A_{(\pi_1,\pi_2,...,\pi_m)(I)}^{(\pi_1,\pi_2,...,\pi_m)(J)}(v_{(\pi_1,\pi_2,...,\pi_m)(I)})$$

since the action of $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$ is by the permutation representation. The result follows from linearly independence and equating the scalars. □

Therefore, we can describe a basis of $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$ by describing $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$-orbits on

$$\begin{aligned}\Omega :=& \{[n_1]\times[n_2]\times\cdots\times[n_m]\}^{\times 2k}\\ =& \left\{\begin{bmatrix}(i_1,j_1,...,s_1),(i_2,j_2,...,s_2),...,(i_k,j_k,...,s_k)\\ (i_{k+1},j_{k+1},...,s_{k+1}),(i_{k+2},j_{k+2},...,s_{k+2}),...,(i_{2k},j_{2k},...,s_{2k})\end{bmatrix}\ \middle|\ \begin{matrix}1\le i_1,i_2,...,i_{2k}\le n_1\\ 1\le j_1,j_2,...,j_{2k}\le n_2\\ \vdots\\ 1\le s_1,s_2,...,s_{2k}\le n_m\end{matrix}\right\}.\end{aligned}$$

The direct product group $S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}$ acts on Ω by

$$(\pi_1,\pi_2,...,\pi_m)\cdot\begin{bmatrix}(i_1,j_1,...,s_1),(i_2,j_2,...,s_2),...,(i_k,j_k,...,s_k)\\ (i_{k+1},j_{k+1},...,s_{k+1}),(i_{k+2},j_{k+2},...,s_{k+2}),...,(i_{2k},j_{2k},...,s_{2k})\end{bmatrix}$$

$$=\begin{bmatrix}(\pi_1(i_1),\pi_2(j_1),...,\pi_m(s_1)),(\pi_1(i_2),\pi_2(j_2),...,\pi_m(s_2)),...,(\pi_1(i_k),\pi_2(j_k),...,\pi_m(s_k))\\ (\pi_1(i_{k+1}),\pi_2(j_{k+1}),...,\pi_m(s_{k+1})),(\pi_1(i_{k+2}),\pi_2(j_{k+2}),...,\pi_m(s_{k+2})),...,(\pi_1(i_{2k}),\pi_2(j_{2k}),...,\pi_m(s_{2k}))\end{bmatrix}.$$

The number of $S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}$-orbits on Ω gives the dimension of the centralizer $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ (see, [4]). Each orbit corresponds to a basis element T of the algebra $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$. The $S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}$-orbits are in one-to-one correspondence with the set partitions of $[2k]$. But as we will see , the $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$-orbits are not necessarily in $1-1$ correspondence with the equivalence relation on the set $[2k]$.

Let $\sim_{l_1},\sim_{l_2},...,\sim_{l_m}$ denote the equivalence relations with $l_1,l_2,...,l_m$ classes on the set $[2k]$ respectively. There are $S(2k,l_1),S(2k,l_2),...,S(2k,l_m)$ such equivalence relations corresponding to $l_1,l_2,...,l_m$ classes respectively. Hence, there are $\prod_{j=1}^{m}S(2k,l_j)$ basis elements T of $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ corresponding to $(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})$. Let $T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})}$ denote one such basis element in $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$.

Let $T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})}\in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ be a basis element of type $\prod_{j=1}^{m}S(2k,l_j)$, so that the entries of $T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})}$ have indices partitioned according to $(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})$. Define

$$\Omega(T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})}) = \left\{\begin{bmatrix}(i_1,j_1,...,s_1),(i_2,j_2,...,s_2),...,(i_k,j_k,...,s_k)\\ (i_{k+1},j_{k+1},...,s_{k+1}),(i_{k+2},j_{k+2},...,s_{k+2}),...,(i_{2k},j_{2k},...,s_{2k})\end{bmatrix}\ \middle|\ \begin{matrix}1\le i_1,i_2,...,i_{2k}\le n_1\\ 1\le j_1,j_2,...,j_{2k}\le n_2\\ \vdots\\ 1\le s_1,s_2,...,s_{2k}\le n_m\end{matrix}\right\},$$

where $i_1,i_2,...,i_{2k}$ are partitioned according to $\sim_{l_1}$, $j_1,j_2,...,j_{2k}$ are partitioned according to $\sim_{l_2}$, ..., $s_1,s_2,...,s_{2k}$ are partitioned according to $\sim_{l_m}$.

Then $\Omega(T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})})$ describes the position of nonzero entries in $T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})}$. For $\alpha\in\Omega(T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})})$, let O_α denote the $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$-orbit of α. Observe that $|\Omega(T_{(\sim_{l_1},\sim_{l_2},...,\sim_{l_m})})| = (n_1!n_2!\cdots n_m!)/(n_1-l_1)!(n_2-l_2)!\cdots(n_m-l_m)!$ and that

$|O_\alpha| = (n_1!n_2!\cdots n_m!)/2^m|(A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m})_\alpha|$, where $(A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m})_\alpha \subseteq A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$ is the stabilizer of α. If $\Omega(T_{(\sim_{l_1},\sim_{l_2},\ldots,\sim_{l_m})})$ is the disjoint union of two $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$-orbits, then we say that $T_{(\sim_{l_1},\sim_{l_2},\ldots,\sim_{l_m})}\in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ splits when lifted to $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$.

It is difficult to compute and understand the dimension formula of the centralizer algebra $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$. Therefore, we first compute the dimensions of the centralizer for $m=2$ and $m=3$, and then we give the dimension formula for $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$. Now, we compute the dimensions of $End_{A_{n_1}\times A_{n_2}}(W^{\otimes k})$ and $End_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})$.

Proposition 3.2. *i) If* $2<n_1,n_2\le 2k$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{n_i-2}\prod_{j=1}^{2}S(2k,l_j)+2\sum_{x\in X_1^2}S(2k,n_{x_1}-1)\left(\sum_{\substack{l_i=1,\\ i\neq x_1\in[2]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1}}^{2}S(2k,l_j)\right)$$

$$+2\sum_{x\in X_1^2}S(2k,n_{x_1})\left(\sum_{\substack{l_i=1,\\ i\neq x_1\in[2]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1}}^{2}S(2k,l_j)\right)+\ 2^2\prod_{j=1}^{2}S(2k,n_j-1)+\ 2^2\sum_{x\in X_1^2}S(2k,n_{x_1})\prod_{\substack{j=1,\\ j\neq x_1}}^{2}$$

$$S(2k,n_j-1)+2^2\prod_{j=1}^{2}S(2k,n_j).$$

ii) If $n_1,n_2=2k+1$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k-1}\prod_{j=1}^{2}S(2k,l_j)+2S(2k,2k)\sum_{x\in X_1^2}\left(\sum_{l_{x_1}=1}^{2k-1}S(2k,l_{x_1})\right)+2^2$$

$$S(2k,2k)S(2k,2k).$$

iii) If $n_1,n_2\ge 2k+2$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k}\prod_{j=1}^{2}S(2k,l_j) = [B(2k)]^2.$$

Proof. i) We know that, $dimEnd_{S_{n_1}\times S_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{n_i}\prod_{j=1}^{2}S(2k,l_j)$. Observe that

$$dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{n_i}c_{l_1,l_2}\prod_{j=1}^{2}S(2k,l_j),$$

where c_{l_1,l_2} is the number of disjoint $A_{n_1}\times A_{n_2}$-orbits that $\Omega(T_{(\sim_{l_1},\sim_{l_2})})$ comprises when $T_{(\sim_{l_1},\sim_{l_2})}$ is of type $\prod_{j=1}^{2}S(2k,l_j)$ (c_{l_1,l_2} is only depends on l_1,l_2). We will see that $c_{l_1,l_2}\in\{1,2,2^2\}$. In other words, $T_{(\sim_{l_1},\sim_{l_2})}\in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ of type $\prod_{j=1}^{2}S(2k,l_j)$ either splits as $T_{(\sim_{l_1},\sim_{l_2})} = T^-_{(\sim_{l_1},\sim_{l_2})}+T^+_{(\sim_{l_1},\sim_{l_2})}$ into a sum of two basis element of $End_{A_{n_1}\times A_{n_2}}(W^{\otimes k})$ when lifted to $End_{A_{n_1}\times A_{n_2}}(W^{\otimes k})$, or remains a basis element of type $\prod_{j=1}^{2}S(2k,l_j)$ in $End_{A_{n_1}\times A_{n_2}}(W^{\otimes k})$.

Let $\alpha\in\Omega(T_{(\sim_{n_1},\sim_{n_2})})$ where $T_{(\sim_{n_1},\sim_{n_2})}\in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ is of type $\prod_{j=1}^{2}S(2k,n_j)$. Then the only element of $A_{n_1}\times A_{n_2}$ that fixes all of the n_1 different entries in A_{n_1}, n_2 different entries in A_{n_2} of α is the identity. Therefore $|O_\alpha| = (n_1!n_2!)/2^2$. Since $|\Omega(T_{(\sim_{n_1},\sim_{n_2})})| = n_1!n_2!$, $c_{n_1,n_2} = 2^2$. Similarly, $c_{n_1,n_2-1} = c_{n_1-1,n_2} = c_{n_1-1,n_2-1} = 2^2$. Let $\alpha\in\Omega(T_{(\sim_{l_1},\sim_{n_2})})$ where $T_{(\sim_{l_1},\sim_{n_2})}\in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ is of type $S(2k,l_1)S(2k,n_2)$, $1\le l_1\le n_1-2$. Observe that $(A_{n_1}\times A_{n_2})_\alpha\cong A_{n_1-l_1}\times\{e_2\}$. Therefore $|O_\alpha| = (n_1!n_2!/2^2)/((n_1-l_1)!/2) = (n_1!n_2!)/2(n_1-l_1)!$. Since

$|\Omega(T_{(\sim_{l_1},\sim_{n_2})})| = (n_1!n_2!)/(n_1-l_1)!$, so $c_{l_1,n_2} = 2,\ (1 \le l_1 \le n_1-2)$. Similarly, $c_{l_1,n_2-1} = 2\ (1 \le l_1 \le n_1-2)$, $c_{n_1,l_2} = c_{n_1-1,l_2} = 2\ (1 \le l_2 \le n_2-2)$.

Now, let $\alpha \in \Omega(T_{(\sim_{l_1},\sim_{l_2})})$ where $T_{(\sim_{l_1},\sim_{l_2})} \in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ is of type $\prod_{j=1}^{2} S(2k,l_j)$, $1 \le l_j \le n_j-2,\ \forall j \in [2]$. Observe that $(A_{n_1}\times A_{n_2})_\alpha \cong A_{n_1-l_1}\times A_{n_2-l_2}$. Therefore $|O_\alpha| = (n_1!n_2!/2^2)/((n_1-l_1)!(n_2-l_2)!/2^2) = (n_1!n_2!)/(n_1-l_1)!(n_2-l_2)! = |\Omega(T_{(\sim_{l_1},\sim_{l_2})})|$, so $c_{l_1,l_2} = 1\ (1 \le l_j \le n_j-2, \forall j \in [2])$, hence the result follows.

ii) Recall that in this case, $dimEnd_{S_{n_1}\times S_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k} \prod_{j=1}^{2} S(2k,l_j)$. We assume that $dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k} c_{l_1,l_2}\prod_{j=1}^{2} S(2k,l_j)$ and as by (i), the only element $T_{(\sim_{l_1},\sim_{l_2})} \in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ that can split into one the types of $\prod_{j=1}^{2} S(2k,n_j), S(2k,n_{x_1})\prod_{\substack{j=1,\\ j\neq x_1}}^{2} S(2k,n_j-1)\Big(x\in X_1^2\Big), \prod_{j=1}^{2} S(2k,n_j-1), S(2k,n_{x_1})\prod_{\substack{j=1,\\ j\neq x_1}}^{2} S(2k,l_j)$ $\Big(x\in X_1^2\Big), S(2k,n_{x_1}-1)\prod_{\substack{j=1,\\ j\neq x_1}}^{2} S(2k,l_j)\Big(x\in X_1^2\Big)$, where $1 \le l_j \le n_j-2 = 2k-1, \forall j \in [2]$. Since $n_1 = n_2 = 2k+1$. Therefore, c_{n_1,n_2}, c_{n_1,n_2-1}, c_{n_1-1,n_2}, c_{n_1,l_2}, c_{l_1,n_2} $(1 \le l_1,l_2 \le 2k-1)$ do not appear in the sum above, only $T_{(\sim_{l_1},\sim_{l_2})} \in End_{S_{n_1}\times S_{n_2}}(W^{\otimes k})$ of type $S(2k,n_1-1)S(2k,n_2-1), S(2k,n_1-1)S(2k,l_2), S(2k,l_1)S(2k,n_2-1)$ splits. Thus $c_{2k,2k} = 2^2, c_{2k,l_2} = c_{l_1,2k} = 2$, and $c_{l_1,l_2} = 1$, where $1 \le l_1,l_2 \le 2k-1$.

iii) In this case, $dimEnd_{S_{n_1}\times S_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k}\prod_{j=1}^{2} S(2k,l_j) = [B(2k)]^2$. Since $n_1,n_2 \ge 2k+2$, so $c_{n_1,n_2}, c_{n_1,n_2-1}, c_{n_1-1,n_2}, c_{n_1-1,n_2-1}, c_{n_1,l_2}, c_{l_1,n_2}, c_{n_1-1,l_2}, c_{l_1,n_2-1}\ (1 \le l_1,l_2 \le 2k)$ do not appear in the sum $dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[2]}}^{2k} c_{l_1,l_2}\prod_{j=1}^{2} S(2k,l_j)$, and therefore $c_{l_1,l_2} = 1\ (1 \le l_1,l_2 \le 2k)$. Hence, the result follows. □

Proposition 3.3. *i) If* $2 < n_1,n_2,n_3 \le 2k$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{n_i-2}\prod_{j=1}^{3} S(2k,l_j) + 2\sum_{x\in X_1^3} S(2k,n_{x_1}-1)\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1\in[3]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,l_j)\Bigg)$$
$$+2\sum_{x\in X_1^3} S(2k,n_{x_1})\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1\in[3]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,l_j)\Bigg) + 2^2\sum_{x\in X_2^3} S(2k,n_{x_1}-1)S(2k,n_{x_2}-1)\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1,x_2\in[3]}}^{n_i-2}$$
$$\prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{3} S(2k,l_j)\Bigg) + 2^2\sum_{y\in Y_{1,1}^3} S(2k,n_{x_1})S(2k,n_{y_1}-1)\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1,y_1\in[3]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1,y_1}}^{3} S(2k,l_j)\Bigg) + 2^2\sum_{x\in X_2^3}$$
$$S(2k,n_{x_1})S(2k,n_{x_2})\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1,x_2\in[3]}}^{n_i-2}\prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{3} S(2k,l_j)\Bigg) + 2^3\prod_{j=1}^{3} S(2k,n_j-1) + 2^3\sum_{x\in X_1^3} S(2k,n_{x_1})$$
$$\prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,n_j-1) + 2^3\sum_{x\in X_2^3} S(2k,n_{x_1})S(2k,n_{x_2})\prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{3} S(2k,n_j-1) + 2^3\prod_{j=1}^{3} S(2k,n_j).$$

ii) If $n_1,n_2,n_3 = 2k+1$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{2k-1}\prod_{j=1}^{3} S(2k,l_j) + 2S(2k,2k)\sum_{x\in X_2^3}\Bigg(\sum_{\substack{l_{x_j}=1,\\ j\in[2]}}^{2k-1}\prod_{j=1}^{2} S(2k,l_{x_j})\Bigg)$$

$$+\,2^2S(2k,2k)S(2k,2k)\sum_{x\in X_1^3}\left(\sum_{l_{x_1}=1}^{2k-1}S(2k,l_{x_1})\right)+2^3S(2k,2k)S(2k,2k)S(2k,2k).$$

iii) If $n_1,n_2,n_3\geq 2k+2$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})=\sum_{\substack{l_i=1,\\ i\in[3]}}^{2k-1}\prod_{j=1}^{3}S(2k,l_j)=[B(2k)]^3.$$

Proof. i) In this case $dimEnd_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})=\sum_{\substack{l_i=1,\\ i\in[3]}}^{n_i}\prod_{j=1}^{3}S(2k,l_j)$. Observe that,

$$dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})=\sum_{\substack{l_i=1,\\ i\in[3]}}^{n_i}c_{l_1,l_2,l_3}\prod_{j=1}^{3}S(2k,l_j),$$

where c_{l_1,l_2,l_3} is the number of disjoint $A_{n_1}\times A_{n_2}\times A_{n_3}$-orbits that $\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})})$ comprises when $T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}$ is of type $\prod_{j=1}^{3}S(2k,l_j)$ (c_{l_1,l_2,l_3} is only depends on l_1,l_2,l_3). We will see that $c_{l_1,l_2,l_3}\in\{1,2,2^2,2^3\}$. In other words, $T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}\in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ of type $\prod_{j=1}^{3}S(2k,l_j)$ either splits as $T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}=T^-_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}+T^+_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}$ into a sum of two basis element of $End_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})$ when lifted to $End_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})$, or remains a basis element of type $\prod_{j=1}^{3}S(2k,l_j)$ in $End_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})$.

Suppose that $\alpha\in\Omega(T_{(\sim_{n_1},\sim_{n_2},\sim_{n_3})})$ where $T_{(\sim_{n_1},\sim_{n_2},\sim_{n_3})}\in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ is of type $\prod_{j=1}^{3}S(2k,n_j)$. Then the only element of $A_{n_1}\times A_{n_2}\times A_{n_3}$ that fixes all of the n_1 different entries in A_{n_1}, n_2 different entries in A_{n_2}, n_3 different entries in A_{n_3} of α is identity. Hence $|O_\alpha|=(n_1!n_2!n_3!)/2^3$. Since $|\Omega(T_{(\sim_{n_1},\sim_{n_2},\sim_{n_3})})|=n_1!n_2!n_3!$, $c_{n_1,n_2,n_3}=2^3$. Like wise, $c_{n_1,n_2,n_3-1}=c_{n_1,n_2-1,n_3}=c_{n_1-1,n_2,n_3}=c_{n_1,n_2-1,n_3-1}=c_{n_1-1,n_2,n_3-1}=c_{n_1-1,n_2-1,n_3}=c_{n_1-1,n_2-1,n_3-1}=2^3$.

Let $\alpha\in\Omega(T_{(\sim_{l_1},\sim_{n_2},\sim_{n_3})})$ where $T_{(\sim_{l_1},\sim_{n_2},\sim_{n_3})}\in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ is of type $S(2k,l_1)\prod_{j=2}^{3}S(2k,n_j)$, $1\leq l_1\leq n_1-2$. Observe that $(A_{n_1}\times A_{n_2}\times A_{n_3})_\alpha\cong A_{n_1-l_1}\times\{e_2\}\times\{e_3\}$. Therefore $|O_\alpha|=(n_1!n_2!n_3!/2^3)/((n_1-l_1)!/2)=(n_1!n_2!n_3!)/2^2(n_1-l_1)!$. Since $|\Omega(T_{(\sim_{l_1},\sim_{n_2},\sim_{n_3})})|=(n_1!n_2!n_3!)/(n_1-l_1)!$, so $c_{l_1,n_2,n_3}=2^2$, $(1\leq l_1\leq n_1-2)$. Similarly, $c_{l_1,n_2-1,n_3}=c_{l_1,n_2,n_3-1}=c_{l_1,n_2-1,n_3-1}=2^2$, $(1\leq l_1\leq n_1-2)$, $c_{n_1,l_2,n_3}=c_{n_1,l_2,n_3-1}=c_{n_1-1,l_2,n_3}=c_{n_1-1,l_2,n_3-1}=2^2$, $(1\leq l_2\leq n_2-2)$, $c_{n_1,n_2,l_3}=c_{n_1-1,n_2,l_3}=c_{n_1,n_2-1,l_3}=c_{n_1-1,n_2-1,l_3}=2^2$, $(1\leq l_3\leq n_3-2)$.

Let $\alpha\in\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3})})$ where $T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3})}\in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ is of type $S(2k,n_3)\prod_{j=2}^{3}S(2k,l_j), 1\leq l_1\leq n_1-2, 1\leq l_2\leq n_2-2$. Observe that $(A_{n_1}\times A_{n_2}\times A_{n_3})_\alpha\cong A_{n_1-l_1}\times A_{n_2-l_2}\times\{e_3\}$. Hence $|O_\alpha|=(n_1!n_2!n_3!/2^3)/((n_1-l_1)!(n_2-l_2)!/2^2)=(n_1!n_2!n_3!)/2(n_1-l_1)!(n_2-l_2)!$. Since $|\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3})})|=(n_1!n_2!n_3!)/(n_1-l_1)!(n_2-l_2)!$, so we get $c_{l_1,l_2,n_3}=2$, $(1\leq l_1\leq n_1-2, 1\leq l_2\leq n_2-2)$. Similarly, $c_{l_1,l_2,n_3-1}=2$, $(1\leq l_1\leq n_1-2, 1\leq l_2\leq n_2-2)$, $c_{l_1,n_2,l_3}=c_{l_1,n_2-1,l_3}=2$, $(1\leq l_1\leq n_1-2, 1\leq l_3\leq n_3-2)$, $c_{n_1,l_2,l_3}=c_{n_1-1,l_2,l_3}=2$, $(1\leq l_2\leq n_2-2, 1\leq l_3\leq n_3-2)$.

Now, let $\alpha\in\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})})$ where $T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})}\in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ is of type $\prod_{j=1}^{3}S(2k,l_j), 1\leq l_j\leq n_j-2, \forall j\in[3]$. Note that $(A_{n_1}\times A_{n_2}\times A_{n_3})_\alpha\cong A_{n_1-l_1}\times A_{n_2-l_2}\times A_{n_3-l_3}$. Hence $|O_\alpha|=(n_1!n_2!n_3!/2^3)/((n_1-l_1)!(n_2-l_2)!(n_3-l_3)!/2^3)=(n_1!n_2!n_3!)/(n_1-l_1)!(n_2-l_2)!(n_3-l_3)!=|\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})})|$, so $c_{l_1,l_2,l_3}=1$, $(1\leq l_j\leq n_j-2,\forall j\in[3])$, and the result follows.

ii) Recall that in this case $dimEnd_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{2k} \prod_{j=1}^{3} S(2k,l_j)$. We assume that $dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{2k} c_{l_1,l_2,l_3} \prod_{j=1}^{3} S(2k,l_j)$ and as by (i), the only $T_{(\sim_{l_1},\sim_{l_2},\sim_{l_3})} \in End_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k})$ that can split into one the types of $\prod_{j=1}^{3} S(2k,n_j), S(2k,n_{x_1})S(2k,n_{x_2}) \prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{3} S(2k,n_j-1)\Big(x\in X_2^3\Big), S(2k,n_{x_1}) \prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,n_j-1)\Big(x\in X_1^3\Big), \prod_{j=1}^{3} S(2k,n_j-1)$, $S(2k,n_{x_1})S(2k,n_{x_2}) \prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{3} S(2k,l_j)\Big(x \in X_2^3\Big), S(2k,n_{x_1})S(2k,n_{y_1}-1) \prod_{\substack{j=1,\\ j\neq x_1,y_1}}^{3} S(2k,l_j)\Big(y \in Y_{1,1}^3\Big), S(2k,n_{x_1}-1)S(2k,n_{x_2}-1) \prod_{j=1,j\neq x_1,x_2}^{3} S(2k,l_j)\Big(x\in X_2^3\Big)$, $S(2k,n_{x_1}) \prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,l_j)\Big(x \in X_1^3\Big)$, $S(2k,n_{x_1}-1) \prod_{\substack{j=1,\\ j\neq x_1}}^{3} S(2k,l_j)\Big(x \in X_1^3\Big)$, where $1\leq l_j \leq n_j-2 = 2k-1, \forall j\in[3]$. Since $n_1=n_2=n_3=2k+1$. Hence, $c_{n_1,n_2,n_3}, c_{n_1,n_2-1,n_3-1}, c_{n_1-1,n_2,n_3-1}, c_{n_1-1,n_2-1,n_3}, c_{n_1,n_2,n_3-1}$, $c_{n_1,n_2-1,n_3}, c_{n_1-1,n_2,n_3}, c_{n_1,n_2,l_3}, c_{n_1,l_2,n_3}, c_{l_1,n_2,n_3}, c_{n_1,n_2-1,l_3}, c_{n_1-1,n_2,l_3}, c_{n_1,l_2,n_3-1}, c_{n_1-1,l_2,n_3}$, $c_{l_1,n_2,n_3-1}, c_{l_1,n_2-1,n_3}, c_{n_1,l_2,l_3}, c_{l_1,n_2,l_3}, c_{l_1,l_2,n_3}(1\leq l_1,\ l_2,\ l_3\leq 2k-1)$ do not appear in the sum above. Thus, $c_{2k,2k,2k}=2^3, c_{2k,2k,l_3}=c_{2k,l_2,2k}=c_{l_1,2k,2k}=2^2, c_{2k,l_2,l_3}=c_{l_1,2k,l_3}=c_{l_1,l_2,2k}=2$, and $c_{l_1,l_2,l_3}=1$, where $1\leq l_1,\ l_2,\ l_3\leq 2k-1$.

iii) In this case $dimEnd_{S_{n_1}\times S_{n_2}\times S_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{2k} \prod_{j=1}^{3} S(2k,l_j) = [B(2k)]^3$. Since $n_1,n_2,n_3 \geq 2k+2$, so $c_{l_1,l_2,l_3}=1,\ (1\leq l_1,l_2,l_3\leq 2k)$, and all other coefficients do not appear in the sum $dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[3]}}^{2k} c_{l_1,l_2,l_3} \prod_{j=1}^{3} S(2k,l_j)$. The result follows. □

4 The Dimension Formula of $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$

In this section, we give the dimension formula for the centralizer $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$ by using the index sets X_t^m and $Y_{t,s}^m$. Also, we demonstrate that for $n_1,n_2,...,n_m\geq 2k+2$, the centralizer is isomorphic to the tensor product partition algebra $P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)$.

Let $[m]=\{1,2,...,m\}$ and let t,s are positive integers. Now, define the index sets X_t^m with $t\leq m$ and $Y_{t,s}^m$ with $2\leq t+s\leq m-1$ as follows:

$$X_t^m = \Big\{x=(x_1,x_2,...,x_t)\ \Big|\ x_1<x_2<\cdots<x_t \text{ and } 1\leq x_i\leq m, \forall\ 1\leq i\leq t\Big\} \text{ and}$$

$$Y_{t,s}^m = \left\{ y=(x_1,x_2,...,x_t,y_1,y_2,...,y_s)\ \middle|\ \begin{array}{l} x_1<x_2<\cdots<x_t,\ y_1<y_2<\cdots<y_s \text{ and} \\ 1\leq x_i\neq y_j\leq m, \forall\ 1\leq i\leq t, 1\leq j\leq s \end{array}\right\}.$$

Proposition 4.1. *i) If* $2<n_1,n_2,...,n_m\leq 2k$, *then*

$$dimEnd_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[m]}}^{n_i-2} \prod_{j=1}^{m} S(2k,l_j) + 2\sum_{x\in X_1^m} S(2k,n_{x_1}-1)\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1\in[m]}}^{n_i-2} \prod_{\substack{j=1,\\ j\neq x_1}}^{m} S(2k,l_j)\Bigg) + 2\sum_{x\in X_1^m} S(2k,n_{x_1})\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1\in[m]}}^{n_i-2} \prod_{\substack{j=1,\\ j\neq x_1}}^{m} S(2k,l_j)\Bigg) + 2^2\sum_{x\in X_2^m} S(2k,n_{x_1}-1)S(2k,n_{x_2}-1) \Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1,x_2\in[m]}}^{n_i-2} \prod_{\substack{j=1,\\ j\neq x_1,x_2}}^{m} S(2k,l_j)\Bigg) + 2^2\sum_{y\in Y_{1,1}^m} S(2k,n_{x_1})S(2k,n_{y_1}-1)\Bigg(\sum_{\substack{l_i=1,\\ i\neq x_1,y_1\in[m]}}^{n_i-2} \prod_{\substack{j=1,\\ j\neq x_1,y_1}}^{m} S(2k,l_j)\Bigg) +$$

$$2^2 \sum_{x \in X_2^m} S(2k, n_{x_1}) S(2k, n_{x_2}) \Bigg(\sum_{\substack{l_i=1, \\ i \neq x_1, x_2 \in [m]}}^{n_i-2} \prod_{\substack{j=1, \\ j \neq x_1, x_2}}^{m} S(2k, l_j) \Bigg) + 2^3 \sum_{x \in X_3^m} S(2k, n_{x_1}-1) S(2k, n_{x_2}-1)$$
$$S(2k, n_{x_3}-1) \Bigg(\sum_{\substack{l_i=1, \\ i \neq x_1, x_2, x_3 \in [m]}}^{n_i-2} \prod_{\substack{j=1, \\ j \neq x_1, x_2, x_2}}^{m} S(2k, l_j) \Bigg) + 2^3 \sum_{y \in Y_{1,2}^m} S(2k, n_{x_1}) S(2k, n_{y_1}-1) S(2k, n_{y_2}-1)$$
$$\Bigg(\sum_{\substack{l_i=1, \\ i \neq x_1, y_1, y_2 \in [m]}}^{n_i-2} \prod_{\substack{j=1, \\ j \neq x_1, y_1, y_2}}^{m} S(2k, l_j) \Bigg) + 2^3 \sum_{y \in Y_{2,1}^m} S(2k, n_{x_1}) S(2k, n_{x_2}) S(2k, n_{y_1}-1) \Bigg(\sum_{\substack{l_i=1, \\ i \neq x_1, x_2, y_1 \in [m]}}^{n_i-2}$$
$$\prod_{\substack{j=1, \\ j \neq x_1, x_2, y_1}}^{m} S(2k, l_j) \Bigg) + 2^3 \sum_{x \in X_3^m} S(2k, n_{x_1}) S(2k, n_{x_2}) S(2k, n_{x_3}) \Bigg(\sum_{\substack{l_i=1, \\ i \neq x_1, x_2, x_3 \in [m]}}^{n_i-2} \prod_{\substack{j=1, \\ j \neq x_1, x_2, x_2}}^{m} S(2k, l_j) \Bigg)$$
$$+ \cdots\cdots + 2^m \prod_{j=1}^{m} S(2k, n_j-1) + 2^m \sum_{x \in X_1^m} S(2k, n_{x_1}) \prod_{\substack{j=1, \\ j \neq x_1}}^{m} S(2k, n_j-1) + 2^m \sum_{x \in X_2^m} S(2k, n_{x_1})$$
$$S(2k, n_{x_2}) \prod_{\substack{j=1, \\ j \neq x_1, x_2}}^{m} S(2k, n_j-1) + \cdots\cdots + 2^m \prod_{j=1}^{m} S(2k, n_j).$$

ii) If $n_1, n_2, ..., n_m = 2k+1$, *then*

$$dimEnd_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1, \\ i \in [m]}}^{2k-1} \prod_{j=1}^{m} S(2k, l_j) + 2S(2k, 2k) \sum_{x \in X_{m-1}^m} \Bigg(\sum_{\substack{l_{x_j}=1, \\ j \in [m-1]}}^{2k-1} \prod_{j=1}^{m-1}$$
$$S(2k, l_{x_j}) \Bigg) + 2^2 S(2k, 2k) S(2k, 2k) \sum_{x \in X_{m-2}^m} \Bigg(\sum_{\substack{l_{x_j}=1, \\ j \in [m-2]}}^{2k-1} \prod_{j=1}^{m-2} S(2k, l_{x_j}) \Bigg) + \cdots\cdots + 2^{m-1}$$
$$\underbrace{S(2k, 2k) \cdots S(2k, 2k)}_{m-1\text{-times}} \sum_{x \in X_1^m} \Bigg(\sum_{l_{x_1}=1}^{2k-1} S(2k, l_{x_1}) \Bigg) + 2^m \underbrace{S(2k, 2k) \cdots S(2k, 2k)}_{m\text{-times}}.$$

iii) If $n_1, n_2, ..., n_m \geq 2k+2$, *then*

$$dimEnd_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1, \\ i \in [m]}}^{2k} \prod_{j=1}^{m} S(2k, l_j) = [B(2k)]^m.$$

Proof. i) Consider $dimEnd_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1, \\ i \in [m]}}^{n_i} \prod_{j=1}^{m} S(2k, l_j)$. Note that,

$$dimEnd_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1, \\ i \in [m]}}^{n_i} c_{l_1, l_2, \ldots, l_m} \prod_{j=1}^{m} S(2k, l_j),$$

where $c_{l_1, l_2, \ldots, l_m}$ is the number of disjoint $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$-orbits that $\Omega(T_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})})$ comprises when $T_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})}$ is of type $\prod_{j=1}^{m} S(2k, l_j)$. We will see that $c_{l_1, l_2, \ldots, l_m} \in \{1, 2, 2^2, ..., 2^m\}$. In other words, $T_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})} \in End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k})$ of type $\prod_{j=1}^{m} S(2k, l_j)$ either splits as $T_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})} = T^-_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})} + T^+_{(\sim_{l_1}, \sim_{l_2}, \ldots, \sim_{l_m})}$ into a sum of two basis element of $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$ when lifted to $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$, or remains a basis element of type $\prod_{j=1}^{m} S(2k, l_j)$ in $End_{A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}}(W^{\otimes k})$.

Let $\alpha \in \Omega(T_{(\sim_{n_1}, \sim_{n_2}, \ldots, \sim_{n_m})})$ where $T_{(\sim_{n_1}, \sim_{n_2}, \ldots, \sim_{n_m})} \in End_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}}(W^{\otimes k})$ is of type $\prod_{j=1}^{m} S(2k, n_j)$. Then the only element of $A_{n_1} \times A_{n_2} \times \cdots \times A_{n_m}$ that fixes all of the n_1 different entries in A_{n_1}, n_2 different entries in A_{n_2}, ..., n_m different entries in A_{n_m} of α is identity. Therefore $|O_\alpha| = (n_1! n_2! \cdots n_m!)/2^m$. Since $|\Omega(T_{(\sim_{n_1}, \sim_{n_2}, \ldots, \sim_{n_m})})| = n_1! n_2! \cdots n_m!$, $c_{n_1, n_2, \ldots, n_m} = 2^m$. Similarly, $c_{n_1, n_2, \ldots, n_m} = 2^m$, when any one of the $n_i, \forall i \in [m]$ is replaced by $n_i{-}1$, $c_{n_1,n_2,\ldots,n_m} = 2^m$, when any two of the $n_i, \forall i \in [m]$ are replaced by $n_i{-}1$, ..., $c_{n_1,n_2,\ldots,n_m} = 2^m$, when all of the $n_i, \forall i \in [m]$ are replaced by $n_i{-}1$.

Let $\alpha \in \Omega(T_{(\sim_{l_1},\sim_{n_2},\ldots,\sim_{n_m})})$ where $T_{(\sim_{l_1},\sim_{n_2},\ldots,\sim_{n_m})} \in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ is of type $S(2k,l_1)\prod_{j=2}^{m} S(2k,n_j)$, $1 \le l_1 \le n_1{-}2$. Observe that $(A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m})_\alpha \cong A_{n_1-l_1}\times\{e_2\}\times\cdots\times\{e_m\}$. Hence $|O_\alpha| = (n_1!n_2!\cdots n_m!/2^m)/((n_1{-}l_1)!/2) = (n_1!n_2!\cdots n_m!)/2^{m-1}(n_1{-}l_1)!$. Since $|\Omega(T_{(\sim_{l_1},\sim_{n_2},\ldots,\sim_{n_m})})| = (n_1!n_2!\cdots n_m!)/(n_1{-}l_1)!$, so $c_{l_1,n_2,\ldots,n_m} = 2^{m-1}\ (1 \le l_1 \le n_1-2)$. Similarly, $c_{l_1,n_2,\ldots,n_m} = 2^{m-1}(1 \le l_1 \le n_1{-}2)$, when any one of the $n_i, \forall i \neq 1 \in [m]$ is replaced by $n_i{-}1$, $c_{l_1,n_2,\ldots,n_m} = 2^{m-1}(1 \le l_1 \le n_1{-}2)$, when any two of the $n_i, \forall i \neq 1 \in [m]$ are replaced by $n_i{-}1$, ..., $c_{l_1,n_2,\ldots,n_m} = 2^{m-1}(1 \le l_1 \le n_1{-}2)$, when all of the $n_i, \forall i \neq 1 \in [m]$ are replaced by $n_i{-}1$.

In this similar way, we can find other $c_{l_1,l_2,\ldots,l_m} = 2^{m-1}(1 \le l_i \le n_i - 2)$ by fixing exactly one $l_i, \forall i \neq 1 \in [m]$ one by one.

Let $\alpha \in \Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3},\ldots,\sim_{n_m})})$ where $T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3},\ldots,\sim_{n_m})} \in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ is of type $S(2k,l_1)S(2k,l_2)\prod_{j=3}^{m} S(2k,n_j)$, $1 \le l_1 \le n_1-2$, $1 \le l_2 \le n_2-2$. Observe that $(A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m})_\alpha \cong A_{n_1-l_1}\times A_{n_2-l_2}\times\{e_3\}\times\cdots\times\{e_m\}$. Therefore $|O_\alpha| = (n_1!n_2!\cdots n_m!/2^m)/((n_1{-}l_1)!(n_2{-}l_2)!/2^2) = (n_1!n_2!\cdots n_m!)/2^{m-2}(n_1{-}l_1)!(n_2{-}l_2)!$. Since $|\Omega(T_{(\sim_{l_1},\sim_{l_2},\sim_{n_3},\ldots,\sim_{n_m})})| = (n_1!n_2!\cdots n_m!)/(n_1-l_1)!(n_2-l_2)!$, so $c_{l_1,l_2,n_3,\ldots,n_m} = 2^{m-2}\ (1 \le l_1 \le n_1{-}2, 1 \le l_2 \le n_2{-}2)$. Similarly, $c_{l_1,l_2,n_3,\ldots,n_m} = 2^{m-2}\ (1 \le l_1 \le n_1{-}2, 1 \le l_2 \le n_2{-}2)$, when any one of the $n_i, \forall i \neq 1,2 \in [m]$ is replaced by $n_i{-}1$, $c_{l_1,l_2,n_3,\ldots,n_m} = 2^{m-2}\ (1 \le l_1 \le n_1{-}2, 1 \le l_2 \le n_2{-}2)$, when any two of the $n_i, \forall i \neq 1,2 \in [m]$ are replaced by $n_i{-}1$, ..., $c_{l_1,l_2,n_3,\ldots,n_m} = 2^{m-2}\ (1 \le l_1 \le n_1{-}2, 1 \le l_2 \le n_2{-}2)$, when all of the $n_i, \forall i \neq 1,2 \in [m]$ are replaced by $n_i{-}1$.

In this similar way, we can find other $c_{l_1,l_2,\ldots,l_m} = 2^{m-2}\ (1 \le l_i \le n_i - 2)$ by fixing exactly two $l_i, \forall i \in [m]$ one by one.

Likewise, $c_{l_1,l_2,\ldots,l_m} = 2^{m-3}(1 \le l_i \le n_i - 2)$, when exactly three $l_i, \forall i \in [m]$ are fixed one by one, $c_{l_1,l_2,\ldots,l_m} = 2^{m-4}(1 \le l_i \le n_i - 2)$, when exactly four $l_i, \forall i \in [m]$ are fixed one by one, ..., $c_{l_1,l_2,\ldots,l_m} = 2(1 \le l_i \le n_i{-}2)$, when exactly $m{-}1$ $l_i, \forall i \in [m]$ are fixed one by one.

Finally, let $\alpha \in \Omega(T_{(\sim_{l_1},\sim_{l_2},\ldots,\sim_{l_m})})$ where $T_{(\sim_{l_1},\sim_{l_2},\ldots,\sim_{l_m})} \in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ is of type $\prod_{j=1}^{m} S(2k,l_j)$, $1 \le l_i \le n_i{-}2, \forall i \in [m]$. Note that $(A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m})_\alpha \cong A_{n_1-l_1} \times A_{n_2-l_2}\times\cdots\times A_{n_m-l_m}$. Hence $|O_\alpha| = (n_1!n_2!\cdots n_m!/2^m)/((n_1{-}l_1)!\cdots(n_m{-}l_m)!/2^m) = (n_1!n_2!\cdots n_m!)/(n_1{-}l_1)!(n_2{-}l_2)!\cdots(n_m{-}l_m)! = |\Omega(T_{(\sim_{l_1},\ldots,\sim_{l_m})})|$, so $c_{l_1,l_2,\ldots,l_m} = 1, 1 \le l_i \le n_i - 2, \forall i \in [m]$, and the result follows.

ii) We know that in this case $dimEnd_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[m]}}^{2k}\prod_{j=1}^{m} S(2k,l_j)$. Assume that $dimEnd_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k}) = \sum_{\substack{l_i=1,\\ i\in[m]}}^{2k} c_{l_1,l_2,\ldots,l_m}\prod_{j=1}^{m} S(2k,l_j)$, and by what we saw in (i), the only $T_{(\sim_{l_1},\sim_{l_2},\ldots,\sim_{l_m})} \in End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ that can split into one the types of $\prod_{j=1}^{m} S(2k,n_j), S(2k,n_{x_1})\cdots S(2k,n_{x_{m-1}})\prod_{\substack{j=1,\\ j\neq x_1,\ldots,x_{m-1}}}^{m} S(2k,n_j{-}1)\big(x \in X^m_{m-1}\big), S(2k,n_{x_1})\cdots S(2k,n_{x_{m-2}})\prod_{\substack{j=1,\\ j\neq x_1,\ldots,x_{m-2}}}^{m} S(2k,n_j{-}1)\big(x \in X^m_{m-2}\big), \ldots\ldots, S(2k,n_{x_1})\prod_{\substack{j=1,\\ j\neq x_1}}^{m} S(2k,n_j{-}1)\big(x \in X^m_1\big)$ $, \prod_{j=1}^{m} S(2k,n_j{-}1), S(2k,n_{x_1})\cdots S(2k,n_{x_{m-1}})\prod_{\substack{j=1,\\ j\neq x_1,\ldots,x_{m-1}}}^{m} S(2k,l_j)\big(x \in X^m_{m-1}\big), S(2k,n_{x_1})\cdots S(2k,n_{x_{m-2}})S(2k,n_{y_1}{-}1)\prod_{\substack{j=1,\\ j\neq x_1,\ldots,x_{m-2},y_1}}^{m} S(2k,l_j)\big(y \in Y^m_{m-2,1}\big), S(2k,n_{x_1})\cdots S(2k,n_{x_{m-3}})$

$S(2k,n_{y_1}-1)S(2k,n_{y_2}-1) \prod_{\substack{j=1,\\ j\neq x_1,\dots,x_{m-3},y_1,y_2}}^{m} S(2k,l_j)\big(y\in Y^m_{m-3,2}\big)$, ...,$S(2k,n_{x_1}-1)S(2k,n_{x_2}-1)$ $\cdots S(2k,n_{x_{m-1}}-1) \prod_{\substack{j=1,\\ j\neq x_1,\dots,x_{m-1}}}^{m} S(2k,l_j)\big(x\in X^m_{m-1}\big)$, ..., $S(2k,n_{x_1}) \prod_{\substack{j=1,\\ j\neq x_1}}^{m} S(2k,l_j)\big(x\in X^m_1\big)$, $S(2k,n_{x_1}-1) \prod_{\substack{j=1,\\ j\neq x_1}}^{m} S(2k,l_j)\Big(x\in X^m_1\Big)$, where $1\leq l_j\leq n_j-2=2k-1, \forall j\in[m]$. Since $n_1=n_2=\cdots=n_m=2k+1$. Therefore, $c_{l_1,l_2,\dots,l_m}$ do not appear in the sum above if at least one of the $l_i=n_i, \forall i\in[m]$. Thus, $c_{2k,2k,\dots,2k}=2^m$, $c_{l_1,2k,\dots,2k}=c_{2k,l_2,2k,\dots,2k}=\cdots=c_{2k,2k,\dots,2k,l_m}=2^{m-1}$, $c_{l_1,l_2,2k,\dots,2k}=c_{l_1,2k,l_3\dots,2k}=\cdots=c_{2k,\dots,2k,l_{m-1},l_m}=2^{m-2}$, ..., $c_{l_1,l_2,\dots,l_{m-1},2k}=\cdots=c_{2k,l_2,\dots,l_m}=2$, and $c_{l_1,l_2,\dots,l_m}=1$, where $1\leq l_1,l_2,\dots,l_m\leq 2k-1$.

iii) In this case $dimEnd_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})=\sum_{\substack{l_i=1,\\ i\in[m]}}^{2k}\prod_{j=1}^{m}S(2k,l_j)=[B(2k)]^m$. Since n_1, $n_2,\dots,n_m\geq 2k+2$, so $c_{l_1,l_2,\dots,l_m}=1,\ 1\leq l_i\leq 2k(n_i-2), \forall i\in[m]$ and all other coefficients do not appear in the sum $dimEnd_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})=\sum_{\substack{l_i=1,\\ i\in[m]}}^{l_i=2k} c_{l_1,l_2,\dots,l_m}\prod_{j=1}^{m}S(2k,l_j)$. The result follows. □

Illustration of Proposition 4.1:

1. Let $m=2$ then $X^2_1=\Big\{x=(x_1)\ \Big|\ 1\leq x_1\leq 2\Big\}=\Big\{x=1,2\Big\}$.

i) If $2<n_1,n_2\leq 2k$, then $dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k})=\sum_{l_1=1,l_2=1}^{n_1-2,n_2-2}S(2k,l_1)S(2k,l_2)+2S(2k,n_1)\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2S(2k,n_2)\sum_{l_1=1}^{n_1-2}S(2k,l_1)+2S(2k,n_1-1)\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2S(2k,n_2-1)\sum_{l_1=1}^{n_1-2}S(2k,l_1)+2^2S(2k,n_1-1)S(2k,n_2-1)+2^2S(2k,n_1)S(2k,n_2-1)+2^2S(2k,n_1-1)S(2k,n_2)+2^2S(2k,n_1)S(2k,n_2)$.

ii) If $n_1,n_2=2k+1$, then $dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k})=\sum_{l_1=1,l_2=1}^{2k-1,2k-1}S(2k,l_1)S(2k,l_2)+2S(2k,2k)\sum_{l_1=1}^{2k-1}S(2k,l_1)+2S(2k,2k)\sum_{l_2=1}^{2k-1}S(2k,l_2)+2^2S(2k,2k)S(2k,2k)$.

iii) If $n_1,n_2\geq 2k+2$, then $dimEnd_{A_{n_1}\times A_{n_2}}(W^{\otimes k})=\sum_{l_1,l_2=1}^{2k}S(2k,l_1)S(2k,l_2)=[B(2k)]^2$.

2. Let $m=3$ then $X^3_1=\Big\{x=(x_1)\ \Big|\ 1\leq x_1\leq 3\Big\}=\Big\{x=1,2,3\Big\}$,
$X^3_2=\Big\{x=(x_1,x_2)\ \Big|\ x_1<x_2 \text{ and } 1\leq x_i\leq 3, \forall\ 1\leq i\leq 2\Big\}=\Big\{x=(1,2),(1,3),(2,3)\Big\}$and
$Y^3_{1,1}=\Big\{y=(x_1,y_1)\ \Big|\ 1\leq x_1\neq y_1\leq 3\Big\}=\Big\{y=(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\Big\}$.

i) If $2<n_1,n_2,n_3\leq 2k$, then $dim\ End_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})=\sum_{l_1=1,l_2=1,l_3=1}^{n_1-2,n_2-2,n_3-2}S(2k,l_1)S(2k,l_2)S(2k,l_3)+2S(2k,n_1)\sum_{l_2=1,l_3=1}^{n_2-2,n_3-2}S(2k,l_2)S(2k,l_3)+2S(2k,n_2)\sum_{l_1=1,l_3=1}^{n_1-2,n_3-2}S(2k,l_1)S(2k,l_3)+2S(2k,n_3)\sum_{l_1=1,l_2=1}^{n_1-2,n_2-2}S(2k,l_1)S(2k,l_2)+2S(2k,n_1-1)\sum_{l_2=1,l_3=1}^{n_2-2,n_3-2}S(2k,l_2)S(2k,l_3)+2S(2k,n_2-1)\sum_{l_1=1,l_3=1}^{n_1-2,n_3-2}S(2k,l_1)S(2k,l_3)+2S(2k,n_3-1)\sum_{l_1=1,l_2=1}^{n_1-2,n_2-2}S(2k,l_1)S(2k,l_2)+2^2S(2k,n_1-1)S(2k,n_2-1)\sum_{l_3=1}^{n_3-2}S(2k,l_3)+2^2S(2k,n_1-1)S(2k,n_3-1)\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2^2S(2k,n_2-1)S(2k,n_3-1)\sum_{l_1=1}^{n_1-2}S(2k,l_1)$ $+2^2S(2k,n_1)S(2k,n_2-1)\sum_{l_3=1}^{n_3-2}S(2k,l_3)+2^2S(2k,n_1)S(2k,n_3-1)\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2^2S(2k,n_2)$

$S(2k,n_1-1)\sum_{l_3=1}^{n_3-2}S(2k,l_3)+2^2S(2k,n_2)S(2k,n_3-1)\sum_{l_1=1}^{n_1-2}S(2k,l_1)+2^2S(2k,n_3)S(2k,n_1-1)$ $\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2^2S(2k,n_3)S(2k,n_2-1)\sum_{l_1=1}^{n_1-2}S(2k,l_1)+2^2S(2k,n_1)S(2k,n_2)\sum_{l_3=1}^{n_3-2}S(2k,l_3)+$ $2^2S(2k,n_1)S(2k,n_3)\sum_{l_2=1}^{n_2-2}S(2k,l_2)+2^2S(2k,n_2)S(2k,n_3)\sum_{l_1=1}^{n_1-2}S(2k,l_1)+2^3S(2k,n_1-1)S(2k,$ $n_2-1)S(2k,n_3-1)+2^3S(2k,n_1)S(2k,n_2-1)S(2k,n_3-1)+2^3S(2k,n_2)S(2k,n_1-1)S(2k,n_3-1)$ $+2^3S(2k,n_3)S(2k,n_1-1)S(2k,n_2-1)+2^3S(2k,n_1)S(2k,n_2)S(2k,n_3-1)+2^3S(2k,n_1)S(2k,n_3)$ $S(2k,n_2-1)+2^3S(2k,n_2)S(2k,n_3)S(2k,n_1-1)+2^3S(2k,n_1)S(2k,n_2)S(2k,n_3)$.

ii) If $n_1,n_2,n_3=2k+1$, then $dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})=\sum_{l_1,l_2,l_3=1}^{2k-1}S(2k,l_1)S(2k,l_2)S(2k,$ $l_3)+2S(2k,2k)\sum_{l_1,l_2=1}^{2k-1}S(2k,l_1)S(2k,l_2)+2S(2k,2k)\sum_{l_1,l_3=1}^{2k-1}S(2k,l_1)S(2k,l_3)+2S(2k,2k)$ $\sum_{l_2,l_3=1}^{2k-1}S(2k,l_2)S(2k,l_3)+2^2S(2k,2k)S(2k,2k)\sum_{l_1=1}^{2k-1}S(2k,l_1)+2^2S(2k,2k)S(2k,2k)\sum_{l_2=1}^{2k-1}$ $S(2k,l_2)+2^2S(2k,2k)S(2k,2k)\sum_{l_3=1}^{2k-1}S(2k,l_3)+2^3S(2k,2k)S(2k,2k)S(2k,2k)$.

iii) If $n_1,n_2,n_3\geq 2k+2$, then
$dimEnd_{A_{n_1}\times A_{n_2}\times A_{n_3}}(W^{\otimes k})=\sum_{l_1,l_2,l_3=1}^{2k}S(2k,l_1)S(2k,l_2)S(2k,l_3)=[B(2k)]^3$.

Theorem 4.2. $\mathbb{C}[A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}]$ *and* $P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)$ *generate full centralizers of each other in* $End(W^{\otimes k})$. *In particular, for* $n_1,n_2,...,n_m\geq 2k+2$,

(a) $P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)\cong End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$,

(b) $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}$ *generates* $End_{P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)}(W^{\otimes k})$.

Proof. (a) For $n_1,n_2,...,n_m\geq 2k$, $P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)\cong End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$. Then $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})$ is a subalgebra of $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$ by (iii) of proposition 4.1 $dim\ End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})= dim\ End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})(W^{\otimes k})$, for $n_1,n_2,...,n_m\geq 2k+2$. Thus $End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})=End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$, for $n_1,...,n_m\geq 2k+2$. In particular, $P_k(n_1)\otimes P_k(n_2)\otimes\cdots\otimes P_k(n_m)\cong End_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_m}}(W^{\otimes k})=$ $End_{A_{n_1}\times A_{n_2}\times\cdots\times A_{n_m}}(W^{\otimes k})$, for $n_1,n_2,...,n_m\geq 2k+2$.

(b) follows from (a) and the Double Centralizer Theorem (See, [1]). □

5 Conclusion

In this paper, we described a basis for the centralizer algebra and computed the dimension formula for the centralizer over the restricted action on tensor products of its permutation representation. Computing the dimension formula of the centralizer algebra is very difficult here, so we have defined the index sets X_t^m with $t\leq m$ and $Y_{t,s}^m$ with $2\leq t+s\leq m-1$. The dimension of the centralizer algebra has been computed for the basic cases, and then extended for the general case using the index sets. Also, we showed that the centralizer algebra is isomorphic to the tensor product partition algebra for $n_1,n_2,...,n_m\geq 2k+2$.

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