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Solving Unconstrained Problems by Using the Conjugate Gradient Technique

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Abstract One of the most important methods used to solve nonlinear optimization problems is the conjugate gradient method (CGM), that can also be used to solve unconstrained problems efficiently and effectively. (does not require high packing capabilities). In this work, Conjugate gradient (CG) technique is adapted using the β_k method of Polak-Ribiere (PR) and β_k of Liu-Storey (LS) method to find the solution for nonlinear optimization problems based on number of iterations to obtain the solution CPU time, and number of functions evaluation. The results show the efficiency of the new technique.

1 Introduction

Let the next problem be unconstrained

$$F(x), x \in R^n \tag{1.1}$$

where $f : R^n \to R$ is a smooth function, we choose a suitable point $x_0 \in R^n$ to find the solution of (1). CGM generated a series $\{x_k\}$ as [1, 2]:

$$x_{k+1} = x_k + \alpha_k d_k \quad k = 0, 1, 2, \ldots \tag{1.2}$$

where $\alpha_k > 0$ is the length parameter, we will get it through the Armijo technique (ALS) and the d_k direction

$$d_k = -\mu g_k \tag{1.3}$$

where $g_k = \nabla f(x_k)$ and β_k is a scalar [3, 4], Using (ALS) condition to obtain α_k, the step size contents [5]

$$f(x_k + \alpha_k d_k) \leq -\mu \alpha_k g_k^T d_k \|g_k\| \tag{1.4}$$

From our previous information, we will obtain the multiplier β_k using different nonlinear conjugate gradient methods [6, 7, 8]. Some of the famous formulas for β_k that we will use are:

LS (Liu- Storey) $$\beta_k^{LS} = \frac{-g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}} \tag{1.5}$$

PR (Polak-Ribiere) [9] $$\beta_k^{PR} = \frac{g_{K+1}^T y_k}{g_k^T g_k} \tag{1.6}$$

We will consider as $\|\cdot\|$ the Euclidean base symbol and

$$y_{k-1} = g_k - g_{k-1} \tag{1.7}$$

The conjugate gradient (CG) approach is known to have a suitable global convergence and is effective for resolving unconstrained problems [10], however, with some flaws its computations

[11, 12]. In order to solve this issue, we propose a hybrid CG approach in this work, where (β_k) is the expression of $\left(\beta_k^{LS}\right)$ and $\left(\beta_k^{PR}\right)$. The numerical outcomes demonstrated that the hybrid method is more successful than the basic method at solving the system (1.1). In order to solve (1.1) under appropriate conditions, we employed (1.5) and (1.6) [13, 14].

The suggested method takes the parameter β_{k} calculated as a convex mixture of $\beta_{\mathrm{k}}^{\mathrm{PR}}$ and $\beta_{\mathrm{k}}^{\mathrm{LS}}$ such that [15]

$$\beta_k = (1+\theta_K)\,\beta_k^{PR} + \theta_K \beta_K^{lS} \tag{1.8}$$

The order of the remaining work is as follows: In section 2, the parameter θ_k is discovered. In section 3, a few presumptions are presented, and the global convergence is demonstrated in section 4. While Numerical tests and conclusion will be proved in the last [16].

2 Suggestion Parameter β_k

The iteration $x_0, x_1, \ldots$ of the suggested algorithm are computed by (1.2) where $\alpha_k > 0$ is firm by (1.4), d_k is achieved by using (1.3), were

$$\beta_k^H = (1+\theta_K)\,\beta_k^{PR} + \theta_K \beta_K^{LS} = (1+\theta_K)\frac{g_{K+1}^T y_k}{g_k^T g_k} + \theta_K \frac{-g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}}$$

$$d_{k+1} = -g_{k+1} + (1+\theta_K)\frac{g_{K+1}^T y_k}{g_k^T g_k} d_k + \theta_K \frac{-g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}} d_k$$

$$y_k^T d_{k+1} = -y_k^T g_{k+1} + (1+\theta_K)\frac{g_{K+1}^T y_k}{g_k^T g_k} y_k^T d_k + \theta_K \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} y_k^T d_k$$

So, from the conjugacy condition $y_k^T d_{k+1} = 0$

$$0 = -y_k^T g_{k+1} + \frac{g_{K+1}^T y_k}{g_k^T g_k} y_k^T d_k + \theta_K \left[\frac{g_{K+1}^T g_{k+1}}{(g_{k+1}+d_k)^T d_k} - \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}\right] y_k^T d_k$$

Then

$$\theta_k = \frac{\frac{g_{K+1}^T y_k}{y_k^T d_k} - \frac{g_{K+1}^T y_k}{g_k^T g_k}}{\frac{g_{K+1}^T g_{k+1}}{(g_{k+1}+d_k)^T d_k} - \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}} \tag{2.1}$$

So

$$\beta_k = \left(1 + \frac{\frac{g_{K+1}^T y_k}{y_k^T d_k} - \frac{g_{K+1}^T y_k}{g_k^T g_k}}{\frac{g_{K+1}^T g_{k+1}}{(g_{k+1}+d_k)^T d_k} - \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}}\right)\beta_k^{PR} + \frac{\frac{g_{K+1}^T y_k}{y_k^T d_k} - \frac{g_{K+1}^T y_k}{g_k^T g_k}}{\frac{g_{K+1}^T g_{k+1}}{(g_{k+1}+d_k)^T d_k} - \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}}\beta_K^{LS}$$

Algorithm 1 The Suggested Algorithm.

Step 1: Choice $x_0 \in R^n, \varepsilon \in (0,1), \gamma > 0, \rho > 0, d_0 = -f_o = -\nabla f(x), k = 0$.

Step 2: If $\|f_{k-1}\| \leq \varepsilon$, then stop. Otherwise go to Step **Step 3:**.

Step 3: From (1.4) to calculate α_k.

Step 4: $x_{k+1} = x_k + \alpha_k d_k$.
if $\|f_k\| \leq \varepsilon$, then stop. otherwise go to Step **Step 5:**.

Step 5: From (1.3) to calculate d_k.

Step 6: Set $k := k+1$, go to Step **Step 3:**.

3 Global Convergence [17, 18]

Assumption 3.1. H_1 : $f(x)$ is a bounded differentiable function. There exists a constant $a > 0$ such that $\|x\| \le a$ for all $x \in L_1$, where the L is a level set.

H_2 : For the neighborhood N of $L_1, L > 0$, i.e.

$$\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\| \text{ for all } x, y \in N,$$

the gradient $g(x)$ is Lipschitz constant.

Lemma 3.2. *Accept that Assumption 3.1 holds. Suppose (1.2), (1.3) where α_k achieves (1.4) and β_K^H achieves the formulation (1.8), formerly each $k, g_{k+1}^T d_{k+1} < 0$ hold.*

Proof. For $k = 1$ we have $g_1^T d_1 = -g_1^T g_1 = -\|g_1\|^2 < 0$ according to $d_1 = -g_1$. If $k > 1$, consider $g_k^T d_k < 0$ holds at (k) th step, then see if $(k+1)^{th}$ step is achievable.

$$g_{k+1}^T d_{k+1} = -g_{k+1}^T g_{k+1} + (1 + \theta_K) \frac{g_{K+1}^T y_k}{g_k^T g_k} g_{k+1}^T d_k + \theta_K \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} g_{k+1}^T d_k$$

Since $0 \le \theta_K \le 1$, then we get

$$\begin{aligned} g_{k+1}^T d_{k+1} &\le -g_{k+1}^T g_{k+1} + \frac{g_{K+1}^T y_k}{g_k^T g_k} g_{k+1}^T d_k + \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} g_{k+1}^T d_k \\ &= -\|g_{k+1}\|^2 + \frac{\|g_{k+1}\|^2 - g_{K+1}^T g_k}{\|g_k\|^2} g_{k+1}^T d_k + \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} g_{k+1}^T d_k \end{aligned}$$

From $\|g_{k+1}\|^2 > |g_{k+1}^T g_{k+1}|$, we get

$$g_{k+1}^T d_{k+1} \le -\|g_{k+1}\|^2 + \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} g_{k+1}^T \le -\left[1 + \frac{g_{k+1}^T d_k}{g_{k+1}^T d_k + \|d_k\|}\right] \|g_{k+1}\|^2$$

Since $g_{k+1}^T d_k \le g_{k+1}^T d_k + \|d_k\|^2$ this indicates $\frac{g_{k+1}^T d_k}{g_{k+1}^T d_k + \|d_k\|} \le 1$.
Suppose $c = \left(1 - \frac{g_{k+1}^T d_k}{g_{k+1}^T d_k + \|d_k\|}\right) \ge 0$.
So, we get $g_{k+1}^T d_{k+1} \le -c\|g_{k+1}\|^2, c \in [0, 1]$.

That means $g_{k+1}^T d_{k+1} < 0$. □

Lemma 3.3. *Suppose that assumption H_1 and H_2 hold. Consider the suggested algorithm 1 where $0 \le \theta_k \le 1$. Let $\alpha_k > 0$ is achieved by ALS. If $\|s_k\|$ tends to zero and $\exists \eta_1, \eta_2 \ge 0$ such that $|g_k|^2 \ge \eta_1 |s_k|^2; |g_{k+1}|^2 \ge \eta_2 |s_k|^2$, and f is uniformly convex function, then*

$$g_k = 0 \tag{3.1}$$

Theorem 3.4. *Assume Assumptions H_1 and H_2 hold and consider β_k in formulas (1.5) and (1.6), where d_k is a descent direction and $\alpha_k > 0$ is achieved by ALS. If*

$$\sum_{k \ge 1} \frac{1}{\|d_{k+1}\|^2} < \infty$$

Then

$$\inf \|g_k\| = 0 \tag{3.2}$$

Proof.

$$\beta_k^H = (1 + \theta_K)\beta_k^{PR} + \theta_K \beta_K^{LS}$$

$$\beta_k^H = (1 + \theta_K) \frac{g_{K+1}^T (g_{k+1} - g_k)}{g_k^T g_k} + \theta_K \frac{g_{K+1}^T g_{k+1}}{(g_{k+1} + d_k)^T d_k}$$

From strong Armijo state, we catch

$$\beta_k^H \leq (1+\theta_K)\frac{g_{K+1}^T g_{k+1} - g_{K+1}^T g_k}{g_k^T g_k} + \theta_K \frac{g_{K+1}^T g_{k+1}}{\sigma g_K^T d_k + \|d_k\|^2}$$

Using Powell restart inequality

$$\beta_k^H \leq \theta_K \frac{\|g_{k+1}\|^2}{g_K^T d_k + \|d_k\|^2} \leq \frac{\|g_{k+1}\|^2}{g_K^T d_k + \|d_k\|^2} \tag{3.3}$$

Now we have the new direction $d_{k+1} = -g_{k+1} + \beta_k^H d_k$

$$\begin{aligned}
\|d_{k+1}\| &= \left\|-g_{k+1} + \beta_k^H d_k\right\| \\
\|d_{k+1}\|^2 &= \left\|-g_{k+1} + \beta_k^H d_k\right\|^2 \leq \left[\|g_{k+1}\| + \left\|\beta_k^H d_k\right\|\right]^2 \\
&= \|g_{k+1}\|^2 + 2\beta_k^H \|g_{k+1}\| \|d_k\| + \left(\beta_k^H\right)^2 \|d_k\|^2
\end{aligned}$$

From (3.3) we get

$$\begin{aligned}
\|d_{k+1}\|^2 \leq \|g_{k+1}\|^2 + 2\frac{\|g_{k+1}\|^2}{g_K^T d_k + \|d_k\|^2}\|g_{k+1}\| \|d_k\| + \frac{\|g_{k+1}\|^4}{\left(g_K^T d_k + \|d_k\|^2\right)^2}\|d_k\|^2 \\
= \left[1 + \frac{\|g_{k+1}\| \|d_k\|}{g_K^T d_k + \|d_k\|^2}\right]^2 \|g_{k+1}\|^2
\end{aligned}$$

From $\|g_{k+1}\| \leq \|g_{k+1}\|^2$ similarly for direction $\|d_k\| \leq \|d_k\|^2$

$$\|d_{k+1}\|^2 \leq \left[1 + \frac{\|g_{k+1}\|^2 \|d_k\|^2}{g_K^T d_k + \|d_k\|^2}\right]^2 \|g_{k+1}\|^2$$

Divided the ratio terms in RHS by $\|d_k\|^2$, implies

$$\|d_{k+1}\|^2 = \left[1 + \frac{\|g_{k+1}\|^2}{1 + \frac{g_K^T d_k}{\|d_k\|^2}}\right]^2 \|g_{k+1}\|^2 \tag{3.4}$$

From Lemma 3.3 it follows that

$$\begin{aligned}
\frac{g_k^T d_k}{\|d_k\|^2} = \frac{-g_k^T d_k}{-\|d_k\|^2} \geq \frac{\omega \|g_k\|^2}{-\|d_k\|^2} = \frac{\omega \|g_k\|^2}{-\frac{\|s_k\|^2}{\alpha_k^2}} = -\omega\eta_1\alpha_k^2 \\
\|d_{k+1}\|^2 = \left[1 + \frac{\eta_2 \|s_k\|}{1 - \omega\eta_1\alpha_k^2}\right]^2 \eta_2 \|s_k\|
\end{aligned}$$

Let $L = \max\{x_{k+1} - x_k\}$, and since the function bounded, we have

$$\|s_k\| \leq D\|d_{k+1}\|^2 \leq \left[1 + \frac{\eta_2 D}{1 - \omega\eta_1\alpha_k^2}\right]^2 \eta_2 D = \varphi \rightarrow \frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{\varphi}$$

Also

$$\sum_{k\geq 1} \frac{1}{\|d_{k+1}\|^2} \geq \sum_{k\geq 1}\frac{1}{\varphi} \Rightarrow \sum_{k\geq 1}\frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{\varphi}\sum_{k\geq 1} 1 = \frac{1}{\varphi} * \infty = \infty$$

Hence (3.2) holds. □

4 Numerical Testing

To demonstrate the value and effectiveness of the proposed TR1 algorithm, its performance will be evaluated in comparison to two well-known algorithms: TR2 from [11] and TR3 from [12]. The problems are taken directly from [3]. where [2] are the initial spots for these problems.

One computer was used to perform testing with 1.70GHz CPU and 8.00 GB of RAM. All algorithm codes are written in MATLAB R 2018a.

Where $\rho = 0.6, \sigma = 0.3, c = 0.4, \gamma = 0.2$, epsilon $= 10^{-8}$, stop condition is $\|F_{k-1}\| \leq 10^{-8}$, and number of totals iteration exceeds 20000 .

The problems are taken from [3] as follows:

$$P_1 \; : f = 100 * \left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

$$P_2 \; : f = \left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

$$P_3 \; : f = \left(x_2 - x_1^2\right)^3 + (1 - x_1)^2$$

$$P_4 \; : f = \left(x_2 - x_1^2\right)^2 + (1 - x_1)$$

Take the starting points from [2] as follows:

$$x_0 = (10, 10, \ldots, 10)^T, \qquad x_1 = (-10, -10, \ldots, -10)^T$$

$$x_2 = (1, 1, \ldots, 1)^T, \qquad x_3 = (-1, -1, \ldots, -1)^T$$

$$x_4 = \left(1, \frac{1}{2}, \frac{1}{3} \ldots, \frac{1}{n}\right)^T, \qquad x_5 = (0.1, 0.1, \ldots, 0.1)^T$$

$$x_6 = \left(\frac{1}{n}, \frac{2}{n}, \ldots, 1\right)^T.$$

Table 1:

P.	Dim.	S.P.	TTR1			TTR2			TTR3		
			N_i	N_f	CPU	N_i	N_f	CPU	N_i	N_f	CPU
P_1	20000	x_0	49	61	1.32	59	280	1.64	140	322	1.821
	20000	x_1	41	64	0.95	59	310	1.9	150	302	2.853
	20000	x_2	45	68	0.74	57	276	53	178	278	0.843
	20000	x_3	44	78	3	57	276	0.8	168	278	0.903
	20000	x_4	49	79	0.60	99	180	75	99	180	1.921
	20000	x_5	48	68	3	66	234	0.8	137	236	1.531
	20000	x_6	41	60	1.72	83	268	75	103	268	1.453
P_2	20000	x_0	53	91	0.97	89	200	1.89	140	322	1.8
	20000	x_1	51	94	5	119	110	2.6	130	302	2.853
	20000	x_2	50	98	0.96	117	176	55	138	278	0.843
	20000	x_3	57	98	5	117	176	1.8	148	278	1.903
	20000	x_4	58	99	0.78	89	140	49	89	180	1.929
	20000	x_5	58	98	3	126	134	1.9	117	236	1.531
	20000	x_6	66	80	1.15	123	168	93	143	268	1.253
P_3	20000	x_0	75	71	1.393	79	280	3.640	170	322	1.648
	20000	x_1	71	76	0.99	139	310	2.9	130	302	1.377
	20000	x_2	72	68	4	90	276	54	118	270	0.976
	20000	x_3	87	68	0.76	87	271	0.8	130	278	0.888

Continued . . .

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P.	Dim.	S.P.	TTR1			TTR2			TTR3		
			N_i	N_f	CPU	N_i	N_f	CPU	N_i	N_f	CPU
	20000	x_4	64	27	2	89	187	87	80	180	1.765
	20000	x_5	78	58	0.63	98	234	1.7	121	236	1.200
	20000	x_6	47	32	9	73	268	63	123	264	1.323
P_4	20000	x_0	58	71	0.88	89	280	0.99	160	312	0.975
	20000	x_1	69	74	1.00	80	390	1.8	150	302	1.076
	20000	x_2	72	68	8	80	276	64	138	278	0.657
	20000	x_3	37	68	0.98	97	176	0.8	147	278	0.897
	20000	x_4	56	279	7	89	280	76	89	189	0.543
	20000	x_5	58	58	0.67	96	234	0.8	177	236	1.945
	20000	x_6	59	320	5	63	268	76	173	268	1.666

Table 1 shows that when compared to the other methods, the purposed TR! technique requires the fewest number of iterations and function evaluations to reach the answer. It also shows that TR1 recorded the least CPU time. The results o show the effectiveness of the new method and it can be said that the proposed method can be considered acceptable to be used in this field.

5 Conclusion

Conjugate gradient methods (CGM) are seen as very effective techniques to solve a system of nonlinear equations as well as unconstrained optimization problems. Global convergence of this approach is demonstrated in this study by combining the β_k parameters from the Polak- Ribiere (PR) technique with the β_k parameters from the Liu-Store (LS) technique in a modified strategy. The results showed that our new strategy was good and effective for solving optimization problems compared with twoothers well- known methods.

The conjugate gradient methods are thought of as very effective techniques utilized to solve the system of nonlinear equations as well as unconstrained optimization problems. This approach's global convergence is demonstrated in this study by merging the β_k parameters from the Polak- Ribiere (PR) technique with the β_k parameters from the Liu- Storey (LS) technique in a modified strategy. The results showed that our new strategy for solving optimization issues was good and efficient when its performance was compared to two well-known approaches.

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