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Existence and uniqueness results for a nonlinear fractional Volterra integro-differential equation with non-local boundary conditions

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Abstract In this article, we explain the existence and the uniqueness of the solutions associated with the non-linear fractional integro-differential equation of Volterra for the fractional derivative of Caputo of values at local conditional limits.
This work is mainly based on the application of the theory of fixed points in the Banach space. This work will be supported by an example at the end .

1 Introduction

Fractional differential equations are considered as one of the most important fields in mathematics through the last years, and they will also accept especially that they have wide applications in many branches, including: chemistry, physics, biology, engineering,
In this work we look for possible solutions to the non-linear Fractional Volterra equation:

$$ {}^{c}D^{\alpha}u(t) = h(t) + \lambda \int_0^t k(t,s) f(s,u(s))ds, \tag{1.1} $$

with two non-local boundary conditions,

$$ au(0) + bu(1) = \mu_1, \quad \mu_1 \in \mathbb{R}, \tag{1.2} $$

and

$$ cu^{'}(0) + du^{'}(1) = \mu_2, \quad \mu_2 \in \mathbb{R}, \tag{1.3} $$

where ${}^{c}D^{\alpha}u(t)$ denotes the α-fractional order derivative of $u(t)$ in the Caputo sense. The kernel $k(t,s)$ and the function $h(t)$ of these equations are given real-valued functions, a, b, c and d real numbers, and $u(t)$ is the unknown function, λ a fixed numerical parameter.

Many researchers have studied Volterra's integro-differential equation [21, 16, 1, 17] and [9], [10]. The study of Volterra's integro-differentiation equation was about searching for exact solutions (See:[14],[4],[24],[3],...). On the other hand, some researchers searched for numerical methods in order to find approximate solutions to this equation, (See:[6],[8],[23],[25],...).

In order to generalize the integral-differential equations, we generalize the fractional derivation of Caputo for this equation to become under the name: "Fractional Volterra's Integro-Differential Equation" (FVIDE). This fractional equation has been discussed in several works through finding exact and approximate solutions. Among the works: [5], [25],[20] and [7]...

On the other hand, the Volterra fractional integral differential equation was studied with the Caputo-Hadamard fractional derivative (see:[13]), in order to prove the existence and uniqueness of positive solutions to this equation with only an initial condition (see:[2]).

In this work, we generalize the initial condition of the Volterra fractional integral equation with a Caputo fractional derivative to a non-local boundary condition. This condition has been mentioned in several works on partial and fractional differential equations, (see: [12],[11],[22].)

2 Preliminaries

In this section, we give some definitions of the derivative and integral operator of Riemann Liouville as well as the derivative operator of Caputo.[18, 19] And also the definition of fixed point theory.

Definition 2.1. [18] The Riemann-Liouville integral of order $\alpha > 0$ of a function $\varphi : [0, \infty] \longrightarrow \mathbb{R}$ is defined by

$$(I^{\alpha}\varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \ \ t \geq 0, \tag{2.1}$$

where $n-1 < \alpha \leq n$, if $n \in \mathbb{N}$, $D^n = \frac{d^n}{dt^n}$. And $\Gamma(\alpha)$ is the Euler's Gamma function defined by $\Gamma(t) = \int_0^{+\infty} \tau^{t-1}e^{-\tau}d\tau$ with $t \in]0, +\infty]$.

Definition 2.2. [18] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : [0, \infty] \longrightarrow \mathbb{R}$ is defined by

$$(D^{\alpha}\varphi)(t) = \left(\frac{d}{dt}\right)^n \left(I^{n-\alpha}\varphi\right)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \ \ (n = [\alpha]+1, \ t \geq 0). \tag{2.2}$$

Definition 2.3. [18, 19] The Caputo fractional derivative of order $\alpha \in \mathbb{R}_+$ for a function $\varphi \in C^n[a, b]$ is defined as

$$({}^cD^{\alpha}\varphi)(t) = I^{n-\alpha}(D^n\varphi)(t) \ \ for \ \ \alpha > 0, and \ \ ({}^cD^{0}\varphi)(t) = \varphi(t). \tag{2.3}$$

where Γ is gamma Euler's function.

Proposition 2.4. *[18] Let $\alpha > 0$ and $\beta > 0$, for all $\varphi \in Ł^1[a, b]$, we have*
i) ${}^cD_a^{\alpha}(I_a^{\alpha}\varphi)(t) = \varphi(t)$.
ii) $I_a^{\alpha}({}^cD_a^{\alpha}\varphi)(t) = \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a)(t-a)^k}{k!}$ *here,* $\varphi \in C^n[a, b]$ *with* $n-1 < \alpha \leq n$.

Theorem 2.5. *[10, 15] Let X be a Banach space. Suppose the operator $T : X \longmapsto X$ satisfies the contractive condition,*

$$\|Tu - Tv\| \leq \sigma \|u - v\| \ \ for: u, v \in X. \tag{2.4}$$

where, $\sigma < 1$ is a constant. Then T has a unique fixed point in X.

Lemma 2.6. *Let $\varphi \in C[1, 2]$, $a+b \neq 0$ and $c+d \neq 0$, then the fractional boundary value problem (2.5),*

$${}^cD^{\alpha}u(t) = \varphi(t), \ 1 < \alpha \leq 2, \ t \in [0, 1] = J, \tag{2.5}$$

with two non-local boundary conditions,

$$au(0) + bu(1) = \mu_1, \ \ cu'(0) + du'(1) = \mu_2, \quad (\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}, \tag{2.6}$$

has a unique solution (2.7) written as follows

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\varphi(s)ds \\ &+ \frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\varphi(s)ds + \frac{\mu_1}{a+b} - \frac{b\mu_2}{(a+b)(c+d)} \\ &+ \left[-\frac{d}{(c+d)\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2}\varphi(s)ds + \frac{\mu_2}{c+d} \right] t. \end{aligned} \tag{2.7}$$

Proof. The general solution to problem (2.5), ${}^cD^{\alpha}u(t) = \varphi(t)$ is

$$u(t) = I^{\alpha}\varphi\,(t) + c_0 + c_1 t. \tag{2.8}$$

Where $(c_0, c_1) \in \mathbb{R} \times \mathbb{R}$. By the two boundary's conditions (1.2),(1.3) we get,

(i) For condition (1.2), we have: $au(0) + bu(1) = \mu_1$, with: $a + b \neq 0$.

$$a\left(\underbrace{\frac{1}{\Gamma(\alpha)}\int_0^0 (0-s)^{\alpha-1}\varphi(s)ds}_{=0} + c_0 + c_1 \times 0\right) + b\left(\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi(s)ds + c_0 + c_1 \times 1\right) = \mu_1,$$

we obtain

$$(a+b)\,c_0 + bc_1 = \mu_1 - \frac{b}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi(s)ds. \tag{2.9}$$

(ii) For condition (1.3),and by differentiating formula (2.8), we find,

$$u'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\varphi(s)ds + c_1, \tag{2.10}$$

we have: $cu'(0) + du'(1) = \mu_2$, with: $c + d \neq 0$.

$$c\left(\underbrace{\frac{1}{\Gamma(\alpha-1)}\int_0^0 (0-s)^{\alpha-2}\varphi(s)ds}_{=0} + c_1\right) + d\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi(s)ds + c_1\right) = \mu_2,$$

we obtain

$$c_1 = -\frac{d}{(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi(s)ds + \frac{\mu_2}{c+d}. \tag{2.11}$$

According to formula (2.9), we have,

$$(a+b)\,c_0 = -\frac{b}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi(s)ds + \frac{bd}{(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi(s)ds - \frac{b\mu_2}{c+d} + \mu_1,$$

we obtain,

$$c_0 = -\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi(s)ds + \frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi(s)ds + \frac{\mu_1}{(a+b)} - \frac{b\mu_2}{(a+b)(c+d)}. \tag{2.12}$$

Substituting the values of c_0 (2.12) and c_1 (2.11) in (2.8), we obtain the integral equation (2.7). □

3 MAIN RESULTS

Through the result obtained in Lemma 2.6. We substitute the equivalent in formula (1.1) and get the fractional integral equation,

$$\begin{aligned}
u(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(h(s)+\lambda\int_0^s k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& -\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(h(s)+\lambda\int_0^s k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& +\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left(h(s)+\lambda\int_0^s k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& +\left[-\frac{b}{(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left(h(s)+\lambda\int_0^s k(s,\tau)f(\tau,u(\tau)d\tau\right)ds\right]t \\
& +\frac{\mu_2}{c+d}t+\frac{\mu_1}{a+b}-\frac{b\mu_2}{(a+b)(c+d)}.
\end{aligned} \tag{3.1}$$

In order to reach correct results, the formula (3.1) should be changed to the following format. By the fractional integral equation (3.1), we have

$$\begin{aligned}
u(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^t\left(\int_0^s (t-s)^{\alpha-1}k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& -\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds-\frac{b\lambda}{(a+b)\Gamma(\alpha)}\int_0^1\left(\int_0^s (1-s)^{\alpha-1}k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& +\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
& +\frac{bd\lambda}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1\left(\int_0^s (1-s)^{\alpha-2}k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& -\frac{d}{(c+d)\Gamma(\alpha-1)}t\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
& -\frac{d\lambda}{(c+d)\Gamma(\alpha-1)}t\int_0^1\left(\int_0^s (1-s)^{\alpha-2}k(s,\tau)f(\tau,u(\tau)d\tau\right)ds \\
& +\frac{\mu_2}{c+d}t+\frac{\mu_1}{a+b}-\frac{b\mu_2}{(a+b)(c+d)}.
\end{aligned} \tag{3.2}$$

Using Fubini theorem, we get

$$\begin{aligned}
u(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^t\left(\int_s^t (t-\tau)^{\alpha-1}k(\tau,s)d\tau\right)f(s,u(s))ds \\
& -\frac{b}{(a+b)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds-\frac{b\lambda}{(a+b)\Gamma(\alpha)}\int_0^1\left(\int_s^1 (1-\tau)^{\alpha-1}k(\tau,s)d\tau\right)f(s,u(s))ds \\
& +\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
& +\frac{bd\lambda}{(a+b)(c+d)\Gamma(\alpha-1)}\int_0^1\left(\int_s^t (1-\tau)^{\alpha-2}k(\tau,s)d\tau\right)f(s,u(s))ds \\
& -\frac{d}{(c+d)\Gamma(\alpha-1)}t\int_0^1 (1-s)^{\alpha-2}h(s)ds \\
& -\frac{d\lambda}{(c+d)\Gamma(\alpha-1)}t\int_0^1\left(\int_s^t (1-\tau)^{\alpha-2}k(\tau,s)d\tau\right)f(s,u(s))ds \\
& +\frac{\mu_2}{c+d}t+\frac{\mu_1}{a+b}-\frac{b\mu_2}{(a+b)(c+d)}.
\end{aligned} \tag{3.3}$$

By changing the integration modes, we get the following

$$\begin{aligned} u(t) &= \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}h(s)ds - \int_0^1 \frac{b}{(a+b)\Gamma(\alpha)}(1-s)^{\alpha-1}h(s)ds \\ &+ \left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)}t\right]\int_0^1 (1-s)^{\alpha-2}h(s)ds \\ &+ \int_0^t \left(\int_s^t \frac{\lambda}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}k(\tau,s)d\tau + \int_s^1 \frac{-b\lambda}{(a+b)\Gamma(\alpha)}(1-\tau)^{\alpha-1}k(\tau,s)d\tau\right) f(s,u(s))ds \\ &+ \lambda\left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)}t\right]\int_t^1 \left(\int_s^1 (1-\tau)^{\alpha-2}k(\tau,s)d\tau\right) f(s,u(s))ds \\ &+ \frac{\mu_2}{c+d}t + \frac{\mu_1}{a+b} - \frac{b\mu_2}{(a+b)(c+d)}. \end{aligned} \tag{3.4}$$

We define the operator $T{:}X \longrightarrow X$ and X be a Banach space : $X = (C\,[0,1]\,,\mathbb{R})$,
with all continuous real functions on $[0,1]$ to $\mathbb{R}$, and the norm u is $\|u\| = \sup_{t\in J}|u(t)|$,
So, we write $Tu(t)$ as follows

$$\begin{aligned} Tu(t) &= \int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}h(s)ds - \int_0^1 \frac{b}{(a+b)\Gamma(\alpha)}(1-s)^{\alpha-1}h(s)ds \\ &+ \left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)}t\right]\int_0^1 (1-s)^{\alpha-2}h(s)ds \\ &+ \int_0^t \left(\int_s^t \frac{\lambda}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}k(\tau,s)d\tau\right) f(s,u(s))ds \\ &+ \lambda\left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)}t\right]\int_0^1 \left(\int_s^t (1-\tau)^{\alpha-2}k(\tau,s)d\tau\right) f(s,u(s))ds \\ &+ \frac{\mu_2}{c+d}t + \frac{\mu_1}{a+b} - \frac{b\mu_2}{(a+b)(c+d)}. \end{aligned} \tag{3.5}$$

In order to obtain final results in the latter, we present the following hypotheses:
1) Let $J = [0,1]$, $f{:}J\times\mathbb{R} \longrightarrow \mathbb{R}$ be continuous function, so f is a lipschitzian function,
$\exists\delta \geq 0, |f\,(t,x) - f\,(t,y)| \leq \delta\,|x-y|$, for all $(t,x),(t,y) \in J\times\mathbb{R}$.
2) Suppose that,

$$\begin{cases} h\,(s) = (1-s)^{2-\alpha}\,h_1\,(s)\,, \\ k\,(\tau,s) = (1-\tau)^{2-\alpha}\,k_1\,(\tau,s)\,, \end{cases} \tag{3.6}$$

with, $h_1\,(s)$ is a continuous function, and $k_1\,(\tau,s)$ is a continuous function with respect to τ.
From formula of the operator T and according the formula (3.6) we conclude that,

$$\begin{cases} A = \left|\frac{bd}{(a+b)(c+d))}\right| + \left|\frac{d}{c+d}\right|, \\ \|h\| = \sup_{t\in J}\left|(1-s)^{2-\alpha}\,h_1\,(s)\right| \leq \|h_1\|\,, \\ M = \sup_{\tau,s\in J}\left|(1-\tau)^{2-\alpha}\,k_1\,(\tau,s)\right| \leq \sup_{\tau,s\in J}|k_1(\tau,s)| = M'. \end{cases} \tag{3.7}$$

Use of fixed point theorem [1],[10] we demonstrate that the operator T is a contraction operator on space X.
Note that u is a fixed point of the operator $T \Rightarrow T\,(u) = u$ if and only if u is solution of the problem (1.1) with non local boundary conditions attached to the problem.

Theorem 3.1. *Suppose that the previous hypotheses are satisfied. The problem (1.1)-(1.3) have a unique solution if*

$$\sigma = \left[\frac{|\lambda| M^{'} \delta}{\Gamma(\alpha+2)} + \frac{A |\lambda| M^{'} \delta}{2\Gamma(\alpha-1)} \right] < 1. \tag{3.8}$$

Proof. For all $u,v \in X$, According to hypotheses H_1 and H_2, we have

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \Bigg| \int_0^t \left(\int_s^t \frac{\lambda}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} k(\tau,s) d\tau \right) [f(s,u(s)) - f(s,v(s))]\, ds \\
&+ \lambda \left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)} t \right] \int_0^1 (\int_s^t (1-\tau)^{\alpha-2} k(\tau,s) d\tau) [f(s,u(s)) - f(s,v(s))] ds \Bigg| . \\
&= \Bigg| \int_0^t \left(\int_s^t \frac{\lambda}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} (1-\tau)^{2-\alpha} k_1(\tau,s)\, d\tau \right) [f(s,u(s)) - f(s,v(s))]\, ds \\
&+ \lambda \left[\frac{bd}{(a+b)(c+d)\Gamma(\alpha-1)} - \frac{d}{(c+d)\Gamma(\alpha-1)} t \right] \int_0^1 \left(\int_s^t k_1(\tau,s) d\tau \right) [f(s,u(s)) - f(s,v(s))]\, ds \Bigg| . \\
&\leq \frac{|\lambda| M^{'} \delta}{\Gamma(\alpha)} \|u-v\| \int_0^t \left(\int_s^t (t-\tau)^{\alpha-1} d\tau \right) ds + \frac{A |\lambda| M^{'} \delta}{\Gamma(\alpha-1)} \|u-v\| \int_0^1 \left(\int_s^t d\tau \right) ds. \\
&= \frac{|\lambda| M^{'} \delta}{\Gamma(\alpha+2)} \|u-v\| \left[t^{\alpha+1} \right] + \frac{A |\lambda| M^{'} \delta}{2\Gamma(\alpha-1)} \|u-v\| [2t-1] . \\
&\leq \left[\frac{|\lambda| M^{'} \delta}{\Gamma(\alpha+2)} + \frac{A |\lambda| M^{'} \delta}{2\Gamma(\alpha-1)} \right] \|u-v\| .
\end{aligned}$$

and we get

$$\|Tu(t) - Tv(t)\| \leq \sigma \|u-v\| . \tag{3.9}$$

Since $\sigma < 1$, therefore, the operator T is a contraction, we conclude that the problem (2.5) with the non local conditions have a unique solution in all X. □

Example 3.2. Let the fractional integro-differential equation mentioned in formula (1.1), with the two non-local conditions (1.2),(1.3) such that:
$\alpha \in \,]\,1, 2]$, $\lambda = \frac{1}{4}$, $a = 1$, $b = 2$,$c = d = 1$, $\mu_1 = 1 + 2e$, $\mu_2 = 2e$, $f(t, u(t)) = u(t) e^{t^2}$, $k(s,t) = ts$, and $h(t) = \frac{1}{4}t + (4t^2+2)e^{t^2} - \frac{1}{4} t e^{2t^2}$.
From hypothesis H_1, f is a Lipschitzian function, we have,

$$\begin{aligned}
|f(t,u(t)) - f(t,v(t))| &= \left| u(t) e^{t^2} - v(t) e^{t^2} \right| \\
&= \left| e^{t^2} [u(t) - v(t)] \right| \\
&\leq \max_{t \in J=[0,1]} e^{t^2} |u(t) - v(t)| \\
&= e |u(t) - v(t)| .
\end{aligned}$$

So for all $(t,x), (t,y) \in J \times \mathbb{R}$

$$|f(t,u(t)) - f(t,v(t))| \leq e |u(t) - v(t)| .$$

It is clear that,
$\delta = e \approx 2.718$, $M = \sup_{s,t \in J} |k(s,t)| = 1 = M^{'}$, and $\|h\| = \sup_{t \in J} |h(t)| = 4$.
So the fractional integro-differential equation with these data in this example is as follows,

$${}^{c}D^{(\alpha)} u(t) = \frac{1}{4} t + (4t^2+2)e^{t^2} - \frac{1}{4} t e^{2t^2} + \frac{1}{4} \int_0^t 4tsu(s) e^{s^2} ds, \ \alpha \in \,]\,1, 2] . \tag{3.10}$$

- For $\alpha = 2$ in (3.10), $u(x) = e^{x^2}$ is a solution to the following classical equation

$$u^{''}(t) = \frac{1}{4}t + (4t^2+2)e^{t^2} - \frac{1}{4}te^{2t^2} + \frac{1}{4}\int_0^t 4tsu(s)\,e^{s^2}ds, \tag{3.11}$$

with non-local boundary conditions,

$$u(0) + 2u(1) = 1 + 2e, \tag{3.12}$$

$$u^{'}(0) + u^{'}(1) = 2e. \tag{3.13}$$

According to formula (3.7) we calculate A,

$$A = \left|\frac{bd}{(a+b)(c+d))}\right| + \left|\frac{d}{c+d}\right| = \left|\frac{2}{3\times 2}\right| + \left|\frac{1}{2}\right| \approx 0.833.$$

We calculate the value of σ from the example data, with $\alpha = 1.5$, we have

$$\sigma = \left[\frac{|\lambda|\,M^{'}\delta}{\Gamma(\alpha+2)} + \frac{A\,|\lambda|\,M^{'}\delta}{2\Gamma(\alpha-1)}\right] = \left[\frac{0.25\times 1\times 2.718}{\Gamma(3.5)} + \frac{0.833\times 0.5\times 1\times 2.718}{2\Gamma(0.25)}\right]$$
$$\approx 0.3641345704 < 1.$$

By theorem 3.1, we can say that problem (3.10) with non-local conditions (3.12) and (3.13) have a unique solution.

4 Conclusion

In this work, we prove the existence and uniqueness of the Fractional Volterra Integro- Differential Equation solutions of the Caputo sense, with non-local conditions. This work is a generalization of the existence and uniqueness proof of the solutions to the ordinary Volterra integral equations.

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