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EXISTENCE OF GLOBAL SOLUTIONS OF A REACTION-DIFFUSION SYSTEM WITH A CROSS-DIFFUSION MATRIX AND FRACTIONAL DERIVATIVES

Nabila Barrouk and Salim Mesbahi

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Abstract This paper aims to prove the existence of global solutions in time for a parabolic fractional reaction-diffusion model with either Dirichlet or Neumann boundary conditions. The model possesses two key properties: solution positivity and a triangular structure. Notably, this study introduces the assumption that the data is not regular and that the nonlinearities exhibit critical growth concerning the gradient. The employed technique relies on the compact semigroups method. The primary objective is to establish, under suitable hypotheses, that the proposed model exhibits a global solution encompassing a broad range of nonlinearities.

1 Introduction

The theory of reaction-diffusion systems originated in the early 20th century through research on population dynamics, combustion theory, and chemical kinetics. Currently, it is a highly advanced field of study with diverse applications across various scientific and engineering disciplines. Murray [25, 26] presents numerous models in biology, ecology, and applied sciences. For a mathematical analysis of these systems, Alaa and Masbahi [1, 22], Moumeni and Barrouk [23, 24], Batiha et *al.* [4], Lions [21], and the references mentioned therein; where he finds many methods and techniques.

Recently, there has been a growing interest among researchers in fractional differential systems due to their extensive applications in various scientific fields, particularly in biomathematics. Numerous theoretical and experimental studies have been conducted focusing on this subject. These investigations have revealed that certain thermal [5], electrochemical [10], and biological [15], are governed by fractional differential equations. Additional significant applications can be found in the mentioned references, covering a range of scientific disciplines. A noteworthy aspect is that the introduction of fractional differentiation reduces the number of model parameters. While several studies have been conducted on fractional reaction-diffusion systems, most of them have focused on cases where the nonlinearities are independent of the gradient. Examples of such works include those by Alsaedi et *al.* in [2], Besteiro and Rial [7]. However, these discussions primarily pertain to models consisting of two equations with diffusion coefficients and specific reaction functions. This is primarily due to the wide-ranging applications of these models. Furthermore, a valuable resource for surveys and literature pertaining to fractional differential equations can be found in ([6, 27, 16]).

In this study, our focus lies on examining solutions for a parabolic fractional reaction-diffusion system. What sets our work apart is the assumption that the data is not smooth and that the nonlinearities exhibit critical growth in relation to the gradient. Our objective is to establish, under suitable assumptions, that the proposed model possesses a global solution encompassing a wide range of nonlinearities. We ensure the preservation of solution positivity and control over the total mass of the system's components over time, which is achieved through hypotheses (2.1)-(2.3). To prove this, we employ a technique based on the compact semigroups method, along with various estimates and the application of the following theorem.

Theorem 1.1. *Let us recall the following classical boundary eigenvalue system for the fractional power of the Laplacian in Ω with homogeneous Neumann boundary condition*

$$\begin{cases} (-\Delta)^{\alpha}\varphi_k = \lambda_k^{\alpha}\varphi_k & \text{in } \Omega \\ \dfrac{\partial \varphi_k}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ and

$$D\left((-\Delta)^{\alpha}\right) = \left\{ u \in L^2(\Omega),\ \frac{\partial u}{\partial \eta} = 0,\ \left\|(-\Delta)^{\alpha} u\right\|_{L^2(\Omega)} < +\infty \right\}$$

$$\left\|(-\Delta)^{\alpha} u\right\|_{L^2(\Omega)}^2 = \sum_{k=1}^{+\infty} \left|\lambda_k^{\alpha}\langle u, \varphi_k\rangle\right|^2$$

This system has a countable system of eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ and $\lambda_k \to \infty$ as $k \to \infty$, and φ_k be the corresponding eigenvectors for all $k \geq 1$.

So for $u \in D\left((-\Delta)^{\alpha}\right)$, we have

$$(-\Delta)^{\alpha} u = \sum_{k=1}^{+\infty} \lambda_k^{\alpha}\langle u, \varphi_k\rangle \varphi_k$$

and the following integration by parts formula

$$\int_{\Omega} u(x)(-\Delta)^{\alpha} v(x)\,dx = \int_{\Omega} v(x)(-\Delta)^{\alpha} u(x)\,dx \ \text{ for } u, v \in D\left((-\Delta)^{\alpha}\right) \tag{1.1}$$

Proof. See Diagana [11]. □

Let then be the following fractional reaction-diffusion model

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u + d_1(-\Delta)^{\alpha} u = f(t, x, u, v, \nabla u, \nabla v) & \text{in } Q_T \\ \dfrac{\partial v}{\partial t} - b\Delta v + d_2(-\Delta)^{\beta} v = g(t, x, u, v, \nabla u, \nabla v) & \text{in } Q_T \end{cases} \tag{1.2}$$

where $Q_T =]0, T[\times \Omega$, $T > 0$, Ω is a regular and bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$, $N \geq 2$, $u = u(t, x)$, $v = v(t, x)$ for $(t, x) \in Q_T$ and f, g are real functions, the presence of the non local operator $(-\Delta)^{\delta}$, $0 < \delta < 1$ with $\delta = \alpha$ or β, which accounts for the anomalous diffusion [17, 18], means that the sub-populations face some obstacles that slow their movement, and the two constants of diffusion a, b, d_1, d_2 are assumed to be nonnegative, is the most approached by the researchers, f and g are enough regular.

The system (1.2) is assumed to be subject to the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \ \text{ or } \ u = v = 0 \text{ in } \Sigma_T =]0, T[\times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0, x) = u_0(x) \geq 0\ ,\ v(0, x) = v_0(x) \geq 0 \text{ for all } x \in \Omega \tag{1.4}$$

which are assumed to be continuous. The local existence in time of the solution (u, v) is classical. The positivity of the solution stems from the positivity of u_0 and v_0. The inspiration for this system stems from the research conducted in [3] and the work carried out by Larez and Leiva in [20]. The mentioned system (1.2)-(1.4) emerges in the domains of physics, chemistry, and diverse biological phenomena, such as population dynamics, as discussed by Fisher [14]. For u_0 and v_0, we assume the following hypothesis

$$u_0, v_0 \text{ are nonnegative functions in } L^1(\Omega) \tag{1.5}$$

The subsequent sections of this paper are structured as follows: In the following section, we state our main result. Subsequently, we provide essential results that are crucial for comprehending the content of this study. In the fourth section, we present results pertaining to the approximate problem. The final section is dedicated to proving the main result.

2 Statement of the main result

2.1 Assumptions

Our main emphasis is on systems that adhere to the triangular structure, which is commonly observed in various applications. We will specifically focus on positive solutions while disregarding other cases. These two fundamental properties are guaranteed by the following assumptions. The nonnegativity of the solutions is preserved for all time, thus the functions f and g are assumed to be quasipositive, which means that

$$\begin{cases} f(t,x,0,v,0,s) \geq 0\ ,\ g(t,x,u,0,r,0) \geq 0 \\ \forall u,v \geq 0,\ r,s \in \mathbb{R}^N,\ \text{and for a.e. } (t,x) \in Q_T \end{cases} \tag{2.1}$$

The total mass of the components u, v are a priori bounded on any finite interval, which is ensured by the existence of a non-negative constant C independent of (u,v) such that

$$f + g \leq C(u+v) \text{ for all } (u,v) \in \mathbb{R}_+^2, \text{ and for a.e. } (t,x) \in Q_T \tag{2.2}$$

$$f \leq C(u+v) \text{ for all } (u,v) \in \mathbb{R}_+^2, \text{ and for a.e. } (t,x) \in Q_T \tag{2.3}$$

Furthermore, we assume the following assumptions about f and g

$$f,\ g : Q_T \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \text{ are measurable} \tag{2.4}$$

$$f,\ g : \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R} \text{ are locally Lipschitz,} \tag{2.5}$$

namely

$$\begin{aligned} &|f(t,x,u,v,p,q) - f(t,x,\hat{u},\hat{v},\hat{p},\hat{q})| + |g(t,x,u,v,p,q) - g(t,x,\hat{u},\hat{v},\hat{p},\hat{q})| \\ &\leq K(r)(|u-\hat{u}| + |v-\hat{v}| + \|p-\hat{p}\| + \|q-\hat{q}\|) \end{aligned}$$

for a.e. $(t,x) \in Q_T$ and for all $0 \leq |u|, |\hat{u}|, |v|, |\hat{v}|, \|p\|, \|\hat{p}\|, \|q\|, \|\hat{q}\| \leq r$, where $\|.\|$ is the norm in $L^1(\Omega)$.

2.2 The main result

The existence of global solutions for the system (1.2)-(1.4) is equivalent to the existence of (u,v) illustrated by the following main Theorem.

Theorem 2.1. *Suppose that hypotheses (1.5)-(2.5) are satisfied, then there exists (u,v) solution of*

$$\begin{cases} u,v \in C\left([0,+\infty[\,, L^1(\Omega)\right) \\ f(t,x,u,v,\nabla u,\nabla v)\ ,\ g(t,x,u,v,\nabla u,\nabla v) \in L^1(Q_T) \\ u(t) = S_1(t)\,u_0 + \int_0^t S_1(t-s)\, f(s,.,u(s),v(s),\nabla u(s),\nabla v(s))\,ds \\ v(t) = S_2(t)\,v_0 + \int_0^t S_2(t-s)\, g(s,.,u(s),v(s),\nabla u(s),\nabla v(s))\,ds, \\ \text{for all } t \in [0,T[\end{cases} \tag{2.6}$$

where $S_1(t)$ and $S_2(t)$ are contraction semigroups in $L^1(\Omega)$ generated, respectively, by $a\Delta - d_1(-\Delta)^{\alpha}$ and $b\Delta - d_2(-\Delta)^{\beta}$.

In order to establish the validity of this theorem, we will rely on the outcomes derived from the results that will be presented in the subsequent section.

3 Some preliminary results

3.1 Local existence

Lemma 3.1. *Consider the operator A, which is m-dissipative in the Banach space X and serves as the infinitesimal generator of the semigroup $S(t)$. Additionally, let F be a locally Lipschitz function, then for all $u_0 \in X$ there exists $T_{\max} = T(u_0)$ such that the system*

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X) \\ \dfrac{du}{dt} - Au = F(t,x,u,\nabla u) \\ u(0) = u_0 \end{cases} \tag{3.1}$$

admits a unique local solution u verifying

$$u(t) = S(t)u_0 + \int_0^t S(t-s) F(s,.,u(s), \nabla u(s))\, ds, \ \forall t \in [0, T_{\max}]$$

Proof. See Pazy [28]. □

Lemma 3.2. *Let $\theta \in C_0^\infty(Q_T)$, $\theta \geq 0$, then there exists a nonnegative function $\Phi \in C^{1,2}(Q_T)$ solution of the system*

$$\begin{cases} -\Phi_t - d\Delta\Phi = \theta & \text{in } Q_T \\ \Phi(t,x) = 0 & \text{on } \Sigma_T \\ \Phi(T,x) = 0 & \text{in } \Omega, \end{cases}$$

for all $q \in]1,\infty[$, and there exists a nonnegative constant $C(q,\Omega,T)$ independent of θ such that

$$\|\Phi_t\|_{w_q^{2,1}(Q_T)} \leq C(q,\Omega,T)\|\theta\|_{L^q(Q_T)}. \tag{3.2}$$

(see Ladyzenskaja et al. [19]).
By Sobolev's imbedding theorem, we deduce from (3.2) the following estimate:
For all $q \in]1,\infty[$, $q' \in]1,\infty]$ such that

$$q \leq q' \text{ and } 2 - \left(\frac{1}{q} - \frac{1}{q'}\right)(n+2) > 0,$$

there exists a nonnegative constant C independent of θ such that

$$\|\Phi\|_{L^{q'}(Q_T)} \leq C\|\theta\|_{L^q(Q_T)}.$$

And for all $\omega_0 \in L^1(\Omega)$ and $h \in L^1(Q_T)$, we have the following equalities

$$\int_{Q_T} (S(t)\omega_0(x))\,\theta dxdt = \int_\Omega \omega_0(x)\,\Phi(0,x)\,dx, \tag{3.3}$$

and

$$\int_{Q_T}\left(\int_0^t S(t-s)\,h(s,x,\omega(s),\nabla\omega(s))\,ds\right)\theta dxdt =$$

$$\int_{Q_T} h(s,x,\omega(s),\nabla\omega(s))\,\Phi(s,x)\,dxds \tag{3.4}$$

Proof. To prove this Lemma, see Bonafede and Schmitt [8]. □

3.2 Compactness of operator

In this section we will give a compactness result of operator L defining the solution of system (3.1) in the case where $u(0) = 0$ with

$$L(F)(t) = u(t) = \int_0^t S(t-s) F(s,.,u(s), \nabla u(s))\, ds \ , \ \ \forall t \in [0,T]$$

Theorem 3.3 (Dunford–Pettis [12]). *Let $\mathcal{F}$ be a bounded set in $L^1(\Omega)$, then $\mathcal{F}$ has a compact closure in the weak topology $\sigma\left(L^1, L^\infty\right)$ if and only if $\mathcal{F}$ is equi-integrable, that is,*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \int_A |f| < \varepsilon \text{ for all } A \subset \Omega \text{ measurable with } |A| < \delta, \ \forall f \in \mathcal{F}$$

and

$$\left\{\forall \varepsilon > 0, \ \exists \omega \subset \Omega \text{ measurable with } |\omega| < \infty \text{ such that } \int_{\Omega\setminus\omega} |f| < \varepsilon, \ \forall f \in \mathcal{F}\right\}$$

Theorem 3.4. *If for all $t > 0$, the operator $S(t)$ is compact, then L is compact of $L^1([0,T], X)$ in $L^1([0,T], X)$.*

Proof. **Step 1.** We show that $S(\lambda) L : F \to S(\lambda) L(F)$ is compact from $L^1([0,T], X)$, i.e., show that the set $\{S(\lambda) L(F)(t) ; \ \|F\|_1 \leq 1\}$ is relatively compact in $L^1([0,T], X), \forall t \in [0,T]$. Since $S(t)$ is compact then, the application $t \to S(t)$ is continuous of $]0, +\infty[$ in $\mathcal{L}(X)$, therefore

$$\forall \varepsilon > 0, \ \forall \delta > 0, \ \exists \eta > 0, \ \forall 0 \leq h \leq \eta, \ \forall t \geq \delta, \ \|S(t+h) - S(t)\|_{\mathcal{L}(X)} \leq \varepsilon$$

By choosing $\lambda = \delta$, we have for $0 \leq t \leq T-h$

$$S(\lambda) u(t+h) - S(\lambda) u(t) =$$

$$\begin{aligned} &= \int_0^{t+h} S(\lambda + t + h - s) F(s,.,u(s), \nabla u(s))\, ds \\ &\quad - \int_0^t S(\lambda + t - s) F(s,.,u(s), \nabla u(s))\, ds \\ &= \int_t^{t+h} S(\lambda + t + h - s) F(s,.,u(s), \nabla u(s))\, ds + \\ &\quad + \int_0^t \left(S(\lambda + t + h - s) - S(\lambda + t - s)\right) F(s,.,u(s), \nabla u(s))\, ds \end{aligned}$$

We obtain then

$$\|S(\lambda) u(t+h) - S(\lambda) u(t)\|_X$$
$$\leq \int_t^{t+h} \|F(s,.,u(s), \nabla u(s))\|_X\, ds + \varepsilon \int_0^t \|F(s,.,u(s), \nabla u(s))\|_X\, ds$$

We define $v(t)$ by

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\|S(\lambda) v(t+h) - S(\lambda) v(t)\|_1 \leq (h + \varepsilon T) \|F(s,.,u(s), \nabla u(s))\|_1$$

which implies that $\{S(\lambda) v; \ \|F\|_1 \leq 1\}$ is equi-integrable, then

$$\{S(\lambda) L(F)(t) ; \ \|F\|_1 \leq 1\}$$

is relatively compact in $L^1([0,T],X)$ which means that $S(\lambda)L$ is compact.

Step 2. We show that $S(\lambda)L$ converges to L when λ tends to 0 in $L^1([0,T],X)$. We have

$$S(\lambda)u(t)-u(t) = \int_0^t S(\lambda+t-s)F(s,.,u(s),\nabla u(s))\,ds - \int_0^t S(t-s)F(s,.,u(s),\nabla u(s))\,ds$$

So, for $t \geq \delta$, we have

$$\|S(\lambda)u(t)-u(t)\| \leq \int_\delta^t \|S(\lambda+s)-S(s)\|_{\mathcal{L}(X)}\|F(s,.,u(s),\nabla u(s))\|\,ds + 2\int_{t-\delta}^t \|F(s,.,u(s),\nabla u(s))\|\,ds$$

We choose $0<\lambda<\eta$, then

$$\|S(\lambda)u(t)-u(t)\| \leq \varepsilon\int_\delta^t \|F(s,.,u(s),\nabla u(s))\|\,ds + 2\int_{t-\delta}^t \|F(s,.,u(s),\nabla u(s))\|\,ds$$

and for $0\leq t<\delta$, we have

$$\|S(\lambda)u(t)-u(t)\| \leq 2\int_0^t \|F(s,.,u(s),\nabla u(s))\|\,ds$$

Since $F\in L^1(0,T,X)$, we obtain

$$\|S(\lambda)u(t)-u(t)\| \leq (\varepsilon T+2\delta)\|F(s,.,u(s),\nabla u(s))\|_1$$

So, if $\lambda\to 0$ then $S(\lambda)u\to u$ in $L^1([0,T],X)$. The operator L is a uniform limit with compact linear operator between two Banach spaces, then L is compact in $L^1([0,T],X)$. □

Remark 3.5. The semigroup $S(t)$ generated by the operator $a\Delta-d_1(-\Delta)^\delta$ or $b\Delta-d_2(-\Delta)^\delta$ is compact in $L^1(\Omega)$.

4 Approximating scheme

We convert system (1.2)-(1.4) to an abstract first order system in the Banach space $X=\left(L^1(\Omega)\right)^2$. For all $n>0$, we define the functions u_{n_0} and v_{n_0} by

$$u_{n_0}=\min\{u_0,n\} \quad \text{and} \quad v_{n_0}=\min\{v_0,n\}$$

It is clear that u_{n_0} and v_{n_0} verify (1.5), i.e.,

$$u_{n_0} \text{ and } v_{n_0} \text{ are nonnegative functions in } L^1(\Omega)$$

Now, we consider the following problem

$$\begin{cases} \dfrac{\partial w_n}{\partial t}-Aw_n=F(t,x,w_n,\nabla w_n) & \text{in } Q_T \\ \dfrac{\partial w_n}{\partial \eta}=0 \text{ or } w_n=0 & \text{in } \Sigma_T \\ w_n(0,.)=w_{n_0}(.) & \text{on } \Omega \end{cases} \tag{4.1}$$

Here $w_n=(u_n,v_n)^t$, $w_{n_0}=(u_{n_0},v_{n_0})^t$, $F=(f,g)^t$, and the operator A is defined as follows

$$A=\begin{pmatrix} a\Delta-d_1(-\Delta)^\alpha & 0 \\ 0 & b\Delta-d_2(-\Delta)^\beta \end{pmatrix}$$

where

$$\begin{aligned} D(A) \quad : \quad &= \big\{ w_n : w_n \in L^1(\Omega) \times L^1(\Omega) \setminus \\ &\Big(\left(a\Delta - d_1 (-\Delta)^{\alpha} \right) u_n, \Big(b\Delta - d_2 (-\Delta)^{\beta} \Big) v_n \Big) \in L^1(\Omega) \times L^1(\Omega) \\ &\text{and } \frac{\partial w_n}{\partial \eta} = 0 \text{ or } w_n = 0 \Big\}. \end{aligned}$$

4.1 Local existence of the solution of system (4.1)

System (4.1) can be written in the form of system (3.1), thus, if (u_n, v_n) is a solution of (4.1) then it satisfies the integral equations

$$\begin{cases} u_n(t) = S_1(t) u_{n_0} + \int_0^t S_1(t-s) f(s,.,u_n(s), v_n(s), \nabla u_n(s), \nabla v_n(s)) \, ds \\ v_n(t) = S_2(t) v_{n_0} + \int_0^t S_2(t-s) g(s,.,u_n(s), v_n(s), \nabla u_n(s), \nabla v_n(s)) \, ds \end{cases} \tag{4.2}$$

where $S_1(t)$ and $S_2(t)$ are the contractions semigroups in $L^1(\Omega)$ generated, respectively, by $a\Delta - d_1(-\Delta)^{\alpha}$ and $b\Delta - d_2(-\Delta)^{\beta}$.

Theorem 4.1. *There exists $T_M > 0$ and (u_n, v_n) a local solution of (4.1) for all $t \in [0, T_M]$.*

Proof. We know that $S_1(t)$, $S_2(t)$ are contraction semigroups and that F is locally Lipschitz, $0 \leq u_{n_0}, v_{n_0} \leq n$, then there exists $T_M > 0$ such that (u_n, v_n) is a local solution of (4.1) on $[0, T_M]$. □

Theorem 4.2. *Let $u_{n_0}, v_{n_0} \in L^1(\Omega)$, then there exists a maximal time $T_{\max} > 0$ and a unique solution $(u_n, v_n) \in C\left([0, T_{\max}), L^1(\Omega) \times L^1(\Omega)\right)$ of system (4.1), with the alternative :*
- either $T_{\max} = +\infty$,
- or $T_{\max} < +\infty$ and $\lim\limits_{t \to T_{\max}} \left(\|u_n(t)\|_\infty + \|v_n(t)\|_\infty \right) = +\infty$.

Proof. For arbitrary $T > 0$, we define the Banach space

$$E_T := \left\{ (u_n, v_n) \in C\left([0,T], L^1(\Omega) \times L^1(\Omega)\right); \ \|(u_n, v_n)\| \leq 2 \|(u_{n_0}, v_{n_0})\| = R \right\}$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega)}$ and $\|\cdot\|$ is the norm of E_T defined by

$$\|(u_n, v_n)\| := \|u_n\|_{L^\infty([0,T], L^\infty(\Omega))} + \|v_n\|_{L^\infty([0,T], L^\infty(\Omega))}$$

Next, for every $(u_n, v_n) \in E_T$, we define

$$\Psi(u_n, v_n) := \left(\Psi_1(u_n, v_n), \Psi_2(u_n, v_n) \right),$$

where for $t \in [0, T]$

$$\Psi_1(u_n, v_n) = S_1(t) u_{n_0} + \int_0^t S_1(t-s) f(s,.,u_n(s), v_n(s), \nabla u_n(s), \nabla v_n(s)) \, ds$$
$$\Psi_2(u_n, v_n) = S_2(t) v_{n_0} + \int_0^t S_2(t-s) g(s,.,u_n(s), v_n(s), \nabla u_n(s), \nabla v_n(s)) \, ds$$

We will prove the local existence by the Banach fixed point theorem.
• $\Psi : E_T \to E_T$ is well defined. Let $(u_n, v_n) \in E_T$, we obtain by maximum principle

$$\begin{aligned} \|\Psi_1(u_n, v_n)\|_\infty &\leq \|u_{n_0}\|_\infty + C\left(\|u_n\|_\infty + \|v_n\|_\infty\right) T \\ &\leq \|u_{n_0}\|_\infty + C\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right) T \end{aligned}$$

Similarly, we have

$$\|\Psi_2(u_n, v_n)\|_\infty \leq \|v_{n_0}\|_\infty + C\left(\|u_{n_0}\|_\infty + \|v_{n_0}\|_\infty\right) T$$

So we have,

$$\begin{aligned}\left\|\Psi\left(u_n,v_n\right)\right\| &\leq \left\|u_{n_0}\right\|_\infty+\left\|v_{n_0}\right\|_\infty+2C\left(\left\|u_{n_0}\right\|_\infty+\left\|v_{n_0}\right\|_\infty\right)T\\ &\leq 2\left(\left\|u_{n_0}\right\|_\infty+\left\|v_{n_0}\right\|_\infty\right)\end{aligned}$$

By choosing $T \leq \frac{\left\|u_{n_0}\right\|_\infty+\left\|v_{n_0}\right\|_\infty}{CR}$, we get $\Psi\left(u_n,v_n\right)\in E_T$.

• Now, we prove that Ψ is a contraction mapping for (u_n,v_n), $(\tilde{u}_n,\tilde{v}_n)\in E_T$, we have

$$\begin{aligned}\left\|\Psi_1\left(u_n,v_n\right)-\Psi_1\left(\tilde{u}_n,\tilde{v}_n\right)\right\|_\infty &\leq L\int_0^t\left\|\left(u_n,v_n\right)-\left(\tilde{u}_n,\tilde{v}_n\right)\right\|_\infty d\tau\\ &\leq LT\left(\left\|\tilde{v}_n-v_n\right\|_\infty+\left\|\tilde{u}_n-u_n\right\|_\infty\right)\end{aligned}$$

Similarly, we obtain

$$\left\|\Psi_2\left(u_n,v_n\right)-\Psi_2\left(\tilde{u}_n,\tilde{v}_n\right)\right\|_\infty \leq LT\left(\left\|\tilde{v}_n-v_n\right\|_\infty+\left\|\tilde{u}_n-u_n\right\|_\infty\right)$$

These estimates imply that

$$\begin{aligned}\left\|\Psi\left(u_n,v_n\right)-\Psi\left(\tilde{u}_n,\tilde{v}_n\right)\right\|_\infty &\leq 2LT\left(\left\|\tilde{v}_n-v_n\right\|_\infty+\left\|\tilde{u}_n-u_n\right\|_\infty\right)\\ &\leq \frac{1}{2}\left\|\left(u_n,v_n\right)-\left(\tilde{u}_n,\tilde{v}_n\right)\right\|,\end{aligned}$$

for $T \leq \max\left\{\frac{\left\|u_{n_0}\right\|_\infty+\left\|v_{n_0}\right\|_\infty}{CR}, \frac{1}{4L}\right\}$.

Consequently, according to the Banach fixed point theorem, Problem (4.1) has a unique mild solution $(u_n,v_n)\in E_T$.

The solution can be extended on a maximal interval $[0,T_{\text{max}})$, where

$$T_{\text{max}} := \sup\left\{T>0,\ \ (u_n,v_n)\ \text{ is a solution to } (4.1) \text{ in } E_T\right\}.$$

□

For the global existence, we need the fact that the solutions are positive.

4.2 Positivity of the solution of the system (4.1)

Moving forward, we will invoke the comparison principle, a principle that will be frequently employed in the subsequent calculations. Let us consider the following problem:

$$\begin{cases} u_t-a\Delta u+d\left(-\Delta\right)^\alpha u=F\left(t,x,u,\nabla u\right) & \text{in } [0,T]\times\Omega\\ \dfrac{\partial u}{\partial\eta}=0 \text{ or } u=0 & \text{in } [0,T]\times\partial\Omega\\ u(0,x)=u_0\left(x\right) & \text{in } \Omega\end{cases}$$

Theorem 4.3 (Comparison Principle). *Let* $u,\bar{u}\in C([0,T]\times\bar{\Omega})\cap C^{1,2}(]0,T[\times\bar{\Omega})$ *be mild solutions of the problem*

$$\begin{cases} u_t-a\Delta u+d\left(-\Delta\right)^\alpha u-F\left(t,x,u,\nabla u\right)\geq \bar{u}_t-a\Delta\bar{u}+d\left(-\Delta\right)^\alpha\bar{u}-F\left(t,x,\bar{u},\nabla\bar{u}\right) & \textit{in } Q_T\\ \dfrac{\partial u}{\partial\eta}\geq\dfrac{\partial\bar{u}}{\partial\eta} \textit{ or } u\geq\bar{u} & \textit{in } \Sigma_T\\ u(0,x)\geq\bar{u}(0,x) & \textit{in } \Omega\end{cases}$$

where $F:\mathbb{R}\to\mathbb{R}$ *is locally Lipschitz. Then*

$$u(t,x)\geq\bar{u}(t,x)\ ,\ \forall(t,x)\in[0,T]\times\Omega.$$

Proof. We refer the reader to [9] or [13]. □

Lemma 4.4. *Let* (u_n, v_n) *be a solution of system (4.1) such that*

$$u_{n_0}(x) \geq 0 \text{ , } v_{n_0}(x) \geq 0 \text{ in } \Omega$$

then

$$u_n(t,x) \geq 0 \text{ and } v_n(t,x) \geq 0 \text{ , } \forall (t,x) \in Q_T.$$

Proof. Let $\bar{u}_n(t,x) = 0$ in Q_T, then $\dfrac{\partial \bar{u}_n}{\partial t} = 0$, $\Delta \bar{u}_n = 0$, $(-\Delta)^\alpha \bar{u}_n = 0$ and $\nabla \bar{u}_n = 0$. Then, according to the hypothesis (2.1), we obtain

$$\begin{aligned} 0 &= \frac{\partial u_n}{\partial t} - a\Delta u_n + d_1(-\Delta)^\alpha u_n - f(t,x,u_n,v_n,\nabla u_n,\nabla v_n) \\ &\geq \frac{\partial \bar{u}_n}{\partial t} - a\Delta \bar{u}_n + d_1(-\Delta)^\alpha \bar{u}_n - f(t,x,\bar{u}_n,v_n,\nabla \bar{u}_n,\nabla v_n) \end{aligned}$$

and

$$u_n(0,x) = u_{n_0}(x) \geq 0 = \bar{u}_n(0,x)$$

Hence, by the comparison principle, we get $u_n(t,x) \geq \bar{u}_n(t,x)$, therefore $u_n(t,x) \geq 0$. In the same, way we find $v_n(t,x) \geq 0$. □

4.3 Global existence of the solution of system (4.1)

To prove the global existence of the solution of problem (4.1) for all nonnegative t, it suffices to find an estimate of the solution for all $t \geq 0$ according to the alternative. The following Lemma shows us the existence of an estimate of the solution of (4.1) in $L^1(\Omega)$.

Lemma 4.5. *Let* (u_n, v_n) *the solution of system (4.1), then there exists* $M(t)$ *which only depends of* t *such that for all* $0 \leq t \leq T_M$*, we have*

$$\|u_n + v_n\|_{L^1(\Omega)} \leq M(t)$$

We can conclude from this estimate that the solution (u_n, v_n) given by Theorem 4.1 is a global solution.

Proof. We can write system (4.1) in the following form:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - a\Delta u_n + d_1(-\Delta)^\alpha u_n = f(t,x,u_n,v_n,\nabla u_n,\nabla v_n) & \text{in } Q_T \\ \dfrac{\partial v_n}{\partial t} - b\Delta v_n + d_2(-\Delta)^\beta v_n = g(t,x,u_n,v_n,\nabla u_n,\nabla v_n) & \text{in } Q_T \\ \dfrac{\partial u_n}{\partial \eta} = \dfrac{\partial v_n}{\partial \eta} = 0 \text{ or } u_n = v_n = 0 & \text{in } \Sigma_T \\ u_n(0,x) = u_{n_0}(x) \quad , \ v_n(0,x) = v_{n_0}(x) & \text{in } \Omega \end{cases} \tag{4.3}$$

which leads to

$$\frac{\partial}{\partial t}(u_n + v_n) - a\Delta u_n - b\Delta v_n + d_1(-\Delta)^\alpha u_n + d_2(-\Delta)^\beta v_n = f + g$$

By taking into account of (2.2), we have

$$\frac{\partial}{\partial t}(u_n + v_n) - a\Delta u_n - b\Delta v_n + d_1(-\Delta)^\alpha u_n + d_2(-\Delta)^\beta v_n \leq C(u_n + v_n)$$

Let us integrate on Ω and use the integration by parts formula (1.1), it comes

$$\int_\Omega (-\Delta)^\alpha u_n(x)\,dx = 0 \text{ and } \int_\Omega (-\Delta)^\beta v_n(x)\,dx = 0$$

and apply the formula of Green

$$\int_\Omega \Delta u_n(x)\,dx = 0 \text{ and } \int_\Omega \Delta v_n(x)\,dx = 0$$

We obtain

$$\frac{\partial}{\partial t}\int_\Omega (u_n+v_n)\,dx \le C\int_\Omega (u_n+v_n)\,dx$$

So

$$\int_\Omega (u_n+v_n)\,dx \le \exp(Ct)\int_\Omega (u_{n_0}+v_{n_0})\,dx$$

If we put

$$M(t) = \exp(Ct)\,\|u_0+v_0\|_{L^1(\Omega)},$$

it comes

$$\|u_n+v_n\|_{L^1(\Omega)} \le M(t) \quad , \quad 0\le t\le T_M.$$

□

We give the following lemma which shows the existence of estimate of the solution (u_n, v_n) of system (4.1) in $L^1(Q_T)\times L^1(Q_T)$.

Lemma 4.6. *For any solution (u_n, v_n) of (4.1), there exists a constant $K(t)$ depends only on t such that*

$$\|u_n+v_n\|_{L^1(Q_T)} \le K(t)\,\|u_0+v_0\|_{L^1(\Omega)}.$$

Proof. We multiply the first equation of (4.2) by θ in $C_0^\infty(Q)$ with $\theta\ge 0$ and we integrate on Q_T, by using (3.3) and (3.4), we obtain

$$\int_{Q_T} u_n\theta\,dxdt = \int_{Q_T} S_1(t)\,u_{n_0}(x)\,\theta\,dxdt +$$

$$\int_{Q_T}\left(\int_0^t S_1(t-s)\,f(s,.,u_n(s),v_n(s),\nabla u_n(s),\nabla v_n(s))\,ds\right)\theta\,dxdt$$

$$= \int_\Omega u_{n_0}(x)\,\Phi(0,x)\,dx + \int_{Q_T} f(s,.,u_n(s),v_n(s),\nabla u_n(s),\nabla v_n(s))\,\Phi(s,x)\,dxds$$

Also, we find

$$\begin{aligned}\int_{Q_T} v_n\theta\,dxdt &= \int_\Omega v_{n_0}(x)\,\Phi(0,x)\,dx\\ &\quad + \int_{Q_T} g(s,.,u_n(s),v_n(s),\nabla u_n(s),\nabla v_n(s))\,\Phi(s,x)\,dxds\end{aligned}$$

therefore

$$\begin{aligned}&\textstyle\int_{Q_T}(u_n+v_n)\,\theta\,dxdt = \int_\Omega (u_{n_0}(x)+v_{n_0}(x))\,\Phi(0,x)\,dx +\\ &\textstyle\int_Q (f(s,.,u_n(s),v_n(s),\nabla u_n(s),\nabla v_n(s)) +\\ &g(s,.,u_n(s),v_n(s),\nabla u_n(s),\nabla v_n(s)))\Phi(s,x)\,dxds\\ &\le \textstyle\int_\Omega (u_0(x)+v_0(x))\,\Phi(0,x)\,dx + \int_Q C\,(u_n+v_n)\,\Phi(s,x)\,dxds\end{aligned}$$

Using Hölder inequality, we deduce

$$\begin{aligned}\int_{Q_T}(u_n+v_n)\,\theta\,dxdt &\le \|u_0+v_0\|_{L^1(\Omega)}\cdot\|\Phi(0,x)\|_{L^\infty(Q_T)}\\ &\quad + C\,\|u_n+v_n\|_{L^1(Q_T)}\cdot\|\Phi\|_{L^\infty(Q_T)}\\ &\le \left(\|u_0+v_0\|_{L^1(\Omega)} + C\,\|u_n+v_n\|_{L^1(Q_T)}\right)\cdot\|\Phi\|_{L^\infty(Q_T)}\\ &\le \max\{1,C\}\cdot\left(\|u_0+v_0\|_{L^1(\Omega)} + \|u_n+v_n\|_{L^1(Q_T)}\right)\cdot\|\Phi\|_{L^\infty(Q_T)}\\ &\le k_1(t)\left(\|u_0+v_0\|_{L^1(\Omega)} + \|u_n+v_n\|_{L^1(Q_T)}\right)\cdot\|\theta\|_{L^\infty(Q_T)}\end{aligned}$$

where $k_1(t) \geq \max\{c, cC\}$. Since θ is arbitrary in $C_0^\infty(Q_T)$, it comes

$$\|u_n + v_n\|_{L^1(Q_T)} \leq k_1(t)\left(\|u_0 + v_0\|_{L^1(\Omega)} + \|u_n + v_n\|_{L^1(Q_T)}\right)$$

Taking $k(t) = \frac{k_1(t)}{1-k_1(t)}$, we find

$$\|u_n + v_n\|_{L^1(Q_T)} \leq k(t)\,\|u_0 + v_0\|_{L^1(\Omega)}.$$

□

5 Proof of the main result

Now, we will prove the main result of this work, it is Theorem 2.1.

Proof of Theorem 2.1. Let us define the application L by

$$L : (w_0, h) \to S(t)\,w_0 + \int_0^t S(t-s)\,h(s, ., w(s), \nabla w(s))\,ds$$

where $S(t)$ the contraction semigroup generated by the operator $a\Delta - d(-\Delta)^\delta$. According to the previous Theorem 3.4 and since $S(t)$ is compact, then the application L is adding two compact applications in $L^1(Q_T)$. This is how L is compact from $L^1(Q_T) \times L^1(Q_T)$ in $L^1(Q_T)$. Therefore, there is a subsequence $\left(u_{n_j}, v_{n_j}\right)$ of (u_n, v_n) and (u, v) of $L^1(Q_T) \times L^1(Q_T)$ such that $\left(u_{n_j}, v_{n_j}\right)$ converges towards (u, v). Let us now show that $\left(u_{n_j}, v_{n_j}\right)$ is a solution of (4.2), we have

$$\begin{cases} u_{n_j}(t,x) = S_1(t)\,u_{n_0} + \int_0^t S_1(t-s)\,f\left(s, ., u_{n_j}(s), v_{n_j}(s), \nabla u_{n_j}(s), \nabla v_{n_j}(s)\right)ds \\ v_{n_j}(t,x) = S_2(t)\,v_{n_0} + \int_0^t S_2(t-s)\,g\left(s, ., u_{n_j}(s), v_{n_j}(s), \nabla u_{n_j}(s), \nabla v_{n_j}(s)\right)ds \end{cases} \tag{5.1}$$

It suffices to show that (u, v) verifies (2.6). It is clear that if $j \to +\infty$, we have the following limits

$$u_{n_0} \to u_0 \quad \text{and} \quad v_{n_0} \to v_0$$

and

$$\begin{aligned} &f\left(t, x, u_{n_j}, v_{n_j}, \nabla u_{n_j}, \nabla v_{n_j}\right) \to f(t, x, u, v, \nabla u, \nabla v) \quad \text{a.e. in } Q_T \\ &g\left(t, x, u_{n_j}, v_{n_j}, \nabla u_{n_j}, \nabla v_{n_j}\right) \to g(t, x, u, v, \nabla u, \nabla v) \quad \text{a.e. in } Q_T \end{aligned} \tag{5.2}$$

Thus to show that (u, v) verifies (2.6), it remains to show that

$$\begin{aligned} &f\left(t, x, u_{n_j}, v_{n_j}, \nabla u_{n_j}, \nabla v_{n_j}\right) \to f(t, x, u, v, \nabla u, \nabla v) \quad \text{in } L^1(Q_T) \\ &g\left(t, x, u_{n_j}, v_{n_j}, \nabla u_{n_j}, \nabla v_{n_j}\right) \to g(t, x, u, v, \nabla u, \nabla v) \quad \text{in } L^1(Q_T) \end{aligned}$$

We integrate the equations of (4.3) on Q_T by taking (1.1)

$$\int_\Omega 1\,(-\Delta)^\alpha v(x)\,dx = \int_\Omega v(x)\,(-\Delta)^\alpha 1 dx = 0$$

into account, it comes

$$d_1 \int_{Q_T} (-\Delta)^\alpha u_{n_j} dxdt = 0 \;\text{ and }\; d_2 \int_{Q_T} (-\Delta)^\beta v_{n_j} dxdt = 0$$

and

$$-a \int_Q \Delta u_{n_j} dxdt = 0 \;\text{ and }\; -b \int_Q \Delta v_{n_j} dxdt = 0,$$

We have

$$\begin{aligned}\int_\Omega u_{n_j}dx-\int_\Omega u_{n_0}dx &= \int_{Q_T} f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)dxdt\\ \int_\Omega v_{n_j}dx-\int_\Omega v_{n_0}dx &= \int_{Q_T} g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)dxdt\end{aligned}$$

from where

$$-\int_Q f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)dxdt \le \int_\Omega u_0 dx \tag{5.3}$$

$$-\int_Q g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)dxdt \le \int_\Omega v_0 dx \tag{5.4}$$

We denote

$$\begin{aligned}N_n &= C\left(u_{n_j}+v_{n_j}\right)-f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\\ M_n &= C\left(u_{n_j}+v_{n_j}\right)-f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)-g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\end{aligned}$$

It is clear that N_n and M_n are positive according to (2.2) and (2.3) of (5.3) and (5.4), we obtain

$$\begin{aligned}\int_{Q_T} N_n dxdt &\le C\int_{Q_T}\left(u_{n_j}+v_{n_j}\right)dxdt+\int_\Omega u_0 dx\\ \int_{Q_T} M_n dxdt &\le C\int_{Q_T}\left(u_{n_j}+v_{n_j}\right)dxdt+\int_\Omega \left(u_0+v_0\right)dx\end{aligned}$$

the Lemma 4.6 gives us

$$\int_{Q_T} N_n dxdt<+\infty \text{ and } \int_{Q_T} M_n dxdt<+\infty$$

which implies

$$\begin{aligned}\int_{Q_T}\left|f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\right|dxdt &\le C\int_{Q_T}\left(u_{n_j}+v_{n_j}\right)dxdt\\ +\int_{Q_T} N_n dxdt &< +\infty\\ \int_{Q_T}\left|g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\right|dxdt &\le C\int_{Q_T}\left(u_{n_j}+v_{n_j}\right)dxdt\\ +\int_{Q_T} M_n dxdt &< +\infty\end{aligned}$$

Let

$$h_n=N_n+C\left(u_{n_j}+v_{n_j}\right) \text{ and } \Psi_n=M_n+C\left(u_{n_j}+v_{n_j}\right)$$

h_n and Ψ_n are in $L^1(Q_T)$ and positive. Furthermore

$$\begin{aligned}\left|f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\right| \le h_n \quad &\text{a.e. in } Q_T\\ \left|g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\right| \le \Psi_n \quad &\text{a.e. in } Q_T\end{aligned}$$

Let us combine this result with (5.2) and we apply Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}f\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\to f\left(t,x,u,v,\nabla u,\nabla v\right) \quad &\text{in } L^1(Q_T)\\ g\left(t,x,u_{n_j},v_{n_j},\nabla u_{n_j},\nabla v_{n_j}\right)\to g\left(t,x,u,v,\nabla u,\nabla v\right) \quad &\text{in } L^1(Q_T)\end{aligned}$$

By passage to the limit when $j \to +\infty$ of (5.1) in $L^1(Q_T)$, we find

$$u(t) = S_1(t)\,u_0 + \int_0^t S_1(t-s)\,f(s,.,u(s),v(s),\nabla u(s),\nabla v(s))\,ds$$

$$v(t) = S_2(t)\,v_0 + \int_0^t S_2(t-s)\,g(s,.,u(s),v(s),\nabla u(s),\nabla v(s))\,ds$$

Then (u,v) verify (2.6) consequently (u,v) is the solution of the system (1.2)-(1.4). □

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References

[1] N. Alaa, S. Mesbahi and W. Bouarifi, *Global existence of weak solutions for parabolic triangular reaction-diffusion systems applied to a climate model*, An. Univ. Craiova Ser. Mat. Inform., **42(1)**, 80–97, (2015).

[2] A. Alsaedi, M. Kirane and R. Lassoued, *Global existence and asymptotic behavior for a time fractional reaction-diffusion system*, Computers and Mathematics with Applications, **73**, 951–958, (2017).

[3] S. Badraoui, *Approximate controllability of a reaction-diffusion system with a cross-diffusion matrix and fractional derivatives on bounded domains*, Bound. Value Probl.,1–14, (2010).

[4] I. Batiha, N. Barrouk, A. Ouannas and W. Alshanti, *On global existence of the fractional reaction-diffusion system's solution*, Int. J. Anal. Appl., **21(1)**, 11, (2023).

[5] J. L. Battaglia, J. C. Batsale, L. Le Lay, A. Oustaloup, and O. Cois, *Heat flux stimation through inverted non integer identification models*, International Journal of Thermal Sciences, **39**, 374–389, (2000).

[6] M. Benyoub and S. Benaissa, *Monotone iterative method for weighted fractional differential equations in Banach space*, Palest. J. Math., **9(1)**, 118–125, (2020).

[7] A. Besteiro and D. Rial, *Global existence for vector valued fractional reaction-diffusion equations*, arXiv:1805.09985 [math.AP], 2018.

[8] S. Bonafede and D. Schmitt, *Triangular reaction-diffusion systems with integrable initial data*, Nonlinear Anal., **33**, 785–841, (1998).

[9] X. Cabre and J. Roquejoffre, *The influence of fractional diffusion in Fisher-KPP equation*, Preprint, arXiv:1202.6072v1, (2012).

[10] R. Darling and J. Newmann, *On the short behaviour of porous interaction electrodes*, Journal of The Electrochemical Society, **144**, 3057–3063, (1997).

[11] T. Diagana, *Some remarks on some strongly coupled reaction-diffusion equations*, Journal für die reine und angewandte Mathematik, (2003).

[12] N. Dunford and J.T. Schwartz, *Linear operators* (3 vol.), Interscience, (1972).

[13] H. Engler, *On the speed of spread for fractional reaction-diffusion equations*, Int. J. Differ. Equ., 2010 Art. ID 315421, 16 pp.

[14] R. Fisher, *The wave of advance of advantageous genes*, Annals of Eugenics, **7(4)**, 355–369, (1937).

[15] G. Gonzàlez-Parra, A.J. Arenas and B. M. Chen-Charpentier, *A fractional order epidemic model for the simulation of outbreaks of influenza A (H1N1)*, Math. Methods Appl. Sci., **37(15)**, 2218–2226, (2014).

[16] S. Ibrahim and S. A. Murad, *Solution of fractional differential equations with some existence and stability results*, Palest. J. Math., **12(1)**, (2023).

[17] M. Ilic, F. Liu, I. Turner, and V. Anh, *Numerical approximation of a fractional-in-space diffusion equation (ii)- with nonhomogeneous boundary conditions*, Fract. Calc. Appl. Anal., **9**, 333–349, (2006).

[18] G. Karch, *Nonlinear evolution equations with anomalous diffusion, Qualitative Properties of Solutions to Partial Differential Equations*, vol. 5 of Jindřich Nečas Cent. Math. Model. Lect. Notes, Matfyzpress, Prague, 25-68, (2009).

[19] O.A. Ladyzenskaya, V.A. Solonnikov and N.N. Uralceva, *Linear and quasilinear equations of parabolic type, Translations of mathematical ms*, volonograph. 23, AMS; Providence, R. I, 1968.

[20] H. Larez and H. Leiva, *Interior controllability of a* 2×2 *reaction-diffusion system with cross-diffusion matrix,* Bound. Value Probl., **2009**, (2009).

[21] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier Villars, (1969).

[22] S. Mesbahi and N. Alaa, *Existence result for triangular reaction-diffusion systems with L^1 data and critical growth with respect to the gradient*, Mediterr. J. Math., **10**, 255–275, (2013).

[23] A. Moumeni and N. Barrouk, *Existence of global solutions for systems of reaction-diffusion with compact result*, International Journal of Pure and Applied Mathematics, **102(2)**, 169–186, (2015), .

[24] A. Moumeni and N. Barrouk, *Triangular reaction-diffusion systems with compact result*, Global Journal of Pure and Applied Mathematics, **11(6)**, 4729–4747, (2015).

[25] J. D. Murray, *Mathematical Biology I : An Introduction, volume I*, Springer-Verlag, 3rd edition, (2003).

[26] J. D. Murray, *Mathematical Biology II : Spatial Models and Biochemical Applications, volume II*, Springer-Verlag, 3rd edition, (2003).

[27] B. A. Tayyan, A. H. Sakka, *Symmetries and exact solutions of conformable fractional partial differential equations*, Palest. J. Math., **9(1)**, (2020).

[28] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, (1983).

Author information

Nabila Barrouk, Faculty of Science and Technology, Department of Mathematics, Mohamed Cherif Messaadia University, P.O.Box 1553, Souk Ahras 41000, Algeria.
E-mail: n.barrouk@univ-soukahras.dz

Salim Mesbahi, Department of Mathematics, Faculty of Sciences, Ferhat Abbas University, Setif, 19000, Algeria.
E-mail: salimbra@gmail.com

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