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A primal-dual large-update interior-point algorithm for symmetric cone optimization based on a new class of kernel functions

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Abstract In this paper, we propose a new primal-dual interior-point algorithm for solving symmetric cone optimization (SCO) problems based on a class of kernel functions (KFs). The latter belongs to the newly introduced hyperbolic type [30, 31, 14]. To the best of our knowledge, this is the first interior-point algorithm for SCO which is based on a hyperbolic barrier term. Using the machinery of Euclidean Jordan algebra, the complexity analysis for large-update primal-dual interior-point methods (IPMs) based on this class of KFs yields an $\mathcal{O}\left(pr^{\frac{p+1}{2p}}\log\frac{r}{\epsilon}\right)$ iteration bound which meets the currently best iteration complexity for SCO by choosing a special value of the parameter p. We consolidate these theoretical results by performing numerical experiments on some NETLIB problems to compare the proposed algorithm with other existing algorithms. The practical performance of the proposed algorithm seems very promising as we got the smallest iteration numbers in almost 60% of the realized experiments.

1 Introduction

SCO is a broad class of convex optimization that contains linear optimization (LO), second-order cone optimization (SOCO) and semidefinite optimization (SDO) as special cases. SCO is the optimization of a linear objective function over the intersection of an affine space and a symmetric cone subject to linear equality constraints.

Many methods were proposed to solve SCO problems. IPMs rank among the most efficient methods both theoretically and practically. IPMs were first introduced by Karmarkar [18] for LO problems. However, Nesterov and Nemirovskii [22] were the first to attempt solving SCO problems using IPMs. They developed a complete theory applicable to the whole class of convex optimization problems. They proved the polynomial complexity for several types of IPMs and related their efficiency to the existence of a certain type of barrier depending on the problem structure, a so-called self-concordant barrier. Subsequently, Nesterov and Todd [23, 24] offered a theoretical foundation of effective IPMs for convex optimization problems expressed in conic form, when the cone and its associated barrier are self-scaled. Following that, Güler [12] applied the Euclidean Jordan algebraic tools in IPMs which was due to the fact that the family of the self-scaled cones is identical to the set of symmetric cones. After that, Faybusovich [9] generalized IPMs for SDO problems to SCO problems using the Euclidean Jordan algebras. Other extensions of primal-dual IPMs for SCO were done by Monteiro and Zhang [21], Schmieta and Alizadeh [29] and Vieira [34]. The latter generalized the kernel-based IPM approach of Bai et al. [4] to SCO.

Inspired by the works of [19, 34], we propose a new primal-dual IPM for solving SCO problems based on the following KF

$$\psi(t) = \frac{t^2-1}{2} - \frac{1}{\cosh^p(2)} \int_1^t \cosh^p\left(\frac{2}{x}\right) dx, \ \forall t > 0, \tag{1.1}$$

with $p \geq 1$. The definition of the barrier term of this KF is based on the hyperbolic cosine function. For other uses of this function, we refer the reader to [1], where the authors introduced a new definition of the fractional derivative using the hyperbolic cosine function. It's worth noting that the function ψ belongs to the newly introduced hyperbolic type of KFs [30, 31, 14, 32, 15, 16]. In all the previous references, the presented primal-dual IPMs were proposed to solve LO and SDO problems. This makes our method the first extension of IPMs based on hyperbolic barrier terms to SCO problems.

We structure our paper as follows. In Section 2 we briefly recall some important definitions and results on Euclidean Jordan algebras and symmetric cones that are needed in the paper. In addition, we give the basic concepts of IPMs for SCO. In Section 3, we give some properties of the new KF including exponential convexity. After that, we derive the iteration bound of our algorithm for large-update methods in Section 4. Our computational experiments are presented in Section 5. Finally, some concluding remarks follow in Section 6.

2 Preliminaries

2.1 Euclidean Jordan algebra

In order to ensure the integrity of this paper, we give some definitions and results on Euclidean Jordan Algebras. For a comprehensive study, we refer the reader to [7, 33].

Definition 2.1. Let $(\mathcal{J}, \circ)$ be an $n-$dimensional space over $\mathbb{R}$ and $\circ : (x, y) \mapsto x \circ y$ be a bilinear map from $\mathcal{J} \times \mathcal{J}$ to $\mathcal{J}$. Then $(\mathcal{J}, \circ, \langle ., . \rangle)$ is an Euclidean Jordan Algebra (EJA) if it satisfies the following conditions:

- $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$ (commutative).
- $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ for all $x, y \in \mathcal{J}$, where $x^2 = x \circ x$ (Jordan's Axiom).
- There exists an inner product such that $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathcal{J}$.

We call $x \circ y$ the Jordan product of x and y and define the trace inner product for all $x, y \in \mathcal{J}$ as follows

$$\langle x, y \rangle := \text{tr}(x \circ y).$$

The norm induced by the inner product $\langle ., . \rangle$ is called the Frobenius norm, which is defined for every $x \in \mathcal{J}$ by

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{\text{tr}(x^2)}.$$

- If there exists an element e such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$, then e is called the multiplicative identity element of the EJA $\mathcal{J}$.
- For any $x \in \mathcal{J}$, let k be the smallest integer such that the set $\{e, x, ..., x^k\}$ is linearly dependent. Then, k is the degree of x denoted deg(x). The rank of $\mathcal{J}$, simply denoted by r, is the maximum of deg(x) for all $x \in \mathcal{J}$.
- A nonzero element $c \in \mathcal{J}$ is said to be idempotent if $c^2 = c$. A set of primitive idempotents $\{c_1, ..., c_r\}$ is called a Jordan frame if $c_i \circ c_j = 0$, $i \neq j \in \{1, ..., r\}$ and $c_1 + ... + c_r = e$.
- For an EJA $\mathcal{J}$, the corresponding cone of squares $\mathcal{K} := \{x^2 : x \in \mathcal{J}\}$ is indeed a symmetric cone. A cone is symmetric if and only if it is the cone of squares of some EJA.

The following theorem provided a spectral decomposition of elements of an EJA.

Theorem 2.2 (Theorem III.1.2 in [7]). *Let $\mathcal{J}$ be an EJA with rank r, and $x \in \mathcal{J}$. Then, there exists a Jordan frame $\{c_1, ..., c_r\}$ and real numbers $\lambda_1(x), ..., \lambda_r(x)$ such that*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i.$$

Each $\lambda_i(x)$ is called an eigenvalue of x.

Using the previous theorem, we can rewrite the Frobenius norm as follows

$$\|x\| = \sqrt{\sum_{i=1}^{r} \lambda_i^2(x)}, \; \forall x \in \mathcal{J}.$$

Now, for any element $x \in \mathcal{J}$, we define the Lyapunov transformation $L(x)$ defined from $\mathcal{J}$ to $\mathcal{J}$ as follows

$$L(x)y := x \circ y, \; \forall y \in \mathcal{J}.$$

Furthermore, we define the quadratic representation for each x in $\mathcal{J}$ as follows:

$$P(x) := 2L(x)^2 - L(x^2),$$

where $L(x)^2 = L(x)L(x)$.

2.2 Primal-dual IPM for SCO

In this subsection, we describe the basic of IPMs for SCO. Let $\mathcal{J}$ be an $n-$dimensional EJA with rank r and $\mathcal{K}$ its associated symmetric cone. The primal -dual pair of SCO problems are given by

$$(P) \quad \min \{\langle c, x\rangle : \; Ax = b, x \in \mathcal{K}\},$$

$$(D) \quad \max \{b^T y : \; A^T y + s = c, s \in \mathcal{K}, \; y \in \mathbb{R}^m\},$$

where $c \in \mathcal{J}$, $b \in \mathbb{R}^m$, and $A = [a_1; \ldots; a_m]$ is the matrix corresponding to the linear transformation that maps x to the m-vector whose i^{th} component is $\langle a_i, x\rangle$. The vectors a_i, $i = 1, \ldots, m$ lie in $\mathcal{J}$ and assumed to be linearly independent. The equations $Ax = b$ and $A^T y + s = c$ mean that for all $i = 1, \ldots, m$, $\langle a_i, x\rangle = b_i$ and $\sum_{i=1}^{m} y_i a_i + s = c$ respectively.

First, we assume that both (P) and (D) satisfy the interior-point condition (IPC) ; that is, there exists $(x^0, y^0, s^0) \in \mathring{\mathcal{K}} \times \mathbb{R}^m \times \mathring{\mathcal{K}}$ with $\mathring{\mathcal{K}}$ denotes the interior of the symmetric cone $\mathcal{K}$, such that

$$Ax^0 = b, \;\; A^T y^0 + s^0 = c.$$

It was shown in [22, 9] that the IPC ensures the existence of optimal solutions $\bar{x}$ and $(\bar{y}, \bar{s})$ of (P) and (D) such that

$$\langle c, \bar{x}\rangle - b^T \bar{y} = \langle \bar{x}, \bar{s}\rangle = 0.$$

This optimal solution can be obtained by solving the following system of equations

$$\begin{cases} Ax = b, x \in \mathcal{K}, \\ A^T y + s = c, s \in \mathcal{K}, \\ x \circ s = 0. \end{cases} \tag{2.1}$$

Replacing the last equation in (2.1) by the parameterized equation $x \circ s = \mu e$, where $\mu > 0$, we arrive at the following system

$$\begin{cases} Ax = b, x \in \mathcal{K}, \\ A^T y + s = c, s \in \mathcal{K}, \\ x \circ s = \mu e, \quad \mu > 0. \end{cases} \tag{2.2}$$

Since the (IPC) is satisfied, the perturbed system (2.2) has a unique solution for each $\mu > 0$ denoted by (x_μ, y_μ, s_μ). The set of unique solutions $\{(x_\mu, y_\mu, s_\mu) : \mu > 0\}$ forms a well-behaved curve, called the central path. If $\mu \to 0$, then the limit of the central path exists and yields optimal solutions for (P) and (D).

Applying the damped Newton method to the perturbed system (2.2), produces the following system for the search direction $(\Delta x, \Delta y, \Delta s) \in \mathcal{J} \times \mathbb{R}^m \times \mathcal{J}$

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s \circ \Delta x + x \circ \Delta s = \mu e - x \circ s. \end{cases}$$

In general, x and s do not operator commute, i.e., $L(x)L(s) \neq L(s)L(x)$. This implies that the above system does not always have a unique solution (see [9]). A well-known way to overcome this difficulty was proposed by Schmieta and Alizadeh in 2003 [29], where they replaced the last equation in system (2.2) by

$$P(u)x \circ P(u)^{-1}s = \mu e,$$

with $u \in \mathring{\mathcal{K}}$, and $P(.)$ the quadratic representation of $\mathcal{K}$.

Applying Newton's method to the obtained system, we arrive at the following system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ P(u)\Delta x \circ P(u)^{-1}s + P(u)x \circ P(u)^{-1}\Delta s = \mu e - P(u)x \circ P(u)^{-1}s. \end{cases} \tag{2.3}$$

In this paper, we will be using the Nesterov-Todd (NT) scaling scheme that was introduced by Nesterov and Todd for self-scaled cones in [23, 24] and generalized to SCOs by Faybusovich in [10], where we focus on the scaling point $u = w^{-\frac{1}{2}}$, with w called the NT scaling point of x and s defined as follows

$$w := P(x)^{\frac{1}{2}}(P(x)^{\frac{1}{2}}s)^{-\frac{1}{2}} = P(s)^{-\frac{1}{2}}(P(s)^{\frac{1}{2}}x)^{\frac{1}{2}}.$$

Let's set

$$v = \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} = \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}}, \quad d_x = \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad d_s = \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}.$$

Putting $\overline{A} = [\overline{a_1}; \ldots; \overline{a_m}]$ with $\overline{a_i} = \sqrt{\mu}P(w)^{\frac{1}{2}}a_i$ for all $i = 1, \ldots, m$, system (2.3) is then rewritten in the following form

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\frac{\Delta y}{\sqrt{\mu}} + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{2.4}$$

Let $v = \sum_{i=1}^{r} \lambda_i(v)c_i$ be the spectral decomposition of v with respect to the Jordan frame $\{c_1, .., c_r\}$. It has been observed that the right-hand side in the last equation of (2.4) is equal to minus gradient of the classical logarithmic scaled barrier (proximity) function

$$\Psi(v) = \sum_{i=1}^{r} \psi_c(\lambda_i(v)),$$

where $\psi_c(t) = \frac{t^2-1}{2} - \log(t)$, is the so-called KF of the barrier function Ψ.
Coming back to system (2.4), we can convert it to

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\frac{\Delta y}{\sqrt{\mu}} + d_s = 0, \\ d_x + d_s = -\nabla\Psi(v), \end{cases}$$

with

$$\nabla\Psi(v) = \psi_c'(v) := \psi_c'(\lambda_1(v))c_1 + \cdots + \psi_c'(\lambda_r(v))c_r.$$

The main idea of kernel based primal-dual IPMs is to replace ψ_c by any strictly convex function $\psi :]0, +\infty[\to [0, +\infty[$ which is minimal at $t = 1$ with $\psi(1) = 0$. For a different ψ, one gets a different search direction. In this work, we replace ψ_c by ψ which was defined previously in (1.1). We also define the associated norm-based proximity measure as follows:

$$\sigma(v) = \frac{1}{2}\|d_x + d_s\| = \frac{1}{2}\|\nabla\Psi(v)\|. \tag{2.5}$$

The generic primal-dual IPM for SCO is summarized in Algorithm 1.

Algorithm 1 : Generic primal-dual interior-point algorithm for SCO

Input
a threshold parameter $\tau \geq 1$;
an accuracy parameter $\epsilon > 0$;
a fixed barrier update parameter
begin
$x := e; s := e; \mu := \mu^0$;
while $r\mu \geq \epsilon$ **do**
  begin (outer iteration)
  $\mu: = (1-\theta)\mu$;
  while $\Psi(v) > \tau$ **do**
    begin (inner iteration)
    $x := x + \alpha\Delta x$;
    $y := y + \alpha\Delta y$;
    $s := s + \alpha\Delta s$;
    $v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}}$; $w := P(x)^{\frac{1}{2}}(P(x)^{\frac{1}{2}}s)^{-\frac{1}{2}}$
  end while(inner iteration)
end while(outer iteration)

3 The new kernel function and its properties

In this section, we provide some easily obtained properties of our new KF defined in (1.1) which are used in the complexity analysis of Algorithm 1. For conveniency, we give the first three derivatives of ψ for all $t > 0$

$$\psi'(t) = t - \frac{\cosh^p\left(\frac{2}{t}\right)}{\cosh^p(2)}, \tag{3.1}$$

$$\psi''(t) = 1 + \frac{2p\sinh\left(\frac{2}{t}\right)\cosh^{p-1}\left(\frac{2}{t}\right)}{\cosh^p(2)t^2}, \tag{3.2}$$

and

$$\psi'''(t) = -\frac{4p}{\cosh^p(2)t^3}\left(\frac{\cosh^p\left(\frac{2}{t}\right)}{t} + \sinh\left(\frac{2}{t}\right)\cosh^{p-1}\left(\frac{2}{t}\right) + \frac{(p-1)\sinh^2\left(\frac{2}{t}\right)\cosh^{p-2}\left(\frac{2}{t}\right)}{t}\right). \tag{3.3}$$

From (3.2), we see that $\psi''(t) \geq 1$, $\forall t > 0$, thus we have the following lemma.

Lemma 3.1 (Lemma 2.1 in [3]). *Let ψ be defined as in (1.1). Then*

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2, \quad \forall t > 0.$$

The next lemma provides useful inequalities for the hyperbolic functions.

Lemma 3.2. *One has*

i) $x\sinh(x) - \cosh(x) \geq 0,\ \forall x \geq c_0$, *where* $1 < c_0 < 2$ *is the solution of the equation* $\coth(x) = x$.

ii) $\frac{1}{2}x^2\sinh(x) - \cosh^2(x) < 0,\ \forall x \geq 0$.

Proof. For the first item, let h be the function defined on $\mathbb{R}$ as follows

$$h(x) = x\sinh(x) - \cosh(x),$$

then,

$$h'(x) = x\cosh(x) > 0,\ \forall x > 0.$$

Thus, the result follows since $h(c_0) = 0$. For the second item, using the Taylor expansion of the hyperbolic sine function, we have

$$\sinh(x) > \frac{x^2}{2},\ \forall x > 0.$$

This implies that

$$\frac{x^2}{2}\sinh(x) - \cosh^2(x) < \sinh^2(x) - \cosh^2(x) = -1 < 0.$$

□

The next lemma reveals some key properties of the new KF.

Lemma 3.3. *Let* ψ *be as defined in* (1.1). *Then,*

(i) $t\psi''(t) - \psi'(t) > 0,\ \forall t > 0$.

(ii) $t\psi''(t) + \psi'(t) > 0,\ \forall t > 0$.

(iii) ψ'' *is monotonically decreasing on* $]0, +\infty[$.

Proof. For the first and second item, from (3.1) and (3.2), we have

$$t\psi''(t) - \psi'(t) = \frac{\cosh^{p-1}\left(\frac{2}{t}\right)}{\cosh^p(2)}\left(\frac{2p}{t}\sinh\left(\frac{2}{t}\right) + \cosh\left(\frac{2}{t}\right)\right) > 0,$$

and

$$\begin{aligned} t\psi''(t) + \psi'(t) &= 2t + \frac{\cosh^{p-1}\left(\frac{2}{t}\right)}{\cosh^p(2)}\left(\frac{2p}{t}\sinh\left(\frac{2}{t}\right) - \cosh\left(\frac{2}{t}\right)\right), \\ &\geq 2t + \frac{\cosh^{p-1}\left(\frac{2}{t}\right)}{\cosh^p(2)}\left(\frac{2}{t}\sinh\left(\frac{2}{t}\right) - \cosh\left(\frac{2}{t}\right)\right) > 0, \end{aligned}$$

using the first item of Lemma 3.2 with $x = \frac{2}{t} \geq 2,\ \forall t \leq 1$.
The inequality for the case $t > 1$, is directly obtained using the fact that $\psi'(t) > 0,\ \forall t > 1$.
The third item follows from (3.3) since $\psi'''(t) < 0, \forall t > 0$. □

Lemma 3.4 (Lemma 2.1.2 in [25]). *A twice differentiable function* $\psi : \mathbb{R}_{++} \to \mathbb{R}$ *verifies property (ii) of Lemma 3.3 if and only if it verifies one of the following equivalent properties*

- $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right),\ \forall t_1, t_2 > 0$.
- *the function* $\xi \mapsto \psi(e^{\xi})$ *is convex.*

We say that ψ *is exponentially convex or shortly e-convex.*

Let $\varrho : [0, +\infty[\longrightarrow [1, +\infty[$ be the inverse function of $\psi(t)$ for $t \geq 1$, and $\rho : [0, +\infty[\longrightarrow]0, 1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for $0 < t \leq 1$. Then we have the following lemma.

Lemma 3.5. *One has*

(i) $1+\sqrt{\frac{2s}{\psi''(1)}} \leq \varrho\left(s\right) \leq 1+\sqrt{2s}, \forall s \in [0,+\infty[$.

(ii) $\cosh\left(\frac{2}{t}\right) \leq \cosh(2)\left(2z+1\right)^{\frac{1}{p}}$ *, for all* $(z,t) \in [0,+\infty[\times]0,1]$ *such that* $z=-\frac{1}{2}\psi'\left(t\right)$.

Proof. The first item can be easily obtained using Lemma 3.1, the fact that

$$\psi(t)=\int_1^t\int_1^x \psi''(y)dydx,$$

and

$$\psi''(t) \geq 1, \ \forall t>0.$$

As for the second item, let $z \geq 0$ and $t \in]0,1]$ such that $z=-\frac{1}{2}\psi'\left(t\right)$, then $\rho\left(z\right)=t$. Using (3.1), we have

$$2z=-t+\frac{\cosh^p\left(\frac{2}{t}\right)}{\cosh^p(2)}.$$

Thus,

$$\begin{aligned}\cosh^p\left(\frac{2}{t}\right) &= \cosh^p(2)(2z+t)\\ &\leq \cosh^p(2)(2z+1).\end{aligned}$$

This completes the proof. □

Theorem 3.6 (Theorem 5.9.12 in [33]). *If* $v \in \mathring{\mathcal{K}}$*, then*

$$\sigma(v) \geq \frac{1}{2}\psi'\left(\varrho(\Psi(v))\right).$$

Corollary 3.7. *Let* $\sigma(v)$ *be defined by* (2.5)*. Then, for any* $v \in \mathring{\mathcal{K}}$*, we have*

$$\sigma(v) \geq \sqrt{\frac{\Psi\left(v\right)}{2}}.$$

Proof. The inequality is obtained using Lemma 3.1, Theorem 3.6 and (2.5). □

Remark 3.8. Through the paper we assume that $\tau \geq 1$. Using Corollary 3.7 and the assumption that $\Psi(v) \geq \tau$, we have

$$\sigma(v) \geq \sqrt{\frac{1}{2}}.$$

4 Analysis of the algorithm

4.1 Growth behavior of the barrier function

We proceed by studying the effect of updating the barrier parameter μ on the value of the function $\Psi(v)$.

Theorem 4.1 (Theorem 5.9.1 in [33]). *For any* $v \in \mathring{\mathcal{K}}$ *and* $\beta > 1$*, we have*

$$\Psi\left(\beta v\right) \leq r\psi\left(\beta\varrho\left(\frac{\Psi\left(v\right)}{r}\right)\right).$$

Assuming that $\Psi\left(v\right) \leq \tau$ just before the $\mu-$update, we get the following upper bound

$$\Psi\left(v_+\right) \leq r\psi\left(\frac{\varrho\left(\frac{\tau}{r}\right)}{\sqrt{1-\theta}}\right), \tag{4.1}$$

where $v_+ = \dfrac{v}{\sqrt{1-\theta}}$ and $0 \leq \theta < 1$.

Corollary 4.2 ([6]). *Let* $0 \leq \theta < 1$, $v_+ = \frac{v}{\sqrt{1-\theta}}$. *If* $\Psi(v) \leq \tau$, *then we have*

$$\Psi(v_+) \leq \frac{\theta r + 2\tau + 2\sqrt{2\tau r}}{2(1-\theta)} := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(v_+)$ during the process of the algorithm.

Proof. Recall that

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t),$$

where $\psi_b(t) = -\frac{1}{\cosh^p(2)} \int_1^t \cosh^p\left(\frac{2}{x}\right) dx$. We can easily verify that $\psi_b(1) = 0$, $\psi_b^{'}(1) = -1$, and ψ_b is monotonically decreasing on $\mathbb{R}_{++}$. Hence, we have

$$\psi(t) \leq \frac{t^2-1}{2}, \ \forall t \geq 1.$$

Putting $t = \dfrac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}} \geq 1$ and using (4.1), we get

$$\Psi\left(v_+\right) \leq \frac{r}{2}\left(\frac{\varrho\left(\frac{\tau}{r}\right)^2}{1-\theta} - 1\right).$$

Moreover, using the first item of Lemma 3.5, we obtain

$$\begin{aligned}\Psi\left(v_+\right) &\leq \frac{r}{2}\left(\frac{\left(1+\sqrt{\frac{2\tau}{r}}\right)^2}{1-\theta} - 1\right) \\ &= \frac{\theta r + 2\tau + 2\sqrt{2\tau r}}{2(1-\theta)}.\end{aligned}$$

□

4.2 Decrease of the proximity during a (damped) Newton step

In this subsection, we want to compute a default step size α such that (x_+, y_+, s_+) defined in Algorithm 1 are feasible and the proximity function decreases sufficiently.

After a damped step, we have

$$\begin{aligned} x_+ &= x + \alpha\Delta x = \sqrt{\mu}P(w)^{\frac{1}{2}}(v + \alpha d_x), \\ s_+ &= s + \alpha\Delta s = \sqrt{\mu}P(w)^{-\frac{1}{2}}(v + \alpha d_s). \end{aligned}$$

Hence, we can rewrite the new scaled vector v_+ as follows

$$v_+ = P(w_+)^{-\frac{1}{2}}P(w)^{\frac{1}{2}}(v + \alpha d_x) = P(w_+)^{\frac{1}{2}}P(w)^{-\frac{1}{2}}(v + \alpha d_s),$$

with

$$w_+ = P(x_+)^{\frac{1}{2}}(P(x_+)^{\frac{1}{2}}s_+)^{-\frac{1}{2}} = P(s_+)^{-\frac{1}{2}}(P(s_+)^{\frac{1}{2}}x_+)^{\frac{1}{2}}.$$

Now we recall the following famous lemma which provides an important similarity property.

Lemma 4.3 (Proposition 5.6 in [33]). *One has*

$$v_+ \sim \left(P(v + \alpha d_x)^{\frac{1}{2}})(v + \alpha d_s)\right)^{\frac{1}{2}}.$$

where the symbol $\sim$ *denotes the similarity relation between two matrices.*

A direct consequence of this lemma is that

$$\Psi(v_+) = \Psi\left((P(v+\alpha d_x)^{\frac{1}{2}})(v+\alpha d_s))^{\frac{1}{2}}\right).$$

Now, we consider the decrease in Ψ as a function of α noted f defined by

$$f(\alpha) = \Psi(v_+) - \Psi(v) = \Psi\left((P(v+\alpha d_x)^{\frac{1}{2}})(v+\alpha d_s))^{\frac{1}{2}}\right) - \Psi(v).$$

The following proposition is a direct consequence of the exponentially convexity property of ψ.

Proposition 4.4 (Theorem 4.3.2 in [33]). *If $x, s \in \mathring{\mathcal{K}}$, one has*

$$\Psi\left((P(x)^{\frac{1}{2}}s)^{\frac{1}{2}}\right) \le \frac{1}{2}(\Psi(x) + \Psi(s)).$$

Using the previous proposition, it follows that

$$\Psi(v_+) \le \frac{1}{2}(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)).$$

Thus

$$f(\alpha) \le \frac{1}{2}(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)) - \Psi(v).$$

For simplicity, we put $\sigma := \sigma(v)$.

Theorem 4.5 (Theorem 5.9.10 in [33]). *Let us set $\bar{\alpha} = \dfrac{1}{\psi''(\rho(2\sigma))}$, as the default step size. Then*

$$f(\bar{\alpha}) \le -\frac{\sigma^2}{\psi''(\rho(2\sigma))}. \tag{4.2}$$

We can obtain the upper bound for the decreasing value of the proximity during an inner iteration by the next theorem.

Theorem 4.6. *If $\bar{\alpha}$ is the default step size and $\sigma \ge 1$, then we have*

$$f(\bar{\alpha}) \le -\frac{\sqrt{2}(\Psi(v))^{\frac{p-1}{2p}}}{72(1+p\cosh(2))}.$$

Proof. From (3.2), we have

$$\psi''(t) = 1 + \frac{2p\sinh\left(\frac{2}{t}\right)\cosh^{p-1}\left(\frac{2}{t}\right)}{\cosh^p(2)t^2}.$$

Using the second item of Lemma 3.2 with $x = \frac{2}{t} \ge 1,\ \forall t \le 1$, we get

$$\psi''(t) \le 1 + \frac{p}{\cosh^p(2)}\cosh^{p+1}\left(\frac{2}{t}\right),\ \forall t \le 1.$$

Let $t = \rho(2\sigma)$. Lemma 3.5 implies that

$$\psi''(\rho(2\sigma)) \le (1+p\cosh(2))(4\sigma+1)^{\frac{p+1}{p}}.$$

Using Remark 3.8 and (4.2), it follows that

$$\begin{aligned} f(\bar{\alpha}) &\le -\frac{\sigma^{\frac{p-1}{p}}}{36(1+p\cosh(2))} \\ &\le -\frac{\sqrt{2}(\Psi(v))^{\frac{p-1}{2p}}}{72(1+p\cosh(2))}, \end{aligned}$$

where the last inequality is obtained using Corollary 3.7. This completes the proof. □

4.3 Iteration complexity

Now, we calculate the number of inner iterations required by the algorithm to return to the situation where $\Psi(v) \leq \tau$ after $\mu-$update. Let us define the value of $\Psi(v)$ after $\mu-$update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_i, i = 1, ..., K$, where K stands for the total number of inner iterations in the outer iteration. The decrease on each inner iteration is given by

$$\Psi_{i+1} \leq \Psi_i - \frac{\sqrt{2}}{72(1+p\cosh(2))}\Psi_i^{\frac{p-1}{2p}}.$$

Lemma 4.7 (Proposition 2.2 in [26]). *Let $t_0, t_1, ..., t_k$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \ k = 0, 1, ..., K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then, $K \leq \left[\frac{t_0^{\gamma}}{\beta\gamma}\right]$.

Now, we derive an upper bound for the total number of inner iterations in an outer iteration.

Lemma 4.8. *One has*

$$K \leq \frac{72\sqrt{2}\,(1+p\cosh(2))\,p}{p+1}(\Psi_0)^{\frac{p+1}{2p}}.$$

Proof. The inequality is obtained applying the previous lemma for $t_k = \Psi_k$, $\gamma = \dfrac{p+1}{2p}$ and $\beta = \frac{\sqrt{2}}{72(1+p\cosh(2))}$. □

We arrive at the final result of this section which summarizes the complexity bound.

Theorem 4.9. *Let Ψ_0 be the value defined in Corollary 4.2 and let $\tau \geq 1$. Then, the total number of iterations to obtain an approximation solution with $r\mu \leq \epsilon$ is bounded by*

$$\left[\frac{72\sqrt{2}\,(1+p\cosh(2))\,p}{p+1}\right]\Psi_0^{\frac{p+1}{2p}}\frac{\log\frac{r}{\epsilon}}{\theta}.$$

Proof. An upper bound for the total number of iterations is obtained by multiplying the upper bound K by the number of barrier parameter updates, which is bounded above by $\frac{1}{\theta}(\log\frac{r}{\epsilon})$ (see [28, Lemma II.17, page 116]). Thus, we obtain the result due to the above lemma. □

For large-update methods with $\tau = \mathcal{O}(r)$ and $\theta = \Theta(1)$, the complexity of the primal-dual interior-point algorithm for SCO based the new KF is $\mathcal{O}\left(pr^{\frac{p+1}{2p}}\log\frac{r}{\epsilon}\right)$ iterations complexity. An interesting choice is $p = \log r$, which gives $\mathcal{O}\left(\sqrt{r}\log r\log\frac{r}{\epsilon}\right)$ iterations complexity.

5 Numerical tests

In our computational study, we consider LO as a special case of SCO. We implemented our algorithm as well as the algorithms based on the KFs provided in Table 1 in MATLAB R2012b. The considered algorithms are all tested on a selection of LO problems from NETLIB [11]. Our experiments are performed on Supermicro dual-2.80 GHz Intel Core i5 server with 4.00 Go RAM for the following parameters $\epsilon = 10^{-8}$, $\tau = n$, and $\theta \in \{0.7, 0.99\}$.

In order to make a fair comparison, we chose a practical step size α as in [17] i.e. $\alpha = \min(\alpha_x, \alpha_s)$, with α_x and α_s defined as follows

$$\alpha_x = \min_{i=1..n}\begin{cases}-\dfrac{x_i}{\Delta x_i} & \text{if } \Delta x_i < 0,\\ 1 & \text{elsewhere.}\end{cases} \qquad \alpha_s = \min_{i=1..n}\begin{cases}-\dfrac{s_i}{\Delta s_i} & \text{if } \Delta s_i < 0,\\ 1 & \text{elsewhere.}\end{cases}$$

Since there are parameters involved in the definition of all the considered KFs, we chose a common value of these parameters: $p = 2$. We also tested our KF for $p = 1, 3, 4$. This left us with 8 different KFs. The summary of results is given in Table 2.

Table 1: Considered KFs with their complexity bounds.

KF	Complexity	Ref.
$\psi_{1,p}(t) = \frac{t^2-1}{2} - \int_1^t \frac{e-1}{(e^x-1)^p} dx,\ p \geq 1$	$\mathcal{O}\left(pn^{\frac{p+1}{2p}} \log \frac{n}{\epsilon}\right)$	[8]
$\psi_{2,p}(t) = \frac{t^2-1}{2} - \int_1^t e^{\frac{p}{x}-p} dx,\ p \geq 1$	$\mathcal{O}\left(p\sqrt{pn} \log \frac{n}{\epsilon}\right)$	[5]
$\psi_{3,p}(t) = \frac{t^2-1}{2} - \int_1^t e^{\frac{1}{x^p}-1} dx,\ p \geq 1$	$\mathcal{O}\left(p\sqrt{n}(\log n)^{\frac{p+1}{p}} \log \frac{n}{\epsilon}\right)$	[2]
$\psi_{4,p}(t) = \frac{t^2-1}{2} - \int_1^t \frac{1}{x^{2p}} e^{\frac{p}{x}-p} dx,\ p \geq 1$	$\mathcal{O}\left(p\sqrt{n}(1+\frac{1}{p}\log n)^2 \log \frac{n}{\epsilon}\right)$	[20]
$\psi_{\text{new},p}(t) = \frac{t^2-1}{2} - \frac{1}{\cosh^p(2)} \int_1^t \cosh^p(\frac{2}{x}) dx,\ \ p \geq 1$	$\mathcal{O}\left(pn^{\frac{p+1}{2p}} \log \frac{n}{\epsilon}\right)$	New

Table 2: Number of iterations for $\theta = \{0.7, 0.99\}$.

Problem	θ	$\psi_{1,2}$	$\psi_{2,2}$	$\psi_{3,2}$	$\psi_{4,2}$	$\psi_{\text{new},1}$	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$
AFIRO	0.7	110	99	110	106	102	**96**	108	106
	0.99	112	108	106	125	**99**	106	119	105
BLEND	0.7	**45**	52	69	73	48	82	47	55
	0.99	75	74	57	58	87	**50**	56	74
NUG6	0.7	74	89	88	107	83	**69**	74	71
	0.99	56	71	67	**51**	70	59	55	71
NUG7	0.7	92	68	86	73	**64**	83	74	75
	0.99	**67**	70	78	80	94	83	84	74
SCTAP2	0.7	68	47	70	**44**	82	78	84	72
	0.99	86	84	71	99	**47**	76	89	73
SCTAP3	0.7	75	50	49	**46**	77	75	79	71
	0.99	47	**45**	46	47	49	**45**	50	49

5.1 Comments

Recall that the numerical results were obtained by performing Algorithm 1 with the KFs defined in Table 1 on six NETLIB problems. For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number. From Table 2 we may conclude a few remarks. First, the iteration numbers of the algorithm based on the new KF depend on the values of the parameter p. In fact, the value $p = 2$ gives better iteration numbers in general. For AFIRO problem, our KF outperformed all other KFs for both values of θ. In all cases, we can observe that the efficiency of each KF depends on the problem under consideration and the value of θ.

Based on these results, we conclude that the algorithm based on the new KF attains the most wins among the considered algorithms.

6 Conclusions and remarks

In this paper, we introduce a new kernel-based interior-point algorithm for symmetric cone programming which includes linear programming, second-order cone programming and semidefinite programming as special cases. The proposed algorithm differs from the existing algorithms, since the barrier term involved in its definition is of hyperbolic type. We prove that, by choosing a special value of the parameter p, our algorithm has the same complexity as the currently best-known IPMs for SCO with large-update methods. Finally, the numerical results show that the newly proposed KF is well promising and outperforms some existing KFs in the literature. In fact, the new algorithm had the smallest number of iterations in most cases.

As a further research we would like to extend our approach to more general optimization problems. Another question of interest is whether the new KF can be used to design an efficient infeasible interior-point algorithm as in [27, 13].

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