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PERFECT CUBE LABELING OF SOME GRAPHS

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Abstract In this paper,I define a new labeling and a new graph called the perfect cube labeling and the perfect cube graph. Let G be a (p, q) graph.A perfect cube labeling of G is a bijection $f : V(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ such that the induced function $g : E(G) \rightarrow N$ given by $g(uv) = (f(u)+f(v))^3$ for every $uv \in E(G)$ are all distinct. A graph which admits perfect cube labeling is called perfect cube graph. In this paper, I discussed that the perfect cube labeling is admitted for some graphs like paths, cycles, star graphs, bi-stars, ladders,coconut trees, and combs.

1 Introduction

A graph labeling is an assignment of labels (integers) to the vertices and/or edges of a graph. Most graph labelings trace their origins to labelings presented by Alexander Rosa in 1967.. A dynamic survey on graph labeling is regularly updated by Joseph A Gallian [5] and it is published by The Electronic Journal of Combinatorics.Labeled graphs have applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal auto correlation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts [6]. A detailed study of applications of graph labeling is given by Bloom and Golomb [2]. Motivated by some of the works [11], [12], and [13] in labeling, I introduced a new labeling technique called Perfect Cube Labeling.The graphs discussed here are finite, connected, un directed and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Also, I will give a brief summary of the definitions that are useful for the present work. In this present work, I aimed to discuss one such labeling known as perfect cube labeling.

Definition 1.1. A walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. However a vertex may appear more than once. Vertices with which walk begins and ends is called its terminal vertices. A walk that begins and ends at the same vertex is called a closed walk. A walk that is not closed is called an open walk. An open walk in which no vertex appears more than once is called a path. A path on n vertices is denoted by P_n.

Definition 1.2. A closed walk in which no vertex appears more than once is called a cycle. A cycle on n vertices is denoted by C_n.

Definition 1.3. A graph is Bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a bi partition of the graph, and X and Y are its parts. If the bipartite graph is simple and every vertex in X is joined to every vertex in Y, then it is called a Complete Bipartite Graph. A star is a complete bipartite graph with $|X| = 1$ or $|Y| = 1$.

Definition 1.4. The n - bi star $B_{n,n}$ obtained from two disjoint copies of $K_{1,n}$ by joining the center vertices by an edge.

Definition 1.5. Ladder graph L_n is a planar undirected graph with $2n$ vertices and $n + 2(n-1)$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_n = P_n X P_2$

Definition 1.6. A Coconut tree CT(m, n) is the graph obtained from the path P_m by appending n new pendant edges at an end vertex of P_m.

Definition 1.7. Comb is a graph obtained by joining a single pendant edge to each vertex of a path.

2 Main Results

Definition 2.1. Let $G = (V(G), E(G))$ be a graph.A perfect cube labeling of G is a bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the induced function $g : E(G) \to N$ given by $g(uv) = (f(u) + f(v))^3$ for every $uv \in E(G)$ are all distinct.

Definition 2.2. Any graph which admits perfect cube labeling is called perfect cube graph.

Theorem 2.3. *The paths P_n are perfect cube graphs.*

Proof. Consider P_n with n vertices Let $u_1, u_2, \ldots . u_n$ be the vertices of the path P_n.Note that P_n has n vertices and $n-1$ edges. The vertex set is $V(P_n) = \{u_i, 1 \le i \le n\}$ and the edge set is $E(P_n) = \{u_i u_{i+1}, 1 \le i \le n-1\}$. Hence the order of the graph is $|V(P_n)| = n$ and the size is $|E(P_n)| = n-1$. Vertex labeling is given by $f(u_i) = i \quad$ for $i = 0, 1, 2, 3 \ldots . n-1$. And the induced function $g : E(G) \to N$ defined by by $g(uv) = (f(u) + f(v))^3 \quad$ for every $uv \in E(G)$ ie. $.g(u_i, u_{i+1}) = 8i^3 + 12i^2 + 6i + 1 \quad i = 0, 1, 2, 3 \ldots n-2$

Thus the vertex labels are $0, 1, 2, 3 \ldots\ldots n-1$ and the edge labels are $1, 27, 125, ---(2n-3)^3$. Since the elements of the edge set are in the increasing order, the path P_n admits perfect cube labeling. Hence all paths P_n are perfect cube graph. □

Example 2.4. Perfect cube labeling of P_5

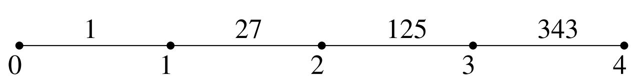

Fig 2.1. perfect cube labeling of P_5

Theorem 2.5. *All Cycles are perfect cube graph*

Proof. Let $u_1, u_2, \ldots . u_n$ be the vertices of the cycle C_n.C_n has n vertices and n edges.The vertex set is $V(C_n) = \{u_i, 1 \le i \le n\}$ and the edge set is $E(C_n) = \{u_i u_{i+1}, 1 \le i \le n-1\} \cup \{u_n u_1\}$.Hence the order of the graph is $|V(C_n)| = n$ and the size is $|E(C_n)| = n$.

Define $f : V(C_n) \to \{0, 1, 2, \ldots n-1\}$ as follows

$$f(u_i) = \begin{cases} 2i-2 & 1 \le i \le \lceil \frac{n}{2} \rceil \\ 2n-2i+1 & \lceil \frac{n}{2} \rceil + 1 \le i \le n \end{cases}$$

And the induced function $g : E(G) \to N$ defined by

Case (i) When n is even

$$g\left(u_i u_{i+1}\right) \text{ by } 2^3, 6^3, \ldots\ldots (2n-6)^3 \quad \text{for} \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1, g\left(u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1}\right) = (2n-3)^3$$

, $g\left(u_n u_1\right) = 1$. Also we can observe that $g\left(u_i u_{i+1}\right)$ are decreasing distinct cubes for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. Hence all edge labels are distinct cubes. Therefore all even cycles are perfect cube graph.

Case (ii) When n is odd
$g\left(u_i u_{i+1}\right)$ by $2^3, 6^3, \ldots\ldots (2n-4)^3$ for $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1, g\left(u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1}\right) = (2n-3)^3$, $g\left(u_n u_1\right) = 1$.Also we can observe that $g\left(u_i u_{i+1}\right)$ are decreasing distinct cubes for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n$. Hence all edge labels are distinct cubes. Therefore all odd cycles are perfect cube graph. From Case (i) and Case (ii) cycles are perfect cube graph for $n \geq 3$ □

Example 2.6. Perfect cube labeling of C_6

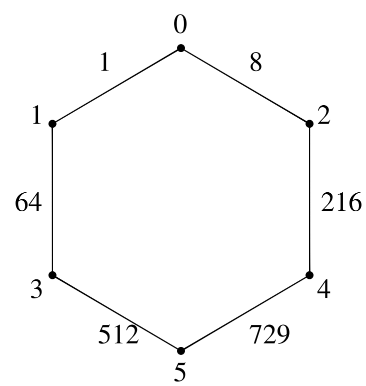

Fig 2.2. perfect cube labeling of C_6

Theorem 2.7. *The Stars $K_{1,n}$ are perfect cube graph*

Proof. Consider $S_n = K_{1,n}$ with $n+1$ vertices. Let u_0 be the apex vertex of the star $K_{1,n}$ and let $u_1, u_2, \ldots . u_n$ be the pendent vertices of the star $K_{1,n} \cdot K_{1,n}$ has $n+1$ vertices and n edges. The vertex set is $V\left(K_{1,n}\right) = \{u_0, u_i, 1 \leq i \leq n\}$ and the edge set is $E\left(K_{1,n}\right) = \{u_0 u_i, 1 \leq i \leq n\}$ Hence the order of the graph is $\left|V\left(K_{1,n}\right)\right| = n+1$ and the size is $\left|E\left(K_{1,n}\right)\right| = n$ Define $f\left(u_0\right) = 0$ and

$$f\left(u_i\right) = i \quad \text{for } i = 1, 2, 3 \ldots n$$

And the induced function $g : E(G) \rightarrow N$ defined by

$$g\left(u_0, u_i\right) = i^3 \quad i = 1, 2, 3 \ldots n$$

Thus the vertex labels are $0, 1, 2, 3 \ldots\ldots n$ and the edge labels are $1, 8, 27, 64, 125, \ldots n^3$ Since the edge labels are increasing consecutive cubes, the star $K_{1,n}$ admits perfect cube labeling. Hence all star $K_{1,n}$ are perfect cube graph. □

Example 2.8. Perfect cube labeling of $K_{1,6}$

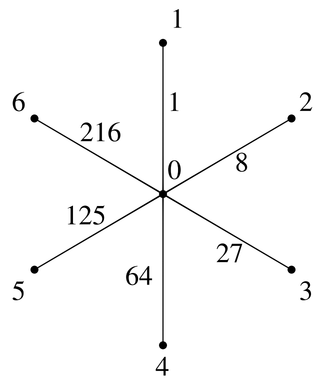

Fig 2.3. Perfect cubee labeling of $K_{1,6}$

Theorem 2.9. *The Bi stars $B_{n,n}$ are perfect cube graph*

Proof. Let u and v be the apex vertices of the bistar and let $u_1, u_2, \ldots .u_n, v_1, v_2, \ldots \ldots v_n$, be the pendent vertices of the bistar $B_{n,n}$. The vertex set is $V(B_{n,n}) = \{u, u_i, 1 \le i \le n\} \cup \{v, v_i, 1 \le i \le n\}$ and the edge set is $\mathrm{E}(B_{n,n}) = \{uu_i, 1 \le i \le n\} \cup \{vv_i, 1 \le i \le n\} \cup \{uv\}$ Hence the order of the graph is $|V(B_{n,n})| = 2n + 2$ and the size is $|E(B_{n,n})| = 2n + 1$. Define $f(u) = 0, f(v) = n + 1$

$$f(u_i) = i \qquad \text{for } i = 1, 2, 3 \ldots n$$
$$f(v_i) = n + 1 + i \qquad \text{for } i = 1, 2, 3, \ldots n$$

ie.. we label the vertex u by $0, v$ by $n + 1, u_1, u_2, \ldots .u_n$ by $1, 2, \ldots n$ and $v_1, v_2, \ldots .v_n$ by $n+ 2, n + 3, \ldots 2n + 1$ And the induced function $g : E(G) \to N$ defined by $g(u, u_i) = i^3$ for $i = 1, 2, 3 \ldots n$ $g(u, v) = (n+1)^3$ $g(v, v_i) = (2n+2+i)^3$ for $i = 1, 2, 3, \ldots n$ are all distinct.

Thus edge labels in the first pendant edge set are $1, 8, 27, 64, 125, ---(n)^3$ and in the second pendant edge set are $(2n + 3)^3, (2n + 4)^3, \ldots (3n + 2)^3$ and the middle edge label is $(n + 1)^3$. Since all these edge labels are distinct all bi stars $B_{n,n}$ are perfect cube graph. □

Example 2.10. Perfect cube labeling of $B_{5,5}$

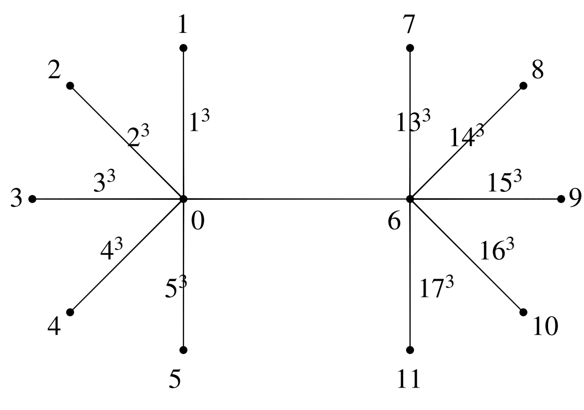

Fig 2.4. perfect cube labeling of a bistar $(B_{5,5})$

Theorem 2.11. *The ladders L_n admits perfect cube labeling*

Proof. Consider ladder graph L_n Let $u_1, u_2, \ldots .u_n$ are the vertices of the one side of the ladder and $v_1, v_2, \ldots .v_n$ be the vertices of the other side of the ladder. Note that L_n has $2n$ vertices and $3n - 2$ edges. The vertex set is $V(L_n) = \{u_i, v_i, 1 \le i \le n\}$ and the edge set is $E(L_n) = \{u_i u_{i+1}, 1 \le i \le n - 1\} \cup \{v_i v_{i+1}, 1 \le i \le n - 1\} \cup \{u_i v_i, 1 \le i \le n\}$

Label the vertices $u_1, u_2, \ldots .u_n$ as $0, 2, 4, \ldots .2n-2$ and $v_1, v_2, \ldots .v_n$ as $1, 3, 5, \ldots .2n-1$. So the first edge set consists of $2^3, 6^3, 10^3, \ldots (4n-6)^3$, the second set consists of $4^3, 8^3, 12^3, \ldots (4n-$

$4)^3$ and the third set has $1^3, 5^3, 9^3, \ldots (4n-3)^3$. All the elements in the edge sets are in the increasing order and distinct. Hence the ladders L_n are perfect cube graphs. □

Example 2.12. Perfect cube labeling of L_5

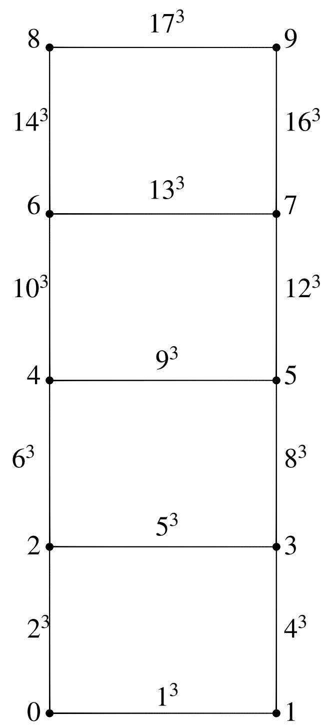

Fig 2.5. Perfect cube labeling of L_5

Theorem 2.13. *The Coconut tree admits perfect cube labeling.*

Proof. Let $u_0, u_1, u_2, \ldots .u_{m-1}$ be the vertices of the path P_m and $v_1, v_2, \ldots .v_n$ be the pendent vertices being adjacent with u_0.
The vertex set is $V(CT(m,n)) = \{u_i, v_j, 0 \le i \le m, 1 \le j \le n\}$ and the edge set is$E(CT(m,n)) = \{u_i u_{i+1}, 0 \le i \le m-2\} \cup \{u_0 v_j, 1 \le j \le n\}$ Label the vertex $u_0, u_1, u_2, \ldots .u_{m-1}$ by $0, 2, 4, 6, \ldots 2(m-1)$ and Label the pendent vertices $v_1, v_2, \ldots .v_n$ by $1, 3, 5, \ldots .2n-1$ Now the edge labels of the first edge set are 2^3, 10^3, 16^3, $..(4m-6)^3$ and the edge weights of the second set are $1^3, 3^3, 5^3, \ldots (2n-1)^3$. These two sets are disjoint sets. Hence the coconut tree admits the perfect cube labeling. □

Example 2.14. Perfect cube labeling of coconut tree $CT(5,6)$

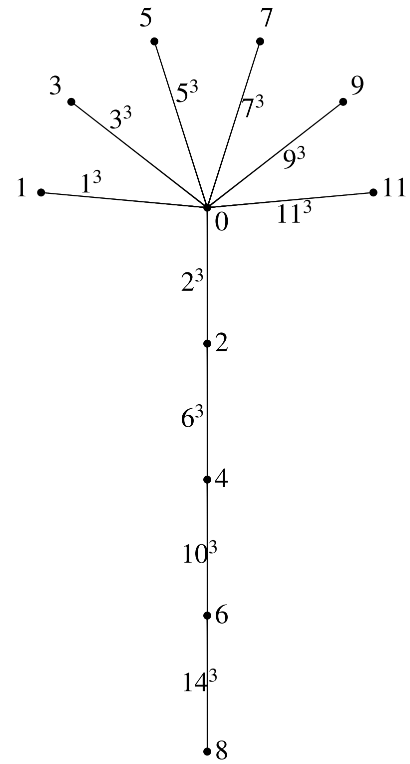

Fig 2.6. Perfect Cube labeling of a coconut tree $CT(5, 6)$

Theorem 2.15. *The* Comb (P_n, K_1) *admits perfect cube labeling.*

Proof. Let $u_1, u_2, \dots .u_n$ be the vertices of the path P_n. Also let $v_1, v_2, \dots .v_n$ be the pendent vertices of the comb graph. (P_n, K_1) has $2n$ vertices and $2n - 1$ edges. The vertex set is $V((P_n, K_1)) = \{u_i, v_i, 1 \le i \le n\}$ and the edge set is $E((P_n, K_1)) = \{u_i u_{i+1}, 1 \le i \le n-1\} \cup \{u_i v_i, 1 \le i \le n\}$ Hence the order of the graph is $|V((P_n, K_1))| = 2n$ and the size is $|E((P_n, K_1))| = 2n - 1$. Define

$$f(u_i) = 2i - 2 \quad \text{for } i = 1, 2, 3 \dots n$$
$$f(v_i) = 2i - 1 \quad \text{for } i = 1, 2, 3, \dots n$$

ie.. we label the vertex $u_1, u_2, \dots .u_n$ by $0, 2, 4, \dots 2n{-}2$ and $v_1, v_2, \dots \dots v_n$ by $1, 3, 5, \dots .2n{-}1$ Now the edge labels of the first edge set are $2^3, 6^3, 10^3, \dots (4n-6)^3$ and the edge labels s of the second edge set are $1^3, 5^3, 9^3, \dots (4n-3)^3$. These two sets are disjoint sets. Hence the comb admits the perfect cube labeling. □

Example 2.16. Perfect cube labeling of comb $P_4 \odot K_1$

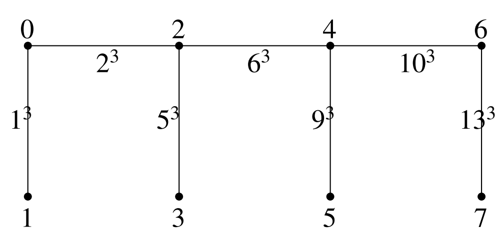

Fig 2.7. Perfect cube labeling of $P_4 \odot K_1$

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