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FCP Δ_0-extensions of rings

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Abstract An extension of commutative rings $R \subseteq S$ is called a Δ_0-extension if any R-submodule of S containing R is an R-subalgebra of S. We characterize FCP Δ_0-extensions, which are a special case of FCP Δ-extensions (the set of all subextensions is stable under the formation of sums) and that we studied in an earlier paper. Using Huckaba-Papick's result which say that a ring extension is a Δ_0-extension if and only if it is a Δ-extension such that any element of S is quadratic over R, a Δ_0-extension is integral. If $R \subseteq S$ is an FCP Δ_0-extension such that R is a local ring, then any R-subalgebra of S is comparable to the seminormalization ${}_S^+R$ and the t-closure ${}_S^tR$ of the extension. The converse holds adding some conditions on ${}_S^+R$ and ${}_S^tR$. The paper ends by considering Δ_0-extensions satisfying another condition as Boolean extensions, pointwise minimal extensions, idealizations, and extensions of the form $R \subseteq R^n$.

1 Introduction and Notation

In this paper, we consider the category of commutative and unital rings. If $R \subseteq S$ is a (ring) extension, we denote by $[R, S]$ the set of all R-subalgebras of S and set $]R, S[:= [R, S] \setminus \{R, S\}$ (with a similar definition for $[R, S[$ or $]R, S]$). For a submodule N of an R-module M, we denote by $[[N, M]]$ the set of all R-submodules of M containing N and set $[[M]] := [[0, M]]$.

When considering the structure of R-submodules of S containing R, we introduce the notion of Δ_0-extensions and the aim of the paper is to study these extensions. This paper is the sequel of a first paper [24] on FCP Δ-extensions and, in fact, a consequence of many papers, three of them published in the PJM: [6], [19] and [21].

A ring extension $R \subset S$ is called a Δ-*extension* by Gilmer and Huckaba [10, Definition page 414] if $T + U \in [R, S]$ for each $T, U \in [R, S]$, which is equivalent to $T + U = TU$ for each $T, U \in [R, S]$ ([24, Proposition 3.4]).

A ring extension $R \subset S$ is called a Δ_0-*extension* by Huckaba and Papick if $T \in [R, S]$ for each R-submodule T of S containing R ([12, Definition, page 430]), that is if $[[R, S]] = [R, S]$. The reader is warned that Δ_0-extensions are called quadratic extensions by Olberding [16, Definition 2.6]. Quadratic extensions in this paper denote a different concept.

A ring extension $R \subset S$ is called *quadratic* if each $t \in S$ satisfies a monic quadratic polynomial over R ([12, Definition, page 430]).

According to Huckaba-Papick's result stated for extensions of integral domains, but still valid for arbitrary extensions, we will greatly use our previous paper [24]. In [8, section 7.2], Fontana, Huckaba and Papick considered Δ_0-extensions of integral domains. Many of their results can be extended to arbitrary extensions.

Proposition 1.1. *[12, Proposition 5] A ring extension is a Δ_0-extension if and only if it is a quadratic Δ-extension. In particular, a Δ_0-extension is integral.*

For an extension $R \subseteq S$, the poset $([R, S], \subseteq)$ is a lattice, where the supremum of any non-void subset is the compositum of its elements, which we call *product* from now on and denote by Π when necessary, and the infimum of any non-void subset is the intersection of its elements. As a general rule, an extension $R \subseteq S$ is said to have some property of lattices if $[R, S]$ has this property. We use lattice definitions and properties described in [15].

The extension $R \subseteq S$ is said to have FIP (for the "finitely many intermediate algebras property") or is an FIP extension if $[R, S]$ is finite. A *chain* of R-subalgebras of S is a set of elements of $[R,S]$ that are pairwise comparable with respect to inclusion. We will say that $R \subseteq S$ is *chained*, also termed a λ-extension by some authors (see [9]), if $[R,S]$ is a chain. We also say that the extension $R \subseteq S$ has FCP (or is an FCP extension) if each chain in $[R,S]$ is finite. Clearly, each extension that satisfies FIP must also satisfy FCP. Dobbs and the authors characterized FCP and FIP extensions [2].

Our main tool will be the minimal (ring) extensions, a concept introduced by Ferrand-Olivier [7]. They are completely known (see Section 2). An extension $R \subset S$ is called *minimal* if $[R,S] = \{R,S\}$. The key connection between the above ideas is that if $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain $\mathcal{C}$ of R-subalgebras of S, $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, with *length* $\ell(\mathcal{C}) := n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. An FCP extension is finitely generated (as an R-algebra), and (module) finite if integral. For any extension $R \subseteq S$, the *length* $\ell[R,S]$ of $[R,S]$ is the supremum of the lengths of chains of R-subalgebras of S. Notice that if $R \subseteq S$ has FCP, then there *does* exist some maximal chain of R-subalgebras of S with length $\ell[R,S]$ [3, Theorem 4.11].

Any undefined material is explained at the end of the section or in the next sections.

Section 2 is devoted to some recalls and results on ring extensions. According to Proposition 1.1, a Δ_0-extension $R \subseteq S$ is integral, so we consider in this paper only integral extensions.

The general properties of Δ_0-extensions are given in Section 3 where the transfer of the Δ_0-property is gotten for several algebraic operations.

In Section 4, we make a more precise study of Δ_0-extensions. A Δ_0-extension $R \subseteq S$ satisfies the following, when FCP (Theorem 4.8): for each $M \in \mathrm{MSupp}(S/R)$, $[R_M, S_M] = [R_M, ({}_S^+R)_M] \cup [({}_S^+R)_M, ({}_S^tR)_M] \cup [({}_S^tR)_M, S_M]$, where ${}_S^tR$ is the t-closure of R in S and ${}_S^+R$ is the seminormalization of R in S (see Definition 2.4). In particular, it gives a characterization of Δ_0-extensions using the canonical decomposition.

The paper ends in Section 5 with some special Δ_0-extensions and examples of Δ_0-extensions. In particular, we consider Boolean extensions, pointwise minimal extensions and idealizations. These special cases allow to characterize more generally some Δ_0-extensions.

We denote by $(R:S)$ the conductor of $R \subseteq S$ and the characteristic of a field k by $\mathrm{c}(k)$.

Finally, $|X|$ is the cardinality of a set X, $\subset$ denotes proper inclusion and, for a positive integer n, we set $\mathbb{N}_n := \{1, \ldots, n\}$.

2 Recalls and results on ring extensions

A *local* ring is here what is called elsewhere a quasi-local ring. As usual, $\mathrm{Spec}(R)$ and $\mathrm{Max}(R)$ are the set of prime and maximal ideals of a ring R. The support of an R-module E is $\mathrm{Supp}_R(E) := \{P \in \mathrm{Spec}(R) \mid E_P \neq 0\}$, and $\mathrm{MSupp}_R(E) := \mathrm{Supp}_R(E) \cap \mathrm{Max}(R)$ (or $\mathrm{Supp}(E)$ and $\mathrm{MSupp}(E)$ if no confusion is possible). If E is an R-module, $\mathrm{L}_R(E)$ (also denoted $\mathrm{L}(E)$) is its length.

A ring morphism $f : R \to S$ (resp. an extension $R \subseteq S$) is said an *i-morphism* (resp. an *i-extension*) if the spectral map ${}^af : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ (resp. the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$) is injective. An integral extension $R \subseteq S$ is an i-extension if and only if the natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective.

If $R \subseteq S$ is a ring extension and $P \in \mathrm{Spec}(R)$, then S_P is both the localization $S_{R\setminus P}$ as a ring and the localization at P of the R-module S. We denote by $\kappa_R(P)$ the residual field R_P/PR_P at P.

The following notions and results are deeply involved in the sequel.

Definition 2.1. [24, Definition 2.1] An extension $R \subset S$ is called M-*crucial* if $\mathrm{Supp}(S/R) = \{M\}$. Such M is called the *crucial (maximal) ideal* $\mathcal{C}(R,S)$ of $R \subset S$.

Theorem 2.2. *[7, Théorème 2.2] A minimal extension is crucial and is either integral ((module)-finite) or a flat epimorphism.*

Three types of minimal integral extensions exist, characterized in the next theorem, (a consequence of the fundamental lemma of Ferrand-Olivier), so that there are four types of minimal extensions, mutually exclusive.

Theorem 2.3. *[2, Theorems 2.2 and 2.3] Let $R \subset T$ be an extension and $M := (R : T)$. Then $R \subset T$ is minimal and finite if and only if $M \in \text{Max}(R)$ and one of the following three conditions holds:*
(a) **inert case***: $M \in \text{Max}(T)$ and $R/M \to T/M$ is a minimal field extension.*
(b) **decomposed case***: There exist $M_1, M_2 \in \text{Max}(T)$ such that $M = M_1 \cap M_2$ and the natural maps $R/M \to T/M_1$ and $R/M \to T/M_2$ are both isomorphisms, or equivalently, there exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 - q \in M$ and $Mq \subseteq M$.*
(c) **ramified case***: There exists $M' \in \text{Max}(T)$ such that ${M'}^2 \subseteq M \subset M'$, $[T/M : R/M] = 2$, and the natural map $R/M \to T/M'$ is an isomorphism, or equivalently, there exists $q \in T \setminus R$ such that $T = R[q]$, $q^2 \in M$ and $Mq \subseteq M$.*
In each of the above cases, $M = \mathcal{C}(R, T)$.

The following definitions are needed for our study.

Definition 2.4. (1) An integral extension $R \subseteq S$ is called *infra-integral* [18] (resp.; *subintegral* [26]) if all its residual extensions $\kappa_R(P) \to \kappa_S(Q)$, (with $Q \in \text{Spec}(S)$ and $P := Q \cap R$) are isomorphisms (resp.; and is an i-extension). An extension $R \subseteq S$ is called *t-closed* (cf. [18]) if the relations $b \in S$, $r \in R$, $b^2 - rb \in R$, $b^3 - rb^2 \in R$ imply $b \in R$. A t-closed FCP extension $R \subseteq S$ is an i-extension [24, Proposition 2.10]. The *t-closure* ${}_S^tR$ of R in S is the smallest element $B \in [R, S]$ such that $B \subseteq S$ is t-closed and the greatest element $B' \in [R, S]$ such that $R \subseteq B'$ is infra-integral. An extension $R \subseteq S$ is called *seminormal* (cf. [26]) if the relations $b \in S$, $b^2 \in R$, $b^3 \in R$ imply $b \in R$. The *seminormalization* ${}_S^+R$ of R in S is the smallest element $B \in [R, S]$ such that $B \subseteq S$ is seminormal and the greatest element $B' \in [R, S]$ such that $R \subseteq B'$ is subintegral. We recall that t-closure and seminormalization commute with localization at arbitrary multiplicative closed sets ([26, Proposition 2.9] and [17, Proposition 3.6]). For a ring extension $R \subseteq S$ and any $P \in \text{Spec}(R)$, we have $({}_S^+R)_P = {}_{S_P}^+R_P$ and $({}_S^tR)_P = {}_{S_P}^tR_P$.

The *canonical decomposition* of an arbitrary ring extension $R \subset S$ is $R \subseteq {}_S^+R \subseteq {}_S^tR \subseteq \overline{R} \subseteq S$, where $\overline{R}$ is the integral closure of R in S.

(2) An extension $R \subset S$ is called *pinched* at the finite chain $\mathcal{C} := \{T_i\}_{i \in \mathbb{N}_n} \subseteq]R, S[$, $n \geq 1$ if $[R, S] = \cup_{i=0}^n [T_i, T_{i+1}]$, where $T_0 := R$ and $T_{n+1} := S$, which means that any element of $[R, S]$ is comparable to the T_i's.

(3) An extension $R \subset S$ is called *simple* if there exists $t \in S \setminus R$ such that $S = R[x]$.

3 General properties of Δ_0-extensions

We recall this first characterization of Δ_0-extensions.

Proposition 3.1. *[12, Remark 6 (ii)] A ring extension $R \subseteq S$ is a Δ_0-extension if and only if $R[s, t] = R + Rs + Rt$ for each $s, t \in S$.*

Many of the results of this section are a copy of similar results of [24].

Proposition 3.2. *Let $R \subseteq S$ be a ring extension. The following statements are equivalent:*

(1) $R \subseteq S$ is a quadratic extension, (resp.; Δ_0-extension).

(2) $R_M \subseteq S_M$ is a quadratic extension, (resp. ; Δ_0-extension) for each $M \in \text{MSupp}(S/R)$.

(3) $R_P \subseteq S_P$ is a quadratic extension, (resp.; Δ_0-extension) for each $P \in \text{Supp}(S/R)$.

(4) $R/I \subseteq S/I$ is a quadratic extension (resp.; Δ_0-extension) for an ideal I shared by R and S.

A simple extension generated by a quadratic element is quadratic.

Proof. We consider first the equivalences for the quadratic properties. Then, (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) are obvious.

Assume that (2) holds with $R_M \subseteq S_M$ quadratic. Let $t \in S$ and $M \in \text{MSupp}(S/R)$. There exist $a, b \in R$, $s \in R \setminus M$ such that $(t/1)^2 + (a/s)(t/1) + (b/s) = 0$, so that $t^2/1 \in (R/M)(t/1) + (R/M) = (Rt + R)_M$. Since this holds for any $M \in \text{MSupp}(S/R)$, it follows that $t^2 \in Rt + R$ and $R \subseteq S$ is quadratic. So, (1) holds.

At last, (1) $\Leftrightarrow$ (4) is obvious.

Assume that $R \subset S$ is simple and let $y \in S$ be a quadratic element generating S over R, so that $S = R[y] = R + Ry$. There exist $a, b \in R$ such that $y^2 = ay + b$. Let $x = \alpha y + \beta \in S$, $\alpha, \beta \in R$. Then $x^2 = (a\alpha + 2\beta)x + \alpha^2 b - a\alpha\beta - \beta^2$ shows that $R \subset S$ is quadratic.

Now, the equivalences for the Δ_0 properties are obvious using Proposition 1.1 and the equivalences for Δ-extensions given in [24, Proposition 3.6] and quadratic extensions gotten here. □

Proposition 3.3. *Let $R \subset S$ be a ring extension. Then $R \subset S$ is a Δ_0-extension if and only if $T \subseteq U$ is a Δ_0-extension for each subextension $[T, U] \subseteq [R, S]$.*

Proof. Obvious, because $[[T, U]] \subseteq [[R, S]]$ for any T-submodule of U containing T. □

Proposition 3.4. *Let $R \subset S$ be a ring extension, $f : R \to R'$ a ring morphism and $S' := R' \otimes_R S$.*

(1) If $f : R \to R'$ is faithfully flat and if $R' \subset S'$ is a Δ_0-extension, then so is $R \subset S$.

(2) If $f : R \to R'$ is a flat ring epimorphism and $R \subset S$ is a Δ_0-extension, then so is $R' \subset S'$.

Proof. (1) The case of a Δ-extension follows from [24, Proposition 3.8]. By considering the ring morphism $\varphi : S \to S'$, and since the inverse image of a subring is a ring, we get the statement for Δ_0-extensions.

(2) As in [24, Proposition 3.8], $R_P \to S_P$ identifies to $R'_Q \to (R' \otimes_R S)_Q$ for $Q \in \mathrm{Spec}(R')$, lying over P in R. □

Given a ring R, its *Nagata ring* $R(X)$ is the localization $R(X) = T^{-1}R[X]$ of the ring of polynomials $R[X]$ with respect to the multiplicatively closed subset T of all polynomials with content R. In [4, Theorem 32], Dobbs and the authors proved that when $R \subset S$ is an extension, whose Nagata extension $R(X) \subset S(X)$ has FIP, the map $\varphi : [R, S] \to [R(X), S(X)]$ defined by $\varphi(T) = T(X)$ is an order-isomorphism. We look at the transfer property of being a Δ_0-extension.

Proposition 3.5. *Let $R \subset S$ be an FCP extension such that $R(X) \subset S(X)$ is a Δ_0-extension. Then, so is $R \subset S$.*

Proof. By [3, Corollary 3.5], we have $S(X) = R(X) \otimes_R S$. Since $R \subset R(X)$ is faithfully flat, an application of Proposition 3.4 gives the result. □

4 Characterization of FCP Δ_0-extensions

Here is a first example of a Δ_0-extension.

Proposition 4.1. *Let $R \subset S$ be a chained FCP extension. Then the following are equivalent:*

(1) $R \subset S$ is a Δ_0-extension;

(2) $R \subset S$ is quadratic;

(3) there exists $y \in S$ quadratic over R such that $S = R[y]$.

Proof. $R \subset S$ is a Δ-extension by [19, Proposition 5.16]. Then (1) $\Leftrightarrow$ (2) by Proposition 1.1. Since $R \subset S$ is simple by [24, Propositions 5.18 and 5.17] or [25, Proposition 2.12], let $y \in S$ be such that $S = R[y]$. If $R \subset S$ is quadratic, so is y. Conversely, assume that there is a quadratic element $y \in S$ such that $S = R[y]$. Then $R \subset S$ is quadratic according to Proposition 3.2. □

A minimal extension is a special case of a chained extension and gives the following.

Proposition 4.2. *A minimal extension $R \subset S$ is a Δ_0-extension if and only if $R \subset S$ is integral, with $[S/(R : S) : R/(R : S)] = 2$ when $R \subset S$ is inert.*

Proof. Since a minimal extension is chained, according to Proposition 4.1, we get that $R \subset S$ is a Δ_0-extension if and only if there exists $y \in S$ quadratic over R such that $S = R[y]$.

Assume that $R \subset S$ is a Δ_0-extension, so that there exists $y \in S$ quadratic over R such that $S = R[y]$. Then, $R \subset S$ is minimal integral, so that $M := (R : S) \in \mathrm{Max}(R)$. Moreover,

$M \in \mathrm{Max}(S)$ when $R \subset S$ is inert by Theorem 2.3. Since y is quadratic, so is its class $\overline{y}$ in $S/M = (R/M)[\overline{y}]$, leading to $[S/(R:S) : R/(R:S)] = 2$.

Conversely, if $R \subset S$ is (minimal) integral, with $[S/(R:S) : R/(R:S)] = 2$ when $R \subset S$ is inert, then $S = R[y]$, for some $y \in S$, which is quadratic when $R \subset S$ is either minimal ramified with $y^2 \in R$, or minimal decomposed with $y^2 - y \in R$ (see Theorem 2.3). When, $R \subset S$ is inert, the first part of the proof gives that the class $\overline{y}$ of y in S/M is such that $S/M = (R/M)[\overline{y}]$ with $\overline{y}$ quadratic, and so is y. In any case, y is quadratic, so that $R \subset S$ is a Δ_0-extension. □

As for Δ-extensions, we are going to characterize Δ_0-extensions by means of the canonical decomposition.

We recall a result of Handelman, cited by Olberding, adapted to our context.

Proposition 4.3. *[16, Lemma 2.8], [11, Lemma 5] Let $K \subset S$ be an FCP Δ_0-extension where K is a field. Then, the K-algebra S is isomorphic to one of the following.*

(1) a field extension of K of degree 2;

(2) a local ring (S, M) such that $M^2 = 0$ with $K \subset S$ subintegral;

(3) K^2;

(4) K^3 with $K \cong \mathbb{Z}/2\mathbb{Z}$.

We will derive from the above proposition a characterization of Δ_0-extensions.

Proposition 4.4. *Let $R \subset S$ be an FCP Δ_0-extension, where (R, M) is a local ring. Then $R \subset S$ is pinched at ${}^{+}_{S}R$ and, if ${}^{+}_{S}R \neq S$, one of the following conditions holds:*

(1) ${}^{+}_{S}R = {}^{t}_{S}R$ and ${}^{t}_{S}R \subset S$ is inert minimal with residual extensions of degree ≤ 2.

(2) ${}^{t}_{S}R = S$ and ${}^{+}_{S}R \subset S$ is a decomposed minimal extension.

(3) ${}^{t}_{S}R = S$, $\ell[{}^{+}_{S}R, S] = 2$ and $R/M \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Since $R \subset S$ is a Δ_0-extension, $[{}^{+}_{S}R, {}^{t}_{S}R]$ and $[{}^{t}_{S}R, S]$ are Δ_0-extensions in view of Proposition 3.3. Since (R, M) is a local ring, so is ${}^{+}_{S}R$. Let N be its maximal ideal.

We begin to show that $R \subset S$ is pinched at ${}^{+}_{S}R$. Assume that there exists some $T \in [R, S] \setminus ([R, {}^{+}_{S}R] \cup [{}^{+}_{S}R, S])$. Set $U := {}^{+}_{T}R \subset {}^{+}_{S}R$. Then, U is a local ring. Let P be its maximal ideal. We have the following diagram:

$$\begin{array}{ccccccccc} & & & & & T & & & \\ & & & & \nearrow & & \searrow & & \\ R & \to & U = {}^{+}_{T}R & \to & & {}^{+}_{S}U & \to & S \\ & & & & & \downarrow & \nearrow & & \\ & & & & & {}^{+}_{S}R & & & \end{array}$$

with $U \subset {}^{+}_{S}U$ subintegral while $U \subset T$ is seminormal. Indeed, $U = {}^{+}_{S}U$ implies $U \subseteq S$ seminormal, so that $T \in [{}^{+}_{S}R, S]$, a contradiction. And $U = T$ implies $T \in [R, {}^{+}_{S}R]$, a contradiction.

Now, let $T_1 \in [U, T]$ be such that $U \subset T_1$ is minimal seminormal (either decomposed or inert) and let $U_1 \in [U, {}^{+}_{S}U]$ be such that $U \subset U_1$ is minimal ramified. Then $P = (U : T_1) = (U : U_1)$, which yields $P = (U : T_1U_1)$, with U/P being a field and $U/P \subset (T_1U_1)/P$ a Δ_0-extension in view of Proposition 3.2. Using Proposition 4.3, we get a contradiction because $U/P \subset (T_1U_1)/P$ is neither subintegral, nor seminormal, according to [5, Propositions 7.4 or 7.6]. Then, there does not exist any $T \in [R, S] \setminus ([R, {}^{+}_{S}R] \cup [{}^{+}_{S}R, S])$.

To conclude, $[R, S] = [R, {}^{+}_{S}R] \cup [{}^{+}_{S}R, S]$ and $R \subset S$ is pinched at ${}^{+}_{S}R$.

We next show that either ${}^{t}_{S}R = S$ or ${}^{t}_{S}R = {}^{+}_{S}R$.

If ${}^{+}_{S}R = S$, then ${}^{+}_{S}R = {}^{t}_{S}R$. Assume now that ${}^{+}_{S}R \neq S$. Since ${}^{+}_{S}R \subset S$ is FCP seminormal, we get that $({}^{+}_{S}R : S) = N$ by [2, Theorem 4.2 and Lemma 4.8]. Moreover, $K := ({}^{+}_{S}R)/N \subset S/N$ is a seminormal Δ_0-extension, where K is a field by Proposition 3.2. We consider the different cases of Proposition 4.3.

In case (1), $K \subset S/N$ is a field extension of degree 2, and then is minimal. Moreover, N is also a maximal ideal of S, so that ${}^{+}_{S}R \subset S$ is minimal inert and ${}^{+}_{S}R = {}^{t}_{S}R$.

In case (2), set $T' := {}_S^tR/N$ and $S' := S/N$, so that $T' = K$, with $K \subset S'$ both t-closed and subintegral, a contradiction since $K \neq S'$. This case does not occur.

In case (3), ${}_S^+R \subset S$ is minimal decomposed, so that $\ell[{}_S^+R, S] = 1$ and then ${}_S^tR = S$.

In case (4), $\ell[{}_S^+R, S] = 2$ and ${}_S^+R \subset S$ is seminormal infra-integral with ${}_S^tR = S$. Moreover, $R/M \cong K = ({}_S^+R)/N \cong \mathbb{Z}/2\mathbb{Z}$ because $R \subseteq {}_S^+R$ is subintegral. □

The previous proposition says that for an FCP Δ_0-extension $R \subset S$ which is not subintegral and where (R, M) is a local ring, either ${}_S^+R \subset S$ is minimal t-closed, or ${}_S^+R \subset S$ is seminormal infra-integral of length ≤ 2. To get a characterization of an FCP Δ_0-extension $R \subset S$, where (R, M) is a local ring, we will consider the two cases for ${}_S^+R \subset S$. Before, we adapt the following lemma from [9, Proposition 4.12].

Lemma 4.5. *Let $R \subset S$ be a simple extension generated by a quadratic element over R. Then, $R \subset S$ is a Δ_0-extension.*

Proof. Set $S = R[y]$, where y is a quadratic element over R. Then, $S = R + Ry$ and it follows from [9, Proposition 4.12] that $R \subset S$ is a Δ_0-extension. □

Let $R \subset S$ be a ring extension. We recall [24, at the beginning of Subsection 2.2] that R is called *unbranched* in S (or $R \subset S$ is unbranched) if $\overline{R}$ is local. An extension $R \subset S$ is said *locally unbranched* if $R_M \subset S_M$ is unbranched for all $M \in \mathrm{MSupp}(S/R)$ and is said *branched* if it is not unbranched.

Proposition 4.6. *Let $R \subset S$ be an integral FCP extension, where R is unbranched in S and such that ${}_S^tR \neq R, S$. Then $R \subset S$ is a Δ_0-extension if and only if $R \subset S$ is a simple extension generated by a quadratic element over R.*

Proof. Assume that $R \subset S$ is a Δ_0-extension. Since S is a local ring, so is R and it follows that ${}_S^+R = {}_S^tR$. In view of Proposition 4.4, we get that $[R, S] = [R, {}_S^+R] \cup [{}_S^+R, S]$ $(*)$, ${}_S^tR \subset S$ is an inert minimal extension and there exists $y \in S$ such that $S = ({}_S^tR)[y]$. Because $R \subset S$ is a Δ_0-extension, it is quadratic, so that y is a quadratic element over R. We show that $S = R[y]$. The condition $(*)$ implies that $R[y] \in [R, {}_S^+R] \cup [{}_S^+R, S]$. If $R[y] \in [R, {}_S^+R]$, then $y \in {}_S^+R = {}_S^tR$ implies that $S = ({}_S^tR)[y] = {}_S^tR$, a contradiction with the assumptions. Then $R[y] \in]{}_S^+R, S]$, whence $S = ({}_S^tR)[y] = ({}_S^+R)[y] \subseteq R[y] \subseteq S$ and therefore $S = R[y]$.

Conversely, assume that $R \subset S$ is a simple extension generated by a quadratic element over R. From Lemma 4.5 we infer that $R \subset S$ is a Δ_0-extension. □

Proposition 4.7. *Let $R \subset S$ be an integral FCP extension, where (R, M) is a local ring branched in S. Then $R \subset S$ is a Δ_0-extension if and only if the following conditions hold:*

(1) $R \subset S$ is infra-integral and pinched at ${}_S^+R$.

(2) $\ell[{}_S^+R, S] \leq 2$ with one of the following conditions:

(a) $\ell[{}_S^+R, S] = 1$ and $R \subset S$ is a simple extension generated by a quadratic element over R.

(b) $\ell[{}_S^+R, S] = 2$, $R/M \cong \mathbb{Z}/2\mathbb{Z}$ and S is generated by a minimal system of two quadratic elements $\{y, z\}$ over R such that $y + z$ is quadratic over R.

Proof. Let $R \subset S$ be an integral FCP extension, with (R, M) a local ring branched in S, so that S is not a local ring. Set $T := {}_S^+R \neq S$. Since T is a local ring, it follows that $T \neq {}_S^tR$ because ${}_S^tR \subseteq S$ is an i-extension (Definition 2.4). Let N be the maximal ideal of T that verifies $N = (T : S)$ by [2, Theorem 4.2 and Lemma 4.8].

Assume first that $R \subset S$ is a Δ_0-extension. In view of Proposition 4.4, $R \subset S$ is pinched at ${}_S^+R$, giving (1) with ${}_S^tR = S$ because ${}_S^+R \neq {}_S^tR$, so that $R \subset S$ is infra-integral, and $T \subset S$ is a seminormal infra-integral extension because S is not local.

Using Proposition 4.4 (2) or (3), we get $\ell[T, S] \leq 2$. If $\ell[T, S] = 1$, then $T \subset S$ is a minimal decomposed extension. In particular, there exists $y \in S$ such that $S = T[y]$ with y quadratic over R. Reasoning as in some part of the proof of Proposition 4.6 we get that $S = R[y]$, because we cannot have $R[y] \subseteq T$; so that $R[y] \in]T, S]$, leading to $R[y] = S$. Then $R \subset S$ is a simple extension generated by a quadratic element over R. Hence (2) (a) holds.

If $\ell[T,S]=2$, then $T/N\cong R/M\cong \mathbb{Z}/2\mathbb{Z}$ by Proposition 4.4 (3) and $T\subset S$ is a seminormal infra-integral extension. Since $N=(T:S)$, we get that $T/N\subset S/N$ is a seminormal infra-integral extension of length 2, with $S/N\cong (\mathbb{Z}/2\mathbb{Z})^3$ by Proposition 4.3. Then, $T\subset S$ is not simple by [22, Corollary 4.22]. In particular, [22, Propositions 2.2 and 2.4] show that there exists a minimal system of generators $y,z\in S$ over T, quadratic over R, such that $S=T[y,z]$. Moreover, $y+z$ is quadratic over R. Mimicking the proof of Proposition 4.6, we get that $R[y],R[z]\in]T,S]$. We claim that $y\not\in R[z]$ and $z\not\in R[y]$. Otherwise, this would imply that $R[y]$ and $R[z]$ are comparable, and so are $T[y]$ and $T[z]$, giving that S is the largest of them, contradicting the minimality of the system of two generators. In particular, $R[y],R[z]\subset R[y,z]$. Then, we have the extensions $T\subset R[y],R[z]\subset R[y,z]\subseteq S$. It follows that $R[y,z]=S$.

Conversely, assume that (1) and (2) hold. If (2) (a) holds, Lemma 4.5 shows that $R\subset S$ is a Δ_0-extension.

Now, assume that (2) (b) holds. Let $y,z\in S$ be a minimal system of two quadratic elements over R such that $S=R[y,z]$ with $y+z$ quadratic over R. We claim that $y,z\in S\setminus T$. If not, we may assume that $y\in T$, so that $S=T[z]$, with z quadratic over R, and then over T. Since $T\subset S$ is seminormal infra-integral, so is $\mathbb{Z}/2\mathbb{Z}\cong T/N\subset S/N=:S'$. Let $\overline{z}$ be the class of z in S'. Then $\overline{z}$ is also quadratic over $k:=\mathbb{Z}/2\mathbb{Z}$. It follows that $S'=k+k\overline{z}$, with $\overline{z}^2=a\overline{z}+b,\ a,b\in k$. We have $a\neq\overline{0}$, because $k\subset S'$ is not minimal ramified. Then, $a=\overline{1}$, so that $\overline{z}^2-\overline{z}\in k$, and $k\subset S'$ is minimal decomposed, a contradiction since $\ell[T,S]=\ell[k,S']=2$. A similar proof shows that $z\not\in T$. We deduce from (1) that $R[y],R[z]\in]T,S[$ with $R[y]\neq R[z]$. Since $|[T,S]|=5$ by [22, Theorem 6.1], we get that $[T,S]=\{T,R[y],R[z],R[y+z],S\}$ because $R[y+z]\neq T,R[y],R[z],S$. Moreover, $R\subset R[y]$ is a Δ_0-extension by Lemma 4.5. Since $R\subset T\subset R[y]$, it follows that $R\subset T$ is a Δ_0-extension.

Let $U,V\in[R,S]=[R,T]\cup[T,S]$. If $U,V\in[R,T]$, then $U+V=UV$ because $R\subset T$ is a Δ_0-extension. If $U,V\in[T,S]$, then $U+V=UV$ since $T\subset S$ is a Δ-extension by [24, Theorem 4.16]. At last, assume, for example, that $U\in[R,T]$ and $V\in[T,S]$. Because of the tower $U\subseteq T\subseteq V$, we get that $U+V=UV=V$. To conclude, $R\subset S$ is a Δ-extension.

Let $x\in S$ so that $R[x]\in[R,T]\cup[T,S]$. If $x\in T$, then, $R[x]\in[R,T]$ and x is quadratic because so is $R\subset T$. If $x\in S\setminus T$, then $R[x]\in]T,S[$, because $R[x]\not\subseteq T$. But $[T,S]=\{T,R[y],R[z],R[y+z],S\}$ yields that $R[x]\in\{R[y],R[z],R[y+z]\}$. It follows that x is quadratic over R, since y,z and $y+z$ are quadratic over R . Hence, $R\subset S$ is a quadratic extension and then a Δ_0-extension by Proposition 1.1. □

Theorem 4.8. *Let $R\subset S$ be an integral FCP extension. Then $R\subset S$ is a Δ_0-extension if and only if, for each $M\in\mathrm{MSupp}(S/R)$, the following conditions hold:*

(1) $R_M\subset S_M$ is pinched at $\{({}_S^+R)_M,({}_S^tR)_M\}$.

(2) $R_M\subseteq({}_S^+R)_M$ is a Δ_0-extension.

(3) If R_M is unbranched in S_M and $S_M\neq({}_S^tR)_M$, then $R_M\subset S_M$ is a simple extension generated by a quadratic element over R_M.

(4) If R_M is branched in S_M, then $R_M\subset S_M$ is infra-integral, $\ell[({}_S^+R)_M,S_M]\leq 2$ and one the following conditions holds:

(a) $\ell[({}_S^+R)_M,S_M]=1$ and $R_M\subset S_M$ is a simple extension generated by a quadratic element over R_M.

(b) $\ell[({}_S^+R)_M,S_M]=2,\ R/M\cong\mathbb{Z}/2\mathbb{Z}$ and S_M is generated by a minimal system of two quadratic elements $\{y,z\}$ over R_M such that $y+z$ is quadratic over R_M.

Proof. Proposition 3.2 says that $R\subset S$ is a Δ_0-extension if and only if so is $R_M\subset S_M$ for any $M\in\mathrm{MSupp}(S/R)$. Moreover, by Definition 2.4, for any $M\in\mathrm{Max}(R)$, we have $({}_S^+R)_M={}_{S_M}^+R_M$ and $({}_S^tR)_M={}_{S_M}^tR_M$.

Therefore, we can reduce to the case where (R,M) is a local ring.

If R is unbranched in S, then S is a local ring such that ${}_S^+R={}_S^tR$. Propositions 4.6, 4.4 and 3.3 give that $R\subset S$ is a Δ_0-extension if and only if (1), (2) and (3) hold when ${}_S^tR\neq R,S$.

If ${}_S^tR=R$, then $R\subset S$ is t-closed and $[R,S]=[{}_S^tR,S]$. If $R\subset S$ is a Δ_0-extension, then $R\subset S$ is a simple extension generated by a quadratic element over R by Proposition 4.4. Conversely, if $R\subset S$ is a simple extension generated by a quadratic element over R, then $R\subset S$

is a Δ_0-extension by Lemma 4.5. Then, $R \subset S$ is a Δ_0-extension if and only if (1), (2) and (3) hold.

If ${}_S^t R = S$, then $R \subset S$ is subintegral, since S is local, so that $[R, S] = [R, {}_S^+ R]$. Then $R \subset S$ is a Δ_0-extension if and only if $R \subset {}_S^+ R$ is a Δ_0-extension if and only if (2) holds ((1) and (3) are trivially satisfied).

Assume that R is branched in S, so that ${}_S^+ R \neq {}_S^t R$. If $R \subset S$ is a Δ_0-extension, then Proposition 4.7 (1) gives that $R \subset S$ is infra-integral, leading to ${}_S^t R = S$ and $R \subset S$ is pinched at ${}_S^+ R$, which is (1). Moreover (2) holds by Proposition 3.3. At last, Proposition 4.7 (2) gives (4).

Conversely, assume that (1), (2) and (4) hold with (R, M) local. By (4), $R \subset S$ is infra-integral, so that ${}_S^t R = S$ and (1) shows that $R \subset S$ is pinched at ${}_S^+ R$. Then, Proposition 4.7 (1) holds. Moreover, (4) implies Proposition 4.7 (2), and $R \subset S$ is a Δ_0-extension. □

Proposition 4.9. *Let $R \subset S$ be a subintegral FCP extension, where (R, M) is a local ring; so that, S is a local ring. Let N be its maximal ideal. Then $R \subset S$ is a Δ_0-extension if and only if one of the following conditions holds:*

(1) $(R : S) \neq M$ and $R \subset S$ is quadratic.

(2) $(R : S) = M$ and $N^2 \subseteq M$.

Proof. Since $R \subset S$ is a subintegral FCP extension, where (R, M) is a local ring, S is a local ring. Let N be its maximal ideal. By Proposition 1.1, $R \subset S$ is a Δ_0-extension if and only if $R \subset S$ is a quadratic Δ-extension. We make a discussion according to $(R : S)$ is M or not.

If $(R : S) \neq M$, then $R \subset S$ is a Δ-extension by [24, Proposition 5.1]. Then, $R \subset S$ is a Δ_0-extension if and only if $R \subset S$ is quadratic.

Assume now that $(R : S) = M$, so that we get the extension $R/M \subset S/M$, where R/M is a field. If $R \subset S$ is a Δ_0-extension, so is $R/M \subset S/M$ by Proposition 3.2, and Proposition 4.3 gives $(N/M)^2 = 0$, which leads to $N^2 \subseteq M$.

Conversely, assume that $N^2 \subseteq M$ with $(R : S) = M$. Since $R \subset S$ is subintegral and (R, M) and (S, N) are local rings, we have $R/M \cong S/N$, so that $S = R + N$. It follows that $R \subset S$ is quadratic because any $x \in S$ is of the form $x = a + n$ with $a \in R$ and $n \in N$, giving $x^2 = 2ax + n^2 - a^2$, where $n^2 \in N^2 \subseteq M$. Then $R \subset S$ is a Δ_0-extension. □

Remark 4.10. When looking at conditions (3) and (4) of Theorem 4.8, we see that, when $R \subset S$ is a Δ_0-extension, then, for each $M \in \mathrm{MSupp}(S/R)$, either $({}_S^t R)_M = ({}_S^+ R)_M$ $(*)$ or $({}_S^t R)_M = S_M$ $(**)$. In case $(*)$, we have $R_M \subset S_M$ pinched at $({}_S^t R)_M$ and in case $(**)$, we have $R_M \subset S_M$ pinched at $({}_S^+ R)_M$.

We may find an example of case $(*)$ where $({}_S^t R)_M \neq R_M, S_M$ in [22, Example 4.10 (1)] coming from an example due to Dobbs-Shapiro [6, Remark 3.4 (h)]. Take $K \subset L$ a field extension of degree 2, so that there exists $y \in L$ such that $L = K[y]$. Set $S := L[X]/(X^2) = L[x]$, where x is the class of X in S, $R := K[x]$ and $T := R[xy]$. Then, $R \subset S$ is a Δ_0-extension with $R \subset T$ minimal ramified and $T \subset S$ minimal inert, because $[R, S] = \{R, T, S\}$ is a chain such that $S = R[y]$ (see Proposition 4.1) because y is quadratic over R.

We may find an example of case $(**)$ where $({}_S^+ R)_M \neq R_M, S_M$ in the next section. We will see in Lemma 5.9 that $R \subset S := R^2$ satisfies case (4) (a) of Theorem 4.8, when R is a local ring. Then, $R \subset S$ is a Δ_0-extension with ${}_S^+ R \subset S$ minimal decomposed. Indeed R^2 is generated over R by $(1, 0)$ which is a quadratic element (see Lemma 4.5).

5 Some special Δ_0-extensions and examples

In this section, we give examples of subintegral Δ_0-extensions with various properties. We also characterize some special types of FCP extensions in order to be Δ_0-extensions.

Let $R \subseteq S$ be an FCP extension, then $[R, S]$ is a complete Noetherian Artinian lattice, R being the least element and S the largest. In the context of the lattice $[R, S]$, some definitions and properties of lattices have the following formulations. (see [15])

(1) $R \subseteq S$ is called *distributive* if intersection and product are each distributive with respect to the other. Actually, each distributivity implies the other [15, Exercise 5, page 33].

(2) Let $T \in [R, S]$. Then, $T' \in [R, S]$ is called a *complement* of T if $T \cap T' = R$ and $TT' = S$.

(3) $R \subseteq S$ is called *Boolean* if $([R, S], \cap, \cdot)$ is a distributive lattice such that each $T \in [R, S]$ has a (necessarily unique) complement.

(4) $R \subseteq S$ is called *arithmetic* if $[R_P, S_P]$ is a chain for each $P \in \mathrm{Spec}(R)$.

(5) $R \subseteq S$ is called *catenarian*, or graded by some authors, if $R \subset S$ has FCP and all maximal chains between two comparable elements have the same length.

Proposition 5.1. *An FCP Δ_0-extension is catenarian.*

Proof. According to Proposition 1.1, an FCP Δ_0-extension is an FCP Δ-extension, and then is catenarian by [24, Proposition 3.14]. □

We begin to characterize Boolean Δ_0-extensions. According to [23, Proposition 3.5], we first consider extensions $R \subset S$ such that R is a local ring.

Proposition 5.2. *Let $R \subset S$ be a Boolean FCP extension, where (R, M) is a local ring. Then $R \subset S$ is a Δ_0-extension if and only if $R \subset S$ is minimal integral, and with $[S/M : R/M] = 2$ when $R \subset S$ is inert.*

Proof. Since $R \subset S$ is Boolean, [23, Theorem 3.30] asserts that one of the following conditions holds because an FCP Boolean extension has FIP:

(1) $R \subset S$ is a minimal extension.

(2) There exist $U, T \in [R, S]$ such that $R \subset T$ is minimal ramified, $R \subset U$ is minimal decomposed and $[R, S] = \{R, T, U, S\}$.

(3) $R \subset S$ is a Boolean t-closed extension.

Assume first that $R \subset S$ is a Δ_0-extension. According to Proposition 4.4, $R \subset S$ is pinched at $\{{}_S^+R, {}_S^tR\}$, so that $\{{}_S^+R, {}_S^tR\} \subseteq \{R, S\}$ because $R \subset S$ being Boolean, $R \subset S$ cannot be pinched at an element different from R and S. Otherwise, this element would not have a complement, a contradiction. If ${}_S^+R \neq S$, then ${}_S^+R = R$ and $R \subset S$ is minimal with $[S/M : R/M] = 2$ when $R \subset S$ is inert by Proposition 4.4. If ${}_S^+R = S$, then $R \subset S$ is subintegral, which implies minimal by (1) because (2) and (3) cannot occur.

Conversely, if $R \subset S$ is minimal integral, with $[S/M : R/M] = 2$ when $R \subset S$ is inert, then $R \subset S$ is a Δ_0-extension by Proposition 4.2. □

Proposition 5.3. *Let $R \subset S$ be an FCP Δ_0-extension. Then $R \subset S$ is distributive if and only if $R \subset S$ is arithmetic.*

Proof. According to Proposition 3.2 and [23, Proposition 2.4], we may assume that (R, M) is a local ring. Assume first that $R \subset S$ is distributive. Let $\varphi : S \to S/R$ be the canonical (surjective) R-module morphism. For $E'_1, E'_2, E'_3 \in [[S/R]]$, set $E_i := \varphi^{-1}(E'_i) \in [[R, S]]$, for $i \in \{1, 2, 3\}$. Then $E_i \in [R, S]$ since $R \subset S$ is a Δ_0-extension, with $E'_i = \varphi(E_i)$. In particular, $E_i + E_j = E_iE_j$ for $i, j \in \{1, 2, 3\}$. But $R \subset S$ is distributive implies that $E_i \cap (E_j + E_k) = E_i \cap E_jE_k = (E_i \cap E_j)(E_i \cap E_k) = (E_i \cap E_j) + (E_i \cap E_k)$ $(*)$ for $i, j, k \in \{1, 2, 3\}$. Applying φ to $(*)$, we get $E'_i \cap (E'_j + E'_k) = (E'_i \cap E'_j) + (E'_i \cap E'_k)$ for $i, j, k \in \{1, 2, 3\}$ showing that any element of $[[S/R]]$ is a distributive R-module. Then, any two elements of $[[S/R]]$ are comparable by [13, Proposition 5.2, p. 119]. Coming back in $[R, S]$, we get that any two elements of $[[R, S]]$ are comparable, and then $[R, S]$ is a chain.

The converse is [19, Proposition 5.18]. □

In Proposition 4.2, we characterized minimal Δ_0-extensions. We now consider Δ_0-properties for pointwise minimal extensions. A ring extension $R \subset S$ is *pointwise minimal* if $R \subset R[t]$ is minimal for each $t \in S \setminus R$. We studied these extensions in a joint work with Cahen in [1]. The properties of pointwise minimal extensions $R \subset S$ allow us to assume that (R, M) is a local ring. In this case, $M = (R : S)$ when $R \subset S$ is integral [1, Theorem 3.2]. In [24, Proposition 5.7], we gave the different conditions for a pointwise minimal FCP extension to be a Δ-extension. Since a Δ_0-extension is a Δ-extension, to get the condition for a pointwise minimal FCP extension to be a Δ_0-extension, it is enough to add the quadratic condition in [24, Proposition 5.7].

Proposition 5.4. *A pointwise minimal FCP extension $R \subset S$ over the local ring (R, M) is a Δ_0-extension if and only if one of the following conditions holds:*

(1) $R \subset S$ *is integral minimal with* $[S/M : R/M] = 2$ *when* $R \subset S$ *is inert.*

(2) $R \subset S$ *is seminormal infra-integral with* $R/M \cong \mathbb{Z}/2\mathbb{Z}$ *and* $|\mathrm{Max}(S)| = 3$.

(3) $R \subset S$ *is subintegral with* $N^2 \subseteq M$, *where* $\mathrm{Max}(S) = \{N\}$.

Proof. Assume first that $R \subset S$ is a Δ_0-extension, and then a Δ-extension. According to [24, Proposition 5.7], we get that one of the following conditions holds:

(1) $R \subset S$ is minimal.

(2) $R \subset S$ is seminormal infra-integral with $|\mathrm{Max}(S)| = 3$.

(3) $R \subset S$ is subintegral with $N^2 \subseteq M$, where $\mathrm{Max}(S) = \{N\}$.

For each of these 3 conditions, we check what is the additional condition satisfied by $R \subset S$ to become a Δ_0-extension.

(1) If $R \subset S$ is minimal, Proposition 4.2 asserts that $R \subset S$ is integral, with $[S/M : R/M] = 2$ when $R \subset S$ is inert.

(2) If $R \subset S$ is seminormal infra-integral with $|\mathrm{Max}(S)| = 3$, then $\ell[R, S] = 2$ by [22, Proposition 4.20]. This implies by Proposition 4.7 that $R/M \cong \mathbb{Z}/2\mathbb{Z}$.

(3) is (3) of the statement.

Conversely, assume that one of conditions (1), (2) or (3) of the statement holds:

If (1) holds, $R \subset S$ is integral minimal with $[S/M : R/M] = 2$ when $R \subset S$ is inert. Hence, $R \subset S$ is a Δ_0-extension by Proposition 4.2.

If (2) holds, $R \subset S$ is seminormal infra-integral with $R/M \cong \mathbb{Z}/2\mathbb{Z}$ and $|\mathrm{Max}(S)| = 3$. Then, $M = (R : S)$ with $M = M_1 \cap M_2 \cap M_3$, where $\mathrm{Max}(S) = \{M_1, M_2, M_3\}$ by [2, Proposition 4.9]. It follows that $S/M \cong \prod_{i=1}^{3} S/M_i \cong (R/M)^3$ and $\ell[R/M, S/M] = 2$ by [22, Proposition 4.20], so that S/M is generated over R/M by a minimal system of two quadratic elements $\{y, z\}$ over R/M such that $y+z$ is quadratic over R/M (for example $y := (1, 0, 0)$ and $z := (0, 1, 0)$ with $y + z = (1, 1, 0)$ also quadratic). This implies that $R \subset S$ is a Δ_0-extension by Proposition 4.7.

If (3) holds, $R \subset S$ is subintegral with $N^2 \subseteq M$, where $\mathrm{Max}(S) = \{N\}$. Since $(R : S) = M$, Proposition 4.9 gives the result. □

An FCP extension $R \subset S$ is said *isotopic FCP* (IFCP) if all minimal subextensions of $R \subset S$ are of the same type. For such extensions which are also Δ_0-extensions and satisfy conditions (2) or (3) of Proposition 5.4, we get the following:

Proposition 5.5. *Let* $R \subset S$ *be an IFCP infra-integral non minimal* Δ_0*-extension where* (R, M) *is a local ring. Assume that* $M = (R : S)$. *Then* $R \subset S$ *is pointwise minimal.*

Proof. Proposition 3.2 implies that $R/M \subset S/M$ is an FCP non minimal Δ_0-extension where R/M is a field. Moreover, $R \subset S$ is pointwise minimal if and only if $R/M \subset S/M$ is pointwise minimal by [1, Proposition 3.1]. Then, we may assume that R is a field (and $M = 0$).

Assume first that $R \subset S$ is seminormal infra-integral. It follows that $S \cong R^3$ with $R \cong \mathbb{Z}/2\mathbb{Z}$ by Proposition 4.3. Then, [1, Proposition 4.14] shows that $R \subset S$ is pointwise minimal because R^3 is a Boolean ring.

Assume now that $R \subset S$ is subintegral. According to Proposition 4.3, we get that S is a local ring (S, N) such that $N^2 = 0$. Then, [1, Proposition 4.16] shows that $R \subset S$ is pointwise minimal because $R \subset S$ is subintegral. □

Propositions 5.4 and 5.5 lead to the following corollary.

Corollary 5.6. *Let* $R \subset S$ *be a seminormal infra-integral FCP and non minimal extension where* (R, M) *is a local ring. Consider the following conditions :*

(1) $R/M = \mathbb{Z}/2\mathbb{Z}$ *and* $S/M \cong (R/M)^3$.

(2) $R \subset S$ *is a* Δ_0*-extension.*

(3) $R \subset S$ *is a pointwise minimal extension.*

Then (1) $\Leftrightarrow$ *(2)* $\Rightarrow$ *(3).*

Proof. Since $R \subset S$ is a seminormal infra-integral FCP and non minimal extension where (R, M) is a local ring, we get that $M = (R : S)$ by [2, Proposition 4.9]. It follows that R/M is a field such that $S/M \cong (R/M)^n$ for some positive integer n. So (2) $\Rightarrow$ (1) comes from Propositions 3.2 and 4.3, by considering the extension $R/M \subset S/M$.

Conversely, if (1) holds, Proposition 4.7 (2)(b) shows that $R \subset S$ is a Δ_0-extension (see the proof of Proposition 5.4).

Now, (2) $\Rightarrow$ (3) by Proposition 5.5. □

Example 5.7. Here is an example of a pointwise minimal extension which is a Δ_0-extension satisfying Proposition 5.4.

Let R be a field and set $S := R[X, Y]/(X^2, Y^2, XY) = R[x, y] = R + Rx + Ry$, where x and y are the classes of X and Y in S. According to [1, Theorem 5.4], $R \subset S$ is pointwise minimal. The maximal ideal of S is $N = Rx + Ry$ with $N^2 = 0$. Then Proposition 5.4 asserts that $R \subset S$ is a Δ_0-extension.

We saw in Corollary 5.6 that in the seminormal infra-integral case, we deal with an extension of the form $R/M \subset (R/M)^3$. We are going to study a more general case of the form $R \subset R^n$, which is an infra-integral extension, using results from [21]. Since we are dealing with FCP extensions, we may consider a local Artinian ring R in view of [21, Proposition 1.4]. We now recall a result which will be useful in the following.

Proposition 5.8. *[21, Proposition 3.2] Let R be a ring with two ideals I and J such that $I, J \neq R$ and $I \cap J = 0$. Then $R \subset R/I \times R/J$ is a Δ_0-extension.*

Lemma 5.9. *Let R be a non-zero ring and n an integer with $n > 1$.*

(1) If $R \subset R^n$ is a Δ-extension, then $n \leq 3$.

(2) $R \subset R^2$ is a Δ_0-extension.

Proof. (1) Since $(R^n)_M = (R_M)^n$ for any maximal ideal M of R, we may assume that R is a local ring. Set $S := R^n$ and $T := {}_S^+R$. Then, $R \subset S$ is infra-integral by [21, Proposition 1.4], with $|\mathrm{Max}(S)| = n$ and $\ell[T, S] = n - 1$ by [2, Lemma 5.4]. Using [24, Corollary 4.20], we get that $n \leq 3$.

(2) It is enough to take $I = J = 0$ in Proposition 5.8. □

When R is not reduced and $n = 3$, [21, Proposition 1.4] says that there is a subintegral part $R \subset {}_{R^3}^+R$ of $R \subset R^3$, so that we cannot use [24, Corollary 4.20]. Here is an example of a Δ-extension $R \subset R^3$, where R is an Artinian local and not reduced ring and which is not a Δ_0-extension.

Example 5.10. Set $R := (\mathbb{Z}/2\mathbb{Z})[T]/(T^2) = (\mathbb{Z}/2\mathbb{Z})[t]$, where t is the class of T in R. Then R is an Artinian local ring which is not reduced and with maximal ideal $M := Rt \neq 0$ such that $M^2 = 0$. In [24, Example 5.10], we show that $R \subset R^3$ is a Δ-extension. We sum up the necessary results in this example. Set $N := M \times M \times M$. Then, $S := {}_{R^3}^+R = R + N$. Let $e_1 := (1, 0, 0)$ and $R_1 := R[e_1] = R + Re_1$. It is also shown that S and R_1 are not comparable. It follows that $R_1 \not\in [R, S] \cup [S, R^3]$ so that $R \subset R^3$ is not pinched at $S = {}_{R^3}^+R$. Then, $R \subset R^3$ is not a Δ_0-extension by Proposition 4.7.

For a Δ_0-extension, we can improve Lemma 5.9.

Proposition 5.11. *Let R be a local Artinian ring, and $n > 1$ an integer. Then $R \subset R^n$ is a Δ_0-extension if and only if either $n = 2$ or $R \cong \mathbb{Z}/2\mathbb{Z}$ with $n = 3$.*

Proof. Lemma 5.9 gives one part of the answer when $n = 2$.

If $R \cong \mathbb{Z}/2\mathbb{Z}$, then $R \subset R^3$ is an infra-integral Δ_0-extension by Proposition 5.6 and [21, Proposition 1.4] since R is a field and $R \subset R^3$ is seminormal.

Conversely, assume that $R \subset R^n$ is a Δ_0-extension, and, in particular, a Δ-extension. Then, $n \leq 3$ by Lemma 5.9. The case $n = 2$ is satisfied by the first part of the proof. Assume that $n = 3$. If R is reduced, then R is a field, so that $R \cong \mathbb{Z}/2\mathbb{Z}$ by Proposition 4.3. We claim that R is reduced when $R \subset R^3$ is a Δ_0-extension. Otherwise, $R \subset R^3$ is not seminormal by

[21, Proposition 1.4]. Set $S := {}_{R^3}^{+}R$. According to Proposition 4.4, $R \subset R^3$ is pinched at S. Since R^2 is not local, $R^2 \notin [R, S]$. Let $\mathcal{B} := \{e_1, e_2, e_3\}$ be the canonical basis of R^3. We can write $R^2 = Re_1 + R(e_2 + e_3)$ (for instance). Let M be the maximal ideal of R. Since R is not reduced, then $M \neq 0$. Let $x \in M \setminus \{0\}$ and set $y := xe_2 \in (M \times M \times M) \setminus R^2$. Recall that $S = R + (M \times M \times M)$ by [21, Proposition 2.8]. This shows that $R^2 \notin [S, R^3]$ because $M \times M \times M \nsubseteq R^2$. Then, $R \subset R^3$ is not a Δ_0-extension, a contradiction. □

Corollary 5.12. *Let R be an Artinian ring and $n > 1$ an integer. Then $R \subset R^n$ is a Δ_0-extension if and only if either $n = 2$ or $n = 3$ with $R_M \cong \mathbb{Z}/2\mathbb{Z}$ for each $M \in \mathrm{Max}(R)$.*

Proof. Use Proposition 3.2 and Proposition 5.11. □

In order to look at properties of Δ_0-extensions related to products of rings, we consider now ring extensions linked to idealization. We have already get the following result:

Proposition 5.13. *[20, Proposition 2.8] Let N be a submodule of an R-module M. Then $R(+)N \subseteq R(+)M$ is a Δ_0-extension.*

We can also generalize a result of Long in [14, Corollary 3.5.6].

Proposition 5.14. *Let $R \subset S$ be a ring extension and M an S-module. Then $R(+)M \subset S(+)M$ is a Δ_0-extension if and only if $R \subset S$ is a Δ_0-extension.*

Proof. M is obviously an R-module. Since [14, Corollary 3.5.6] gives the equivalence for the Δ-extension property, it is enough to check the equivalence for the quadratic property.

Assume that $R \subset S$ is a quadratic extension and let $(s, m) \in S(+)M$, where $s \in S$ and $m \in M$. There exist $a, b \in R$ such that $s^2 = as+b$. Then, $(s, m)^2 = (s^2, 2sm) = (as+b, 2sm) = (a, 0)(s, m) + (b, (2s - a)m)$ shows that $R(+)M \subset S(+)M$ is a quadratic extension.

Conversely, assume that $R(+)M \subset S(+)M$ is a quadratic extension and let $s \in S$. There exist $(a, m), (b, n) \in R(+)M$ such that $(s, 0)^2 = (a, m)(s, 0) + (b, n)$. It is enough to consider the first components to see that $s^2 = as + b$. Then, $R \subset S$ is a quadratic extension. □

In [14], Long considers also extensions coming from bowtie ring (or amalgamated duplication of a ring along an ideal), whose definition we recall. Let R be a ring and I an ideal of R. The bowtie ring $R \bowtie I$ is the set $\{(r, r+i) \mid r \in R,\ i \in I\}$, where the ring operations are defined componentwise. We also generalized his results to Δ_0-extensions.

Proposition 5.15. *Let R be a ring and I an ideal of R. Then $R \subset R \bowtie I$ is a Δ_0-extension.*

Proof. Since [14, Corollary 3.2.6] gives the result for the Δ-extension property, it is enough to check the result for the quadratic property.

Ler $(r, r + i) \in R \bowtie I$, with $r \in R,\ i \in I$. Then, $(r, r + i)^2 = (r^2, r^2 + 2ri + i^2) = (2r + i, 2r + i)(r, r + i) - (r^2 + ri, r^2 + ri)$ shows that $R \subset R \bowtie I$ is a quadratic extension. □

Proposition 5.16. *Let $R \subset S$ be a ring extension and I an ideal shared by S and R. Then $R \bowtie I \subset S \bowtie I$ is a Δ_0-extension if and only if $R \subset S$ is a Δ_0-extension.*

Proof. Since [14, Corollary 3.2.18] gives the equivalence for the Δ-extension property, it is enough to check the equivalence for the quadratic property.

Assume that $R \subset S$ is a quadratic extension and let $(s, s + i) \in S \bowtie I$, where $s \in S$ and $i \in I$. There exist $a, b \in R$ such that $s^2 = as + b$. Then, $(s, s + i)^2 = (s^2, s^2 + 2is + i^2) = (as+b, as+b+2is+i^2) = (a, a+i)(s, s+i) + (b, b+i(s-a))$ shows that $R(+)M \subset S(+)M$ is a quadratic extension.

The converse is obvious as in Proposition 5.14. □

We have a more precise result than Proposition 4.9 for length two subintegral Δ_0-extensions.

Proposition 5.17. *Let $R \subset S$ be a subintegral FCP extension of length two, where (R, M) is a local ring. Then $R \subset S$ is a Δ_0-extension if and only if either $R \subset S$ is pointwise minimal or $(R : S) \neq M$.*

Proof. Since $R \subset S$ is a subintegral extension, S is a local ring. Let N be its maximal ideal so that $S = R+N$. Moreover, $R \subset S$ satisfies one of the following conditions: either $(*)$ $|[R,S]| = 3$ and $R \subset S$ is simple, or $(**)$ $R \subset S$ is pointwise minimal ([22, Propositions 2.2 and 4.16]. We are going to characterize, for each case, when $R \subset S$ is a Δ_0-extension.

In case $(**)$, $(R : S) = M$ by [1, Theorem 3.2] and $N^2 \subseteq M$ according to [1, Propositions 3.9 and 4.16]. Then, Proposition 5.4 shows that $R \subset S$ is a Δ_0-extension.

In case $(*)$, $R \subset S$ is simple and $|[R,S]| = 3$. Then, there is some $y \in N$ such that $S = R[y]$. According to [22, Corollary 4.17], $M^2 \subseteq (R : S) \subseteq M$, $[R,S] = \{R, R+N^2, S\}$ and one of the following condition holds:

(1) $(R : S) = M$, $N^2 \not\subseteq M$ and $N^3 \subseteq M$.

(2) $(R : S) \neq M$, $y^2 \notin R$, $MS = M + N^2 = M + Ry^2 \subset N$ and $MN^2 \subseteq M$.

(3) $(R : S) \neq M$, $y^2 \in R$ and $\dim_{R/M}((M + My)/M) = 1$.

The case (3) implies that $R \subset S$ is a Δ_0-extension by Lemma 4.5 because y is quadratic. In this case, $(R : S) \neq M$.

In case (1), since $(R : S) = M$, we have $R \subset S$ is a Δ_0-extension $\Leftrightarrow$ $R/M \subset S/M$ is a Δ_0-extension. But R/M is a field and S/M is a local ring with maximal ideal N/M. Then, we can use Proposition 4.3. If $R/M \subset S/M$ is a Δ_0-extension, then $(N/M)^2 = 0$, giving $N^2 \subseteq M$, a contradiction with (1). Then, case (1) does not lead to a Δ_0-extension.

In case (2), $(R : S) \neq M$ implies that $My \not\subseteq R$ because $S = R[y]$. Moreover, $y \in N$ and $y^2 \in N^2$ shows that $T := R + N^2 = R + Ry^2 \subset S$. But $MS \subseteq T$ leads to $My \subseteq T$. Set $T' := R + My$. We claim that $T' = T$. We have $R \subset T' \subseteq T$. Since $M^2y^2 \subseteq MN^2 \subseteq M$, we get that $T = T'$ because $[R,S] = \{R,T,S\}$. It follows that $y^2 \in T' = R + My$, so that y is a quadratic element over R and $R \subset S$ is a Δ_0-extension by Proposition 4.1.

To conclude, when $R \subset S$ is simple, $R \subset S$ is a Δ_0-extension if and only if $(R : S) \neq M$. □

We have just see in the proof of Proposition 5.17 the case of a subintegral extension of length two, which is a chain and a Δ_0-extension (case $(*)$). The next example shows that there exists a subintegral extension of length n, for any integer $n > 1$, which is a chain and a Δ_0-extension.

Example 5.18. Set $R := (\mathbb{Z}/2\mathbb{Z})[T]/(T^n)$ and $S := R[Y]/(Y^2 - tY) = R[y]$, where t is the class of T in R, y is the class of Y in S and $n \in \mathbb{N}$, $n \geq 2$. Then R is a SPIR with maximal ideal $M := Rt$. We claim that $R \subset S$ is a subintegral extension. Since $y^2 = ty$, an obvious induction yields that $y^k = t^{k-1}y$ for any integer $k \leq n$. For each $i = 0, \dots, n-1$, set $x_i := t^{n-i}y$ and $R_i := R[x_i]$, so that $R = R_0$. Set also $R_n := S$. We show by induction on $i \in \mathbb{N}_{n-1}$ the following: $R_i = R + Rx_i$ is a local ring with maximal ideal $M_i := Rt + Rx_i$ and $R_{i-1} \subset R_i$ is a minimal ramified extension. First, $R_{i-1} \subseteq R_i$ for $i \geq 1$ because $x_{i-1} = tx_i$. Since $x_1 = t^{n-1}y$, we have $x_1^2 = t^{2n-2}y^2 = t^{n+(n-2)}y^2 = 0$ and $tx_1 = t^n y = 0$, so that $R \subset R_1$ is a minimal ramified extension and R_1 is a local ring with maximal ideal $M_1 := Rt + Rx_1$. The induction hypothesis is fulfilled for $i = 1$. Assume that the induction hypothesis holds for some $i < n$ and any $k \leq i$. Then, $R_i = R + Rx_i$ is a local ring with maximal ideal $M_i := Rt + Rx_i$ and $R_{i-1} \subset R_i$ is a minimal ramified extension. After some calculations, we get that $x_{i+1}^2 = t^{2n-2i-2}y^2 = t^{n-i-1}t^{n-i}y = t^{n-i-1}x_i \in R_i$, $tx_{i+1} = t^{n-i-1+1}y = t^{n-i}y = x_i \in M_i$ and $x_i x_{i+1} = t^{n-i}t^{n-i-1}y^2 = t^{n-i}t^{n-i-1}ty = t^{n-i}x_i \in M_i$. In particular, $R_{i+1} \in [R,S]$. Moreover, $x_{i+1} \notin R_i$ because we cannot have $t^{n-i-1}y = a + bt^{n-i}y$ for any $a, b \in R$. Then $R_i \subset R_{i+1}$ is a minimal ramified extension, so that $R_{i+1} = R_i + R_i x_{i+1} = R + Rx_{i+1}$ is a local ring with maximal ideal $M_{i+1} = M_i + R_i x_{i+1} = Rt + Rx_{i+1}$. The induction hypothesis holds for $i+1$, and then for any $i \leq n-1$. Moreover, $R_{n-1} \subset S$ is also a minimal ramified extension since $x_{n-1} = ty = y^2$. This implies that S is a local ring and $R \subset S$ is a simple subintegral extension generated by the quadratic element y over R, so that $R \subset S$ is a Δ_0-extension of length n by Lemma 4.5.

It remains to show that $[R,S]$ is the chain $\{R_i\}_{i=0}^n$. According to [2, Theorem 4.2], $R \subset S$ has FCP. Then, it is strongly affine by [2, Proposition 3.12] (that is to say that each R-subalgebra of S is a finite-type R-algebra). Then, any $T \in [R,S]$ is of the form $T = R[z_1, \dots, z_m]$. Let $z \in S$. We claim that $R[z]$ is some of the R_i's. Since $z \in S$, we can write $z = a + by$, where $a, b \in R$. If $b \notin M$, then $y \in R[z]$, so that $R[z] = S$. We have $R[z] = R$ when $b = 0$. Assume that $b \in M \setminus \{0\}$. Then, $b = ct^k$ for some $k \in \{1, \dots, n-1\}$ and $c \in R \setminus M$. It follows that $x_{n-k} = c^{-1}(z-a)$, so that $R[z] = R_{n-k}$. Coming back to T and letting x_{i_j} be such that $R[z_j] = R[x_{i_j}]$, we have $T = R[x_{i_1}, \ldots, x_{i_m}] = R[x_{i_l}] = R_{i_l}$, where $i_l = \sup\{i_1, \ldots, i_m\}$. Then, $[R, S]$ is a chain.

Remark 5.19. According to Proposition 4.9, there exists a subintegral extension $R \subset S$ which is chained and is not Δ_0. Take for instance $S := k[Y]/(Y^3) = k[y]$, where k is a field and y is the class of Y in S. Then, $k \subset S$ is a subintegral extension of length two by [2, Lemma 5.4] since the maximal ideal of S is $N := ky + ky^2$. Moreover, $[k, S] = \{k, k[y^2], S\}$ by [22, Theorem 6.1] because $S = k[y]$ is simple and then is not pointwise minimal. It follows that $k \subset S$ is not a Δ_0-extension since $N^2 \neq 0$. In fact, y is not quadratic.

We end this paper by an example of a subintegral Δ_0-extension which does not satisfy any of the precedent cases: simple, pointwise minimal, chained, length two extension. We do not write the calculations which are sometimes tedious, but straightforwad.

Example 5.20. Let $R := (\mathbb{Z}/2\mathbb{Z})[T]/(T^2) = R + Rt$, where t is the class of T in R. Then R is a local ring with maximal ideal $M = Rt$ such that $t^2 = 0$. Set $S := R[X, Y]/(X^2 - tX, Y^2 - tY, XY, t(X - Y)) = R[x, y] = R + Rx + Ry$, where x and y are the classes of X and Y in S. We have the relations $x^2 = tx = ty = y^2$ and $xy = 0$ $(*)$. Set $R_1 := R[tx] = R[ty]$, $R_2 := R[x+y]$, $R_3 := R[tx + x + y]$, $S_1 := R[x]$, $S_2 := R[y]$ and $S_3 := R[tx, x + y]$. We have the following diagram:

$$\begin{array}{ccccccc} & & & & S_1 & & \\ & & & \nearrow & & \searrow & \\ & & R_1 & \rightarrow & S_2 & \rightarrow & S \\ & \nearrow & & \searrow & & \nearrow & \\ R & \rightarrow & R_2 & \rightarrow & S_3 & & \\ & \searrow & & \nearrow & & & \\ & & R_3 & & & & \end{array}$$

In the following, using [2, Theorem 4.2], we get that S is a local ring, with maximal ideal $N := Rt + Rx + Ry$ and $R \subset S$ is a subintegral FCP extension because we prove that $R \subset R_i$, $R_i \subset S_3$ and $S_3 \subset S$ are minimal ramified for any $i \in \{1, 2, 3\}$. To give a sketch of the calculations, we will often have to prove that two elements of S are equal, that is some element $z \in S$ is equal to 0. Writing $z = a + bx + cy$, with $a, b, c \in R$, we get the equation $a + bX + cY = (X^2 - tX)P_1(X, Y) + (Y^2 - tY)P_2(X, Y) + XYP_3(X, Y) + t(X - Y)P_4(X, Y)$ $(**)$. Setting $P_1(X, Y) := \sum_{i,j} a_{i,j}X^iY^j$, $P_2(X, Y) := \sum_{i,j} b_{i,j}X^iY^j$, $P_3(X, Y) := \sum_{i,j} c_{i,j}X^iY^j$ and $P_4(X, Y) := \sum_{i,j} d_{i,j}X^iY^j$, relations $(*)$ and $(**)$ leads to $a = 0$, $b = -ta_{0,0} + td_{0,0}$, $c = -tb_{0,0} - td_{0,0}$, $0 = a_{0,0} - ta_{1,0} + td_{1,0}$, $0 = -ta_{0,1} - tb_{1,0} + c_{0,0} + td_{0,1} - td_{1,0}$, $0 = b_{0,0} - tb_{0,1} - td_{0,1}$. According to the values of b and c, we obtain the following results: $R \subset R_i$, $R_i \subset S_3$ and $R_1 \subset S_i$ are minimal ramified for each $i = 1, 2, 3$, with $R_i \neq R_j$, $S_i \neq S_j$, and $S_3 = R_iR_j$ for each $i, j \in \{1, 2, 3\}$, $i \neq j$. By [5, Proposition 7.6], $S_i \subset S$ is also minimal ramified for each $i = 1, 2, 3$, so that $\ell[R, S] = 3$. Moreover, we get $[R, S] = \{R, R_i, S_i, S\}_{i=1,2,3}$ because we now show that there does not exist some $T \in [R, S] \setminus \{R, R_i, S_i, S\}_{i=1,2,3}$ in two steps. First, such a T cannot verify $R \subset T$ is minimal (ramified), setting $T := R[z]$, for some $z \in S$. By the way, we show that any element of S is quadratic, so that $R \subset S$ is a quadratic extension. Indeed, we may set $z = \alpha x + \beta y$, $\alpha, \beta \in R$. It follows that $z^2 = (\alpha + \beta)tz$. The second step shows that there does not exist $T \neq S_j$ for $j \in \{1, 2, 3\}$ such that $R_i \subset T$ is minimal for some $i \in \{1, 2, 3\}$. Indeed, if such a T, exists, we should have $\ell[R, T] = 2$, and T would contain necessarily some R_i. Since $\ell[R_1, S] = 2$, [22, Theorem 6.1] shows that $|[R_1, S]| = 5$, but $\{R_1, S_i, S\}_{i=1,2,3} \subseteq [R_1, S]$ yields that such a T does not exist in $[R_1, S]$. The same theorem shows that $R_i \subset S$ is a chain for $i = 2, 3$ because $S = R_2[x] = R_3[x]$ and such a T does not exist in $[R_i, S]$ for $i = 2, 3$.

We have already shown that $R \subset S$ is quadratic. Here, $(R : S) \neq M$ since $tx \notin R$. Then, $R \subset S$ is a Δ_0-extension by Proposition 4.9. We may remark that $N^2 = Rtx \not\subseteq M$.

We also get that $R \subset S$ is not a pointwise minimal extension because $R \subset R[x]$ is not minimal. At last, $R \subset S$ is not a simple extension because there does not exist some $z \in S$ such that $S = R[z]$. Of course, $R \subset S$ is neither a chain, nor a length two extension.

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