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Error Estimation and Approximate Solution of Nonlinear Fredholm Integro-Differential Equations

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Abstract The main focus of this paper is to introduce and analyze numerical methods for solving nonlinear Fredholm integro-differential equations. Two specific methods, namely homotopy perturbation and variational iteration methods, are implemented in this study. To assess the accuracy of the proposed schemes, extensive computational testing is conducted. The convergence analysis is performed using contracting mapping, which is a common method for proving the convergence of iterative algorithms. Additionally, the paper explores the error bound of an approximate solution generated from the partial sum of the series. This provides insights into the quality of the approximation.Comparison studies between the proposed methods are also carried out to evaluate their performance. The accuracy of the methods is assessed through numerical examples and compared against existing solutions. To further analyze the errors, l^{∞} and l^{2} norms are used. These norms quantify the differences between the approximate and actual solutions, providing a measure of accuracy. Finally, the efficiency of the approach is evaluated, considering factors such as computational complexity and execution time. The results of the numerical experiments are compared with previous studies and analytical solutions to validate their reliability and compatibility.

1 Introduction

Integro-differential equations (IDEs) play a crucial role in the modeling and analysis of various engineering applications. These equations arise in a wide range of disciplines, including physics, chemistry, biology, and finance. An IDE combines both integrals and derivatives of functions, making them versatile in representing complex phenomena. Nonlinear Fredholm integro-differential equations (NFIDs) are a specific type of IDE that have significant importance in the field of functional analysis. NFIDs find applications in diverse areas such as mechanics, electric fields, biochemistry, and finances. They provide a framework for studying nonlinear behaviors and phenomena that involve interactions between different variables and their derivatives. Given the wide range of real-world problems that can be described by IDEs, including NFIDs, developing efficient and accurate numerical methods for solving these equations is of great importance in many fields of research and engineering.[1, 2, 3, 4, 5].

The second kind of nonlinear Fredholm integro-differential equation that will be considered in this work has the form

$$y''(x) = f(x) + \lambda \int_a^b k(x,t) G\left(y(t)\right) dt, \tag{1.1}$$

with initial conditions

$$y(a) = \alpha,\ y'(a) = \beta,$$

where $k(x,t)$, $f(x)$ are a known functions, G is nonlinear function.

Various methods for solving Fredholm integro-differential equations have been extensively studied by several authors[6, 7, 8, 9]. For instance, Alao et al. [10] employed the Adomian decomposition and variational iteration methods to solve integro-differential equations. Behzadi at el. [2] focused on solving a specific class of nonlinear Volterra-Fredholm integro-differential equations. Moreover, numerous authors have investigated different methods for tackling integro-differential equations, [11, 12, 13]. A. Daraghmeh, N. Qatanani, and H. Jarar'a focused on three numerical methods to solve linear systems of Fredholm integral equations. These methods include the Legendre wavelet method and the Taylor series expansion method. [14]. On the other hand, Singh proposed a novel approach to develop a numerical solution to obtain the solution of linear Volterra-type integral equations. This approach involved obtaining asymptotic approximations to solutions [15]. Furthermore, Jalal and Malik investigated the infinite system of Hammerstein-type integral equations in two variables. Their study aimed to analyze the behavior and properties of these equations [16].
The Homotopy perturbation method (HPM) is a powerful technique that combines the ideas of the homotopy method from topology with the traditional perturbation approach. This coupling allows for the approximation or analytical solution of various problems that arise in different scientific fields.One of the key advantages of HPM is its ability to rapidly converge the solution series in many cases. This leads to efficient and effective solutions with a small number of iterations. Moreover, HPM is known for providing good approximations of solutions, which is beneficial in practical applications. The HPM approach has been extensively used to solve a wide range of problems, particularly those involving integral equations. These include various scientific issues spanning different fields, and specific examples can be found in references [17, 18, 19]. Moreover, HPM has been used to address integral equations in particular, as references [18, 20, 21, 22, 23, 24].
The aim of this work is to utilize both the homotopy perturbation method (HPM) and the veritional iteration method to find approximate solutions Eq.(1.1). The primary objective of this study is to determine a convergence condition for the solutions obtained through these methods. This condition will help ensure that the solutions converge to the true solution of Eq.(1.1). Additionally, the authors seek to estimate the error associated with the approximate solutions obtained using HPM and the veritional iteration method. This error estimation is crucial in assessing the accuracy and reliability of the obtained solutions. By providing a convergence condition and error estimation, the authors aim to establish the validity and effectiveness of the employed methods.
The present study is prepared as follows: In section 2, the Homotopy method with some lemmas are given. in section 3 and 4 the convergence theorem of the proposed methods and extension of nonlinear integro-differential equations are given. The error estimate of proposed methos are proved in section 5. The variational iteration method is explained in section 6. Finally, Some numerical experiments with conclusion are shown in section 7 and 8 respectively.

2 Homotopy Perturbation Method (HPM)

The non-linear operator equation is defined as follows:

$$\mathcal{A}(y) - f(x) = 0, x \in D, \tag{2.1}$$

subject to the boundary conditions

$$\mathcal{B}\left(y, \frac{\partial y}{\partial n}\right) = 0,\ x \in \Omega,$$

where x and y are independent and dependent variables vector, respectively. Let $f(x)$ is a known function, and Ω is the boundary of domain D, and $\mathcal{A}$ is a nonlinear differential operator formed by the sum of operators L and N, where L is a linear operator and N is a non-linear operator as well as $\mathcal{B}$ is a boundary operator.

We construct a homotopy, $z(x,p) : D \times [0,1] \longrightarrow \mathbb{R}$ according to HPM[25], which satisfies

$$\begin{aligned} H(z,p) &= (1-p)\left(L(z) - L(y_0)\right) \\ &+p\left(L(z) + N(z) - f(x)\right) = 0, \end{aligned} \tag{2.2}$$

where y_0 is an initial approximation verifying the boundary conditions, and $p \in [0,1], x \in D$.

The minor embedding parameter is symbolized by p. The solution of Eq.(2.2) expressed as a power series with the parameter p as the following

$$z(x) = z_0(x) + pz_1(x) + p^2 z_2(x) + \cdots . \tag{2.3}$$

Using $p = 1$, the solution for Eq.(2.1) expressed as

$$y(x) = \lim_{p \to 1} z(x) = \sum_{i=0}^{\infty} z_i(x). \tag{2.4}$$

The component of a series solution of Eq.(2.4) are obtained by resolving systems of equations formed by replacing Eq.(2.3) with Eq.(2.2) and comparing the identical powers of p. Let's represent $(j+1)^{th}$ terms of the approximate solution $\hat{y}_j(x)$ as

$$\hat{y}_j(x) = \sum_{m=0}^{j} z_m(x). \tag{2.5}$$

Lemma 2.1. *The convergence of Eq.(2.4) is a solution of Eq.(2.1).*

Let X and Y be a complied normed spaces (Banach space), then the operator $\mathcal{A} : X \to Y$ is a nonlinear contraction mapping if there exist a constant $\alpha \in [0,1)$, such that $\|\mathcal{A}(z) - \mathcal{A}(\hat{z})\| \leq \alpha\|z - \hat{z}\|$ for all $z, \hat{z}$ belong to Y; Furthermore, according to Banach's fixed point theorem, we have the fixed point y, in other words $\mathcal{A}(y) = y$.

If Considering the sequence created by the HPM as the form

$$\hat{y}_j = \mathcal{A}(\hat{y}_{j-1}),\ \hat{y}_{j-1} = \sum_{i=0}^{j-1} y_i,$$

where i is a posative integer number.
Suppose

$$\hat{y}_0 = z_0 = y_0 \in B_x(y), where \left\{\, B_x(y) = y^* \in X : \|y^* - y\| \leq x \right\},$$

then we have

(i) $\|\hat{y}_j - y\| \leq \alpha^j \|z_0 - y\|$,

(ii) $\hat{y}_j \in B_x(y)$,

(iii) $\lim_{j \longrightarrow \infty} \hat{y}_j = y$.

Proof. (i) Using the induction on j, for $j = 1$, we obtain

$$\|\hat{y}_1 - y\| = \|\mathcal{A}(\hat{y}_0) - \mathcal{A}(y)\| \leq \alpha\|z_0 - y\|$$

Let $\|\hat{y}_{j-1} - y\| \leq \alpha^{j-1}\|\hat{y}_0 - y\|$ be an induction hypothesis,then we obtain

$$\begin{aligned} \|\hat{y}_{j-1} - y\| &= \|\mathcal{A}(\hat{y}_{j-1}) - \mathcal{A}(y)\| \\ &\leq \alpha\|\hat{y}_0 - y\| \\ &\leq \alpha\alpha^{j-1}\|z_0 - y\| = \alpha^j\|z_0 - y\|. \end{aligned}$$

(ii) Applying (i), we have

$$\|\hat{y}_j - y\| \leq \alpha^j \|z_0 - y\| \leq \alpha^j x < x,$$

then $\hat{y}_j \in B_x(y)$.

(iii) Since we have $\|\hat{y} - y\| \leq \alpha^j \|z_0 - y\|$ $and \lim_{j\to\infty} \alpha^j = 0$, $then \lim_{j\to\infty} \|\hat{y} - y\| = 0$,

that is

$$\lim_{j\to\infty} \hat{y} = y.$$

□

Before attempting to demonstrate the method's convergence theorem for the nonlinear integral problem, we present a new formula to simplify the proof of the convergence theorem. Recalling Eq.(2.2), yields

$$H(z,p) = [L(z)] + p\,[N(z)] = f(x), \tag{2.6}$$

if $z = \sum_{j=0}^{\infty} p^n z_j, where\; y = z|_{p=1}$, then the homotopy in Eq.(2.6) expressed as

$$H(z,p) = L\left(\sum_{n=0}^{\infty} p^n z_n\right) + p\left[\sum_{n=0}^{\infty} p^n B_n(z_0,\cdots,z_n)\right] = f(x), \tag{2.7}$$

the decomposed polynomials of the nonlinear operator are denoted by $B_n, n = 0,1,2,\cdots$ and defined by

$$N(z) = \sum_{n=0}^{\infty} p^n B_n(z_0,\cdots,z_n). \tag{2.8}$$

Therefore, the decomposed polynomials B_j are given as

$$B_n(z_0,\cdots,z_n) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n} N\left(\sum_{j=0}^{n} p^n z_j\right)\right]_{p=0} \tag{2.9}$$

where, $n = 0,1,2,\cdots$.

3 Assumption

Let $N(z)$ be a non-linear operator and z is described as $z = \sum_{j=0}^{n} p^n z_j$, then

$$(i) \qquad \left[\frac{\partial^n}{\partial p^n} N(z)\right]_{p=0} = \left[\frac{\partial^n}{\partial p^n} N\left(\sum_{j=0}^{\infty} p^n z_j\right)\right]_{p=0} = \left[\frac{\partial^n}{\partial p^n} N\left(\sum_{j=0}^{n} p^n z_j\right)\right]_{p=0},$$

(ii)

$$N(z) = \sum_{n=0}^{\infty} p^n B_n(z_0,\cdots,z_n).$$

Recalling Eq.(2.7) and linear differential operator L along with applying boundary and initial conditions, imply that we have

$$\begin{aligned} z_0(x) + pz_1(x) + p^2z_2(x) + \cdots + L^{-1}\left[pB_0(z_0)\right] + \\ L^{-1}\left[p^2B_1(z_0,z_1)\right] + L^{-1}\left[p^3B_2(z_0,z_1,z_2)\right] + \cdots \\ = L^{-1}\left[f(\rho)\right]. \end{aligned} \quad (3.1)$$

The above components $z_0, z_1, z_2, \cdots$ computed recursively by implemting the coefficients with powers of p, gives

$$\begin{aligned} z_0 &= \psi_0 \\ z_n &= -L^{-1}\left[B_{n-1}(z_0, \cdots, z_{n-1})\right], n = 1, 2, \cdots. \end{aligned} \quad (3.2)$$

The zeroth term ψ_0 reflects the components produced via applying L^{-1} to the original function $f(x)$ under the provided condition. We may now described the $(n+1)$ component's truncated series solution $\hat{u_n}$ of Eq.(2.1) which is defined as

$$\hat{y}_j(x) = \sum_{m=0}^{j} p^m z_m|_{p=1} = \sum_{m=0}^{j} z_m \ . \quad (3.3)$$

The homotopy in Eq.(2.7) is identical to the recurrence relation in Eq.(3.2). Consider the non-linear Fredholm integral equation the second type is formed as

$$y(x) = f(x) + \lambda \int_a^b k(x,t)G\left(y(t)\right)dt. \quad (3.4)$$

The function $f(x)$ in interval $[a,b]$ is a bounded for all x, and the kernel $|k(x,t)| \leq \mu$, is abounded, since there exist a posative integer number β this yields $G(y(t))$ be a Lipschitz continous for

$$|G(y) - G(z)| \leq \beta|y - z|$$

and the operator $N(y)$ is expressed as a nonlinear function G(y(t)). in Eq.(2.1) In this situation, the identity operator is $L = L^{-1} = I$ with the function $G(y(t))$ according to the Eq.(2.9) may be written as

$$G(y(t)) = \sum_{n=0}^{\infty} B_n(z_0, \cdots, z_n), \quad (3.5)$$

where B_n is the sum of decomposed polynomials defined by the formula

$$B_n(z_0, \cdots, z_n) = \frac{1}{n!}\left[\frac{\partial^n}{\partial p^n} G\left(\sum_{j=0}^{n} p^n z_j\right)\right]_{p=0}, n = 0, 1, 2, \cdots. \quad (3.6)$$

From Eq.(3.3) and (3.5), we can write

$$G(\hat{y}_n) = \sum_{i=0}^{n} B_i(z_0, \cdots, z_i), \quad (3.7)$$

where $\hat{y}_n = \sum_{i=0}^{n} z_i(x)$ is the partial sum of the solutions.

The nonlinear Fredholm integral in Eq.(3.4) may be solved by applying the recursive relation in Eq.(3.2) with $L = L^{-1} = I$ and $f(x)$, and that is

$$y(x) = \sum_{i=0}^{\infty} z_i(x), \quad (3.8)$$

where

$$z_0 = f(x)$$
$$z_n = \int_a^b k(x,t)B_{n-1}(z_0,\cdots,z_{n-1})dt, n = 1,2,\cdots. \tag{3.9}$$

Theorem 3.1. *For the nonlinear Fredholm integral Eq.(3.4), the series solution of the Homotopy perturbation method which defined as*

$$y(x) = \sum_{i=0}^{\infty} z_i(x)$$

is converges if $\xi \in [0,1)$ *and* $\max_{a\le x\le b} |z_1| < \infty$, *where* $\xi = \mu\mu_0(\mu_1 - a)$, *where* μ, μ_0, μ_1 *and* a *are constants.*

Proof. To prove that the sequence $\{\hat{y}_n\}_{n=0}^{\infty}$ as partial sum is a Cauchy sequence in a Banach space $(C[a,b], \|.\|_\infty)$.

we suppose that $n \ge m$, and

$$\|\hat{y}_n - \hat{y}_m\|_\infty = \max_{a\le x\le b} |\hat{y}_n - \hat{y}_m| = \max_{a\le x\le b} \left| \sum_{i=m+1}^{n} z_i(x) \right|$$
$$= \max_{a\le x\le b} \left| \sum_{i=m+1}^{n} \int_a^b k(x,t)B_{i-1}dt \right|$$
$$= \max_{a\le x\le b} \left| \int_a^b k(x,t) \sum_{i=m}^{n-1} B_i dt \right|$$

Using Eq.(3.5), we obtain

$$\sum_{i=m}^{n-1} B_i = \left[G(\hat{y}_{n-1}) - G(\hat{y}_{m-1})\right].$$

As a result

$$\|\hat{y}_n - \hat{y}_m\|_\infty = \max_{a\le x\le b} |\hat{u}_n - \hat{u}_m|$$
$$= \left| \int_a^b k(x,t)\left[G(\hat{y}_{n-1}) - G(\hat{y}_{m-1})\right] dt \right|$$
$$\le \max_{a\le x\le b} \int_a^b |k(x,t)| \left|\left[G(\hat{y}_{n-1}) - G(\hat{y}_{m-1})\right]\right| dt$$
$$\le \xi \|\hat{y}_{n-1} - \hat{y}_{m-1}\|.$$

Therefore

$$\|\hat{y}_{m+1} - \hat{y}_m\|_\infty \le \xi \|\hat{y}_m - \hat{y}_{m-1}\|$$
$$\le \xi^2 \|\hat{y}_{m-1} - \hat{y}_{m-2}\|$$
$$\le \cdots$$
$$\le \xi^m \|\hat{y}_1 - \hat{y}_0\|$$

$$\begin{aligned}
\|\hat{y}_n - \hat{y}_m\|_\infty &\le \|\hat{y}_{m+1} - \hat{y}_m\| + \|\hat{y}_{m+2} - \hat{y}_{m+1}\| + \cdots + \|\hat{y}_n - \hat{y}_{n-1}\| \\
&\le \left[\xi^m + \xi^{m+1} + \xi^{m+2} + \cdots + \xi^{n-1}\right] \|\hat{y}_1 - \hat{y}_0\| \\
&= \xi^m \left[1 + \xi + \xi^2 + \cdots + \xi^{n-m-1}\right] \|z_1(x)\| \\
&= \xi^m \left(\frac{1-\xi^{n-m}}{1-\xi}\right) \|z_1(x)\|.
\end{aligned}$$

Therefore $(1-\xi^{n-m}) \le 1$, so we have

$$\|\hat{y}_1 - \hat{y}_0\|_\infty \le \frac{\xi^m}{1-\xi} \max_{a\le x\le b} |z_1(x)|, \tag{3.10}$$

where $0 \le \xi < 1$.

In spite of $|z_1(x)| = \left|\int_a^b k(x,t)G(z_0(t))dt\right| < \infty$ where $k(x,t)$ and $z_0(t) = f(t)$ are bounded, then $\|\hat{y}_n - \hat{y}_m\|_\infty$ tends to zero as m tends to infinity.

The series solution in Eq.(3.8) is a converges, because $\{\hat{y}_n\}_{n=0}^\infty$ in space of continuous functions $C([a,b])$ is Couchy sequence. □

4 Extension of integro-differential equations

The nonlinear Fredholm integro-differential equation is formulated as follows:

$$Ly''(x) = f(x) + \int_a^b k(x,t)G(y(t)dt,\ \mathcal{B}\left(y, \frac{\partial y}{\partial n}\right) = 0, \tag{4.1}$$

L is a linear differential operator.
We extended theorem 3.1 to show that the series solution in Eq.(3.8) obtained from the recursive formulas

$$\begin{aligned}
z_0 &= L^{-1}[f(x)] + \psi_0 \\
z_n &= L^{-1}\left[\int_a^b k(x,t)B_{i-1}(z_0,\cdots,z_{i-1})dt\right], i = 1,2\cdots,
\end{aligned} \tag{4.2}$$

converges to the exact solution $y(x)$ of Eq.(4.1) if it exists, where ψ_0 is the function arising from using the initial or boundary conditions.

5 The Error Estimation of the Integro Differential-Equations

In this section, we will discuss the following theorem demonstrates how to estimate the error of a nonlinear Fredholm integro-differential equation.

Theorem 5.1. *A maximum absolute error for the $m+1$ term's can be truncated the series solution*

$$y(x) = \sum_{i=0}^\infty z_i(x)$$

of the nonlinear Fredholm integro-differential equation can be estimated as

$$\max_{a\le x\le b}\left|y(x) - \sum_{i=0}^m z_i(x)\right| \le \frac{g\xi^m + 1}{\mu_0(1-\xi)},$$

and g is defined as $g = \max_{a\le x\le b} |G(y(x))|$.

Proof. By replacing Eq.(3.10) to the Theorem (3.1), we obtain $\|\hat{y}_n - \hat{y}_m\| \le \frac{\xi^m}{1-\xi} \max_{a\le x\le b} |z_1(x)|$, as a result if $\hat{y}_n$ $convergence\ to\ y(x),\ when\ n$ convergence to the ∞ , so

$$\max_{a\le x\le b} |z_1(x)| \le (L_0 - a)\mu \max_{a\le x\le b} |G(y_0(x))|,$$

then

$$\|y(x) - \hat{y}_m\| \le \frac{\xi^m + 1}{\mu_0(1-\xi)} \max_{a\le x\le b} |G(y(x))|.$$

As a result, the greatest absolute error of the truncated series solution of the $(m+1)$ component in $[a,b]$ can be approximated as

$$\max_{a\le x\le b} \left| y(x) - \sum_{i=0}^{m} z_i(x) \right| \le \frac{g\xi^m + 1}{\mu_0(1-\xi)}.$$

□

6 The Variational Iteration Method

This section aims to provide fast converging approximations of the exact solution for nonlinear Fredholm integro-differential equations. The correction functional for the equation is defined by using this technique [26]. Previous studies on numerical solutions of integral problems have utilized various methods such as the Several numerical solutions of integral problems have been studied by finite element method [27, 28] Adomian decomposition method, Taylor expansion method, Direct computation method, and Bernstein polynomials method. [29, 30]. In this work, the focus is on implementing a variational iteration technique specifically for solving nonlinear Fredholm integro-differential equations. The main idea behind this technique is to generate consecutive approximations of the exact solution that converge rapidly. By defining a suitable correction functional, the technique aims to iteratively improve the accuracy of the approximation. This enables a more efficient and accurate solution of the nonlinear Fredholm integro-differential equation.

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t) \left(y_n^{(i)}(t) - f(t) - \int_a^b k(t,r)G(y_n(r))dr \right) dt.$$

The Lagrange multiplier $\lambda_i(t)$ is calculated by

$$\lambda_i(t) = \frac{(-1)^n (t-x)^{n-1}}{(n-1)!},\ i = 1, 2, \ldots.$$

Using the Taylor series, we can represented the initial conditions as the following

$$y_0(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \ldots$$

7 Numerical Experiments

The section illustrates the performance of a proposed methods, through an implementation based on Matlab programming. The pointwise error is used to measure the error between the numerical and analytical solutions. We denote by E errors term described by

$$E_{VIM} = y(x) - Y_{VIM}.(x)$$

$$E_{HPM} = y(x) - Y_{HPM}.(x).$$

Let us introduce the three accuracy indicators when using space step size h

The pointwise error

$$\varepsilon_{VIM} = |E(x_i)|$$
$$\varepsilon_{HPM} = |E(x_i)|.$$

The l^∞ norm of the error

$$l^\infty(E_{VIM}, h) = \max_{0 \le i \le N} |E_{VIM}(x_i)|.$$

The l^2 norm of the error

$$l^2(E_{VIM}, h) = \sqrt{h \sum_{i=0}^{N} |E_{VIM}(x_i)|^2}.$$

The l^∞ norm of the error

$$l^\infty(E_{HPM}, h) = \max_{0 \le i \le N} |E_{HPM}(x_i)|.$$

The l^2 norm of the error

$$l^2(E_{HPM}, h) = \sqrt{h \sum_{i=0}^{N} |E_{HPM}(x_i)|^2}.$$

Example 7.1. Consider non linear Fredholm integro-differential equation

$$y''(x) = -cosx - \frac{\pi^2 x}{288} + \frac{1}{72}\int_0^\pi xty^2(t)dt, \tag{7.1}$$

with initial conditions $y(0) = 1, y'(0) = 0$, and exact solution $y(x) = cosx$.

The HPM of Eq.(7.1) is given by

$$H(z, p) = z''(x) + cosx + \frac{\pi^2 x}{288} - \frac{1}{72}\int_0^\pi xtz^2(t)dt = 0.$$

By substituting Eq.(3.8) into Eq.(7.1), and comparing the equivalent terms with regard to the powers of p, we obtain

$$\begin{aligned}
p^0 &: z_0''(x) = cosx,\\
p^1 &: z_1''(x) = -\frac{\pi^2 x}{288} + \frac{1}{72}\int_0^\pi xtz_0^2(t)dt,\\
p^2 &: z_2''(x) = \frac{1}{72}\int_0^\pi xt\left(2z_1(t)z_0(t)\right)dt,\\
p^3 &: z_3''(x) = \frac{1}{72}\int_0^\pi xt\left(2z_0(t)z_2(t) + z_1^2(t)\right)dt.
\end{aligned} \tag{7.2}$$

By simple computation of Eq.(7.2), we obtain

$$\begin{aligned}
p^0 &: z_0(x) = cosx,\\
p^1 &: z_1(x) = -\frac{\pi^2 x^2}{1728} + \frac{x^3}{432}\int_0^\pi tz_0^2(t)dt \Longrightarrow z_1(x) = 0,\\
p^2 &: z_2(x) = \frac{x^3}{432}\int_0^\pi t\left(2z_1(t)z_0(t)\right)dt \Longrightarrow z_2(x) = 0,\\
p^3 &: z_3(x) = \frac{x^3}{432}\int_0^\pi t\left(2z_0(t)z_2(t) + z_1^2(t)\right)dt \Longrightarrow z_3(x) = 0.
\end{aligned} \tag{7.3}$$

Using this approach repeatedly, we obtain $z_n(x) = 0, \forall n \geq 1$, then the approximate solution is derived by

$$y(x) = \sum_{i=0}^{\infty} z_i(x) = z_0(x) + z_1(x) + z_2(x) + ...$$

and

$$y(x) = cosx,$$

it is the exact solution.

The correction functional for Eq.(7.1) is used to solve Eq.(7.1) via variational iteration approach as the following

$$y_{n+1}(x) = y_n(x) + \int_0^x (t-x)\left(y_n''(t) + cos(t) + \frac{\pi^2 t}{288} - \frac{1}{72}\int_0^\pi try_n^2(r)dr\right) dt.$$

By setting the initial condition $y_0(x) = 1$, of the second kind Fredholm integro differential equations, replacing choosing into the correction functional yields

$$\begin{aligned} y_0(x) &= 1 \\ y_1(x) &= cosx + 0.00571157662\, x^3 \\ y_2(x) &= cosx - 0.00056929712\, x^3 \\ y_3(x) &= cosx + 0.00126764676\, x^3. \end{aligned}$$

Table 1. Comparison of approximate and exact solution for Example 7.1 .

x_i	$y(x)$	$Y_{VIM}(x)$	$Y_{HPM}(x)$	ε_{VIM}	l^∞_{VIM}	l^2_{VIM}
0	1	1	1	0		
0.1	0.995	0.9950054	0.995	1.27e-06		
0.2	0.98007	0.980076	0.98007	1.014e-05		
0.3	0.95534	0.955370	0.95534	3.423e-05		
0.4	0.92106	0.921142	0.92106	8.113e-05	$N = 20$	$N = 20$
0.5	0.87758	0.877741	0.87758	0.00015846	0.00126764	0.08359371
0.6	0.82534	0.825609	0.82534	0.00027381		
0.7	0.76484	0.765276	0.76484	0.0004348		
0.8	0.69671	0.697355	0.69671	0.00064904		
0.9	0.62161	0.622534	0.62161	0.00092411		
1	0.5403	0.541569	0.5403	0.00126765		

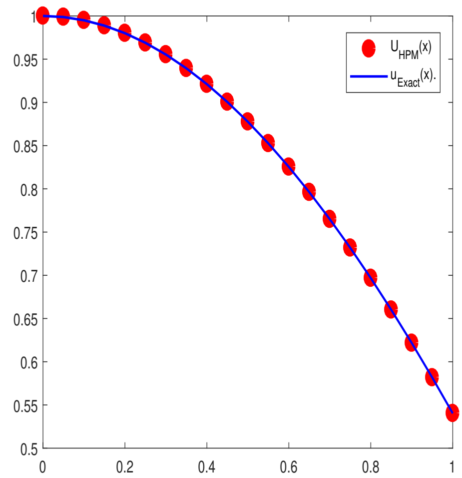

Figure 1. Comparative between the exact and approximate solution of Example 7.1 using HPM.

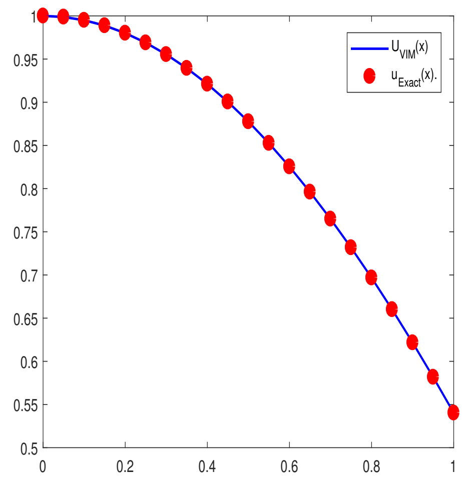

Figure 2. Comparative between the exact and approximate solution of Example 7.1 using VIM.

Example 7.2. Consider second order nonlinear Fredholm integro-differential equation

$$y'''(x) = sinx - x - \int_0^{\pi/2} xty'(t)dt, \tag{7.4}$$

with initial conditions $y(0) = 1, y'(0) = 0 \; and \; y'(0) = -1$, the analytical solution is $y(x) = cosx$.

The HPM of Eq.(7.4) is as the form

$$H(z,p) = z'''(x) - sinx - \left(-x - \int_0^{\pi/2} xtz'(t)dt\right) = 0. \tag{7.5}$$

By replacing the approximate solution provided by Eq.(3.8)) into Eq.(7.5), and comparing the appropriate power components of the embedding parameter p, the recurrence relations that result to the approximate solution are obtained.

$$\begin{aligned}
p^0 &: z_0'''(x) = sinx - x, \\
p^1 &: z_1'''(x) = -x \int_0^{\pi/2} tz_0'(t)dt, \\
p^2 &: z_2'''(x) = -x \int_0^{\pi/2} tz_1'(t)dt, \\
p^3 &: z_3'''(x) = -x \int_0^{\pi/2} tz_2'(t)dt, \\
p^4 &: z_4'''(x) = -x \int_0^{\pi/2} tz_3'(t)dt, \\
p^5 &: z_5'''(x) = -x \int_0^{\pi/2} tz_4'(t)dt.
\end{aligned} \tag{7.6}$$

Finally, an approximation solution may be found by

$$y(x) = \sum_{i=0}^{\infty} z_i(x) = z_0(x) + z_1(x) + z_2(x) + z_3(x) + z_4(x) + z_5(x) + ...,$$

and

$$\begin{aligned}
y(x) = cosx &- 0.0416666666666x^4 - 0.0194366217904x^4 \\
&+ 0.0061958217433x^4 - 0.001975042259x^4 \\
&+ 0.0006295852182x^4 - 0.0002006931979x^4 + \ldots,
\end{aligned}$$

which is the approximate solution.

Now we apply the variational iteration approach to solve Eq.(7.4), and we defined the correction functional of Eq.(7.4) as the following formula

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{-(t-x)^2}{2} \left(y_n'''(t) - sint + t + \int_0^{\frac{\pi}{2}} try_n'(r)dr \right) dt.$$

The initial condition of the Fredholm integro-differential equation is choosing as

$$y_0(x) = 1 - \frac{x^2}{2}$$

, and the consecutive approximations are obtained by putting it into the correction functional.

$$y_0(x) = 1 - \frac{x^2}{2},$$
$$y_1(x) = cosx + 0.012163674792\ x^4,$$
$$y_2(x) = cosx - 0.0038774207559\ x^4,$$
$$y_3(x) = cosx + 0.0012360073723\ x^4,$$
$$y_4(x) = cosx - 0.0003940026942\ x^4,$$
$$y_5(x) = cosx + 0.0001255964377\ x^4.$$

Table 2. Comparison of approximate and exact solution for Example 7.2.

x_i	$y(x)$	$Y_{VIM}(x)$	ε_{VIM}	$Y_{HPM}(x)$	ε_{HPM}	l^∞_{VIM}	l^2_{VIM}	l^∞_{HPM}	l^2_{HPM}
0	1	1	0	1	0				
0.1	0.995	0.995004	1e-08	0.995	0				
0.2	0.9801	0.980066	2e-07	0.98	0.0001				
0.3	0.9553	0.955337	1.02e-06	0.9552	0.0001				
0.4	0.9211	0.921064	3.22e-06	0.9206	0.0005	$N = 20$	$N = 20$	$N = 20$	$N = 20$
0.5	0.8776	0.877590	7.85e-06	0.8764	0.0012	1.25596437e-04	0.02381743	0.01840	0.28844410
0.6	0.8253	0.825351	1.628e-05	0.8229	0.0024				
0.7	0.7648	0.764872	3.016e-05	0.7604	0.0044				
0.8	0.6967	0.696758	5.144e-05	0.6892	0.0075				
0.9	0.6216	0.621692	8.24e-05	0.6095	0.0121				
1	0.5403	0.540427	0.0001256	0.5219	0.0184				

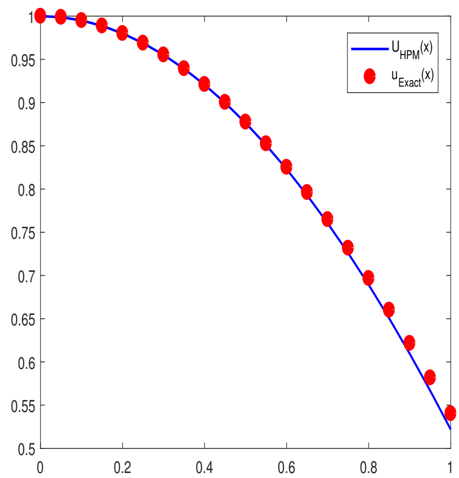

Figure 3. Comparative between the exact and approximate solution of Example 7.2 using HPM.

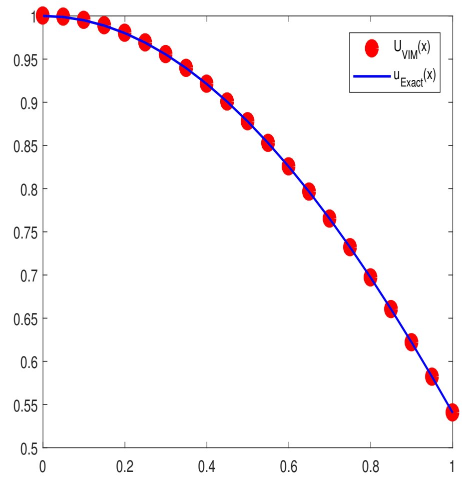

Figure 4. Comparative between the exact and approximate solution of Example 7.2 using VIM.

The proposed methods for finding approximate solutions, namely HPM and VIM, have been compared in Tables 1 and 2. The pointwise error norms for both methods are reported in these tables. The results demonstrate that both methods are efficient and provide solutions that are comparable to analytical solutions. To further evaluate the performance of these methods, the physical behavior of the exact and approximate solutions at different space levels is depicted graphically in Figures 1, 2, 3, and 4. These figures illustrate that the numerical solutions obtained from both HPM and VIM are in good agreement with the exact solutions. However, it is observed that the Homotopy perturbation method (HPM) outperforms the variational iteration method (VIM). This is attributed to the fact that the errors produced by the HPM are much closer to zero. The obtained numerical solutions from HPM also indicate that this method is reliable and yields results that are compatible with the analytical solutions. In summary, the comparative analysis of HPM and VIM in Tables 1 and 2, as well as the graphical illustrations in Figures 1, 2, 3, and 4, demonstrate the efficiency and accuracy of these methods for approximating solutions. The HPM, in particular, stands out as a faster and more reliable method in this study.

8 Conclusion

In this paper, we proposed an efficient numerical scheme for solving nonlinear Fredholm integro-differential equations (NFID) using the homotopy perturbation and variational iteration methods. The proposed method incorporates the contraction mapping technique to handle convergence analysis.The numerical scheme devised in this study proves to be highly effective in solving NFID problems quickly and accurately. Through the calculation of error norms such as l^∞ and l^2 for different spatial levels, the authors demonstrate the reliability and efficiency of their proposed method. Numerical experiments conducted in this study further validate the proposed method, showing that the results obtained are fruitful, powerful, and dependable. Moreover, the performance of the method aligns well with previous results reported in the literature. Overall, we assert that their proposed method represents a significant advancement in the field of solving NFID problems. By offering an efficient numerical approach capable of achieving high levels of accuracy. In our future research, we plan to explore the continuous and discontinuous Galerkin methods in both spatial and temporal dimensions [31, 32, 33, 34, 35, 36] for solving the NFID problem, with the goal of achieving higher levels of accuracy in our solutions. Additionally, we intend to broaden the application of the compact finite difference method to various challenging real-world problems including, integro-differential problems [37, 38, 39, 40].

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