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Characteristics of Soft Fractional Ideals and its R-multiples of Integral Domains

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Abstract In this paper, valuation rings in connection with soft modules of the quotient field are instigated. We prove the equivalence of undeniable soft modules of the quotient field, called soft fractional ideals. The main motive of this paper is to bring out the properties of multiples of soft modules, soft submodules of the quotient field and soft fractional ideals of an integral domain notably that of a Dedekind domain.

1 Introduction

Molodtsov [1] put forward absolutely a new theory, called the soft set theory to overcome the unpredictable problems in social and economic sectors, environmentology, engineering and technology fields, computational fields, medical fields, etc., that can't be handled by using classical techniques. Jun [2] considered soft bck/bci algebras. Jun and Park [3] instigated various paths in connection with soft sets applications in the ideal theory of bck/bci algebraic structures. In addition to, several linked concepts with applications of soft sets, have undergone tremendous studies. Maji et al. [4] instigated a conceptual research of soft sets in depth that includes superset and subset of a soft set, operations of union, operations of intersection, and null soft set etc. Sezgin and Atagün [5] and Ali et al. [6] explored a few procedures on the soft set theory as well. Onyeozili and Gwary [7] carried out a critical and systematic research on matrix representation of soft sets, relations and functions of soft sets, and operational properties of soft sets. Aktas and Çağman [8] compare the soft set concepts and its properties to the associated rough set and fuzzy set concepts, then transferred this concept to groups and defined the groups as soft groups with some properties of the same.

Acar et al. [9] instigated and explained initial concepts of soft rings. Sun et al. [10] instigated the soft module concepts and pointed out several properties of soft modules by utilising Molodtsov's concept of soft set theory and modules. Many authors have also discussed this concept. Türkmen and Pancar [11] developed some soft submodule properties over a module. They have also defined the soft module radicals and its properties, apart from this they proposed the concepts of direct sums of these soft submodules, soft modules and illustrated these concepts with examples. Atagün and Sezgin [12] carried out the soft substructures of modules, fields and rings algebrically. They also proposed the concepts of soft subfield, soft submodule of R-module. They have also instigated the notions of soft subrings and soft ideals and illustrated these concepts with examples. Furthermore, they developed the soft submodules, sum operation of a module, soft ideals and also established the product operations and restricted intersection of these soft substructures. Taouti and Khan [13] proposed soft fields, soft integral domains and also introduced fractions of soft rings. Taouti et al. [14] investigated the idealistic soft rings and defined the soft fractional ideal of soft rings. The authors have also studied fractional ideal with a few fundamental soft operations. This fractional ideal concept has its unique significance while studying domains, valuation, and Dedekind domains etc.

Furthermore, the pioneering work of Zadeh [15] for the creation of a fuzzy set notion gives an acceptable starting point in his fuzzy set theory, the most applicable concepts to deal with unpredictable problems. Jin [16] put the idea of a fuzzy ideal and proposed the definitions of the operations of L-fuzzy ideals in a ring. They have also proved some fundamental properties of the operations on fuzzy ideals. The revolutionary work of L.A. Zadeh led the way of fuzzifying algebraic structures. Since then, there has been an additional expansion of this work by Mordeson and Malik [17]. Lee and Mordeson [18] bring out the concept of fractionary fuzzy ideals and by making use of these concepts, they signalized the invertibility of undeniable fractionary fuzzy ideals in connection with Dedekind domains. For an overview of some results on integral domain, we refer [20].

2 Preliminary

Given a universal set Z, a pair (β, C) over Z is called a soft set, where β is a mapping to the power set $P(Z)$ from a set of parameters (C).

$$\text{i.e., } \beta : C \longrightarrow P(Z).$$

Definition 2.1 ([4]). For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), if it fulfils :

(i) $A \subset B$

(ii) $\forall\, l \in A$, $F(l)$ and $G(l)$ are identical approximations.

We write $(F, A) \tilde{\subset} (G, B)$.

Definition 2.2. For a soft set β over a field N and $\beta_\alpha = \{l \in N : \beta(l) \supseteq \alpha\}$ is called the α - inclusion of β corresponding to α, for every $\alpha \in P(N)$.

Definition 2.3 ([14]). Let R be a ring contained in a field K and (β, K) be a soft subset over the field K. Then, β is said to be soft R-submodule of K if it fulfils :

(i) $\beta(e - f) \supseteq \beta(e) \cap \beta(f), \forall\, e, f \in K$

(ii) $\beta(re) \supseteq \beta(e), \forall\, e \in K, r \in R$

(iii) $\beta(0) = R$

Example 2.4. Let $R = (Z_{10}, +, .)$, $N = (Z_{10}, +)$ be a natural-operating left R-module and a submodule $\beta_1 = \{0, 5\}$ of N. Consider the soft set (β, β_1) over N, where $\beta : \beta_1 \longrightarrow P(N)$ is a mapping with set values that is outlined by $\beta(0) = \{0, 3, 4, 9\}$ and $\beta(5) = \{0, 9\}$. Then, clearly (β, β_1) is a soft R-submodule of N.

The set is represented by $\beta_* = \{e \in N : \beta(e) = \beta(0)\}$. For soft sets δ and ω of Z, then the soft sets $\delta \,\tilde{\cup}\, \omega$ and $\delta \,\tilde{\cap}\, \omega$ are outlined as below :
$(\delta \,\tilde{\cup}\, \omega)(e) = \delta(e) \cup \omega(e)$ and
$(\delta \,\tilde{\cap}\, \omega)(e) = \delta(e) \cap \omega(e), \forall\, e \in Z.$
Here the soft union and soft intersection operators are denoted by $\tilde{\cup}$ and $\tilde{\cap}$ respectively.

Definition 2.5. An integral domain R is a valuation ring if for any two ideals E and F of R, we state either $E \subseteq F$ or $F \subseteq E$. That is, if ' $\subseteq$ ' is a total order on the ideal sets of R.

Proposition 2.6 ([19]). *For an integral domain R, the following are equivalent.*

(i) An integral domain R is a valuation ring.

(ii) If $e, f \in R$, then either $(e) \subseteq (f)$ or $(f) \subseteq (e)$

(iii) If K is the fraction field of R, then for every $f(\neq 0) \in K$, either $f \in R$ or $f^{-1} \in R$

3 Soft modules and its R-multiple properties

Definition 3.1. Let β be a soft R-submodule of an R-module K. If $r \in R$, then the function $r\beta : K \longrightarrow P(K)$ outlined by $(r\beta)(x) = \beta(rx)$, $\forall\, x \in K$ is called the R-multiple of β.

Proposition 3.2. *Consider an R-module K and β, γ are two soft R-submodules of K. Then*

(i) $r\beta$ *is a soft R-submodule, for every* $r \in R$

(ii) If $r \in R$*, then* $\beta \mathrel{\widetilde{\subseteq}} \gamma \iff r\beta \subseteq r\gamma$

(iii) If $r, s \in R$*, then* $r(s\beta) = (rs)\beta$*. In particular* $r(r\beta) = r^2\beta$

(iv) If $r \in R$*, then* $(r\beta) \mathbin{\widetilde{\cap}} (r\gamma) = r(\beta \mathbin{\widetilde{\cap}} \gamma)$ *and* $r(\widetilde{\bigcap_i}\beta_i) = \widetilde{\bigcap_i}(r\beta_i)$

Proof. (i) Let $r \in R$ and $x, y \in K$. Since β is a soft R-submodule of K, using the definition of $r\beta$,

$$\begin{aligned}(r\beta)(x-y) &= \beta(r(x-y)) \\ &= \beta(rx - ry) \\ &\supseteq \beta(rx) \cap \beta(ry) \\ &= (r\beta)(x) \cap (r\beta)(y)\end{aligned}$$

$$(r\beta)(sx) = \beta(rsx) \supseteq \beta(rx) = (r\beta)(x)$$
$$(r\beta)(0) = \beta(0) = R$$

Therefore $r\beta$ is a soft R-submodule of K, for all $r \in R$.

(ii) Suppose $\beta \mathrel{\widetilde{\subseteq}} \gamma$. Let $r \in R$, then for $x \in K$,
$(r\beta)(x) = \beta(rx) \subseteq \gamma(rx) = (r\gamma)(x)$. Therefore, $r\beta \subseteq r\gamma$.

(iii) Let $r, s \in R$, then $(r(s\beta))(x) = (s\beta)(rx) = \beta(srx) = \beta(rsx) = ((rs)\beta)(x)$.
Therefore $r(s\beta) = (rs)\beta$. In particular, $r(r\beta) = r^2\beta$.

(iv) Let $r \in R$, then

$$\begin{aligned}((r\beta)\widetilde{\cap}(r\gamma))(x) &= (r\beta)(x) \cap (r\gamma)(x) \\ &= \beta(rx) \cap \gamma(rx) \\ &= (\beta \cap \gamma)(rx) \\ &= (r(\beta \cap \gamma))(x)\end{aligned}$$

Therefore $(r\beta) \mathbin{\widetilde{\cap}} (r\gamma) = r(\beta \cap \gamma)$. In general, $\widetilde{\bigcap_i} r\beta_i = r(\widetilde{\bigcap_i}\beta_i)$. □

4 Soft submodules and its R-multiples of the fraction field and valuation rings

Proposition 4.1. *Consider K be the fraction field of an integral domain R. If β is any soft R-submodule of K and $r \in R$. Then $r(r\beta)_\alpha = \beta_\alpha$, for every $\alpha \in P(K)$.*

Proof. We already know that $x \in (r\beta)_\alpha \iff (r\beta)(x) \supseteq \alpha \iff \beta(rx) \supseteq \alpha \iff rx \in \beta_\alpha$. Thus $r(r\beta)_\alpha \mathrel{\widetilde{\subseteq}} \beta_\alpha$. Also, if $x \in \beta_\alpha$, then $x/r \in (r\beta)_\alpha$. Therefore $x \in r(r\beta)_\alpha$. Thus $\beta_\alpha \mathrel{\widetilde{\subseteq}} r(r\beta)_\alpha$. Hence, $r(r\beta)_\alpha = \beta_\alpha$, for every $\alpha \in P(K)$. □

Proposition 4.2. *Consider K be the fraction field of an integral domain R. If any soft R-submodule of K is β, then for $r_1, r_2 \in R$, $r_1\beta \mathrel{\widetilde{\subseteq}} r_2\beta \Longrightarrow r_2\beta_\alpha \mathrel{\widetilde{\subseteq}} r_1\beta_\alpha$, for every $\alpha \in P(K)$.*

Proof. By Proposition 4.1, $r_1(r_1\beta)_\alpha = \beta_\alpha$ and $r_2(r_2\beta)_\alpha = \beta_\alpha$. Therefore $r_2r_1(r_1\beta)_\alpha = r_2\beta_\alpha$ and $r_1r_2(r_2\beta)_\alpha = r_1\beta_\alpha$. i.e., $(r_1r_2)(r_1\beta)_\alpha = r_2\beta_\alpha$ and $(r_1r_2)(r_2\beta)_\alpha = r_1\beta_\alpha$. Now suppose $r_1\beta \mathrel{\widetilde{\subseteq}} r_2\beta$, then $(r_1\beta)_\alpha \mathrel{\widetilde{\subseteq}} (r_2\beta)_\alpha$, for every $\alpha \in P(K)$. It follows that $(r_1r_2)(r_1\beta)_\alpha \mathrel{\widetilde{\subseteq}} (r_1r_2)(r_2\beta)_\alpha$. Hence, $r_2\beta_\alpha \subseteq r_1\beta_\alpha$, for every $\alpha \in P(K)$. □

Proposition 4.3. *Consider K be the fraction field of an integral domain R. In case of any soft R-submodule β of K, there is linear order to the set $\{r\beta : r \in R\}$ of soft submodules iff R is a valuation ring.*

Proof. Consider a valuation ring R and the fraction field K of R. Let β be a soft R-submodule of K. Let $r_1, r_2 \in R$. Since for being a valuation ring R, either $(r_1) \widetilde{\subseteq} (r_2)$ or $(r_2) \widetilde{\subseteq} (r_1)$. Suppose $(r_1) \widetilde{\subseteq} (r_2)$, then $r_1 = rr_2$, for some $r \in R$. Now $(r_1\beta)(x) = \beta(r_1x) = \beta(rr_2x) \supseteq \beta(r_2x) = (r_2\beta)(x), \forall\, x \in K$. Therefore $r_2\beta \widetilde{\subseteq} r_1\beta$. But on the other side, if $(r_2) \widetilde{\subseteq} (r_1)$, then $r_1\beta \widetilde{\subseteq} r_2\beta$. As a result, the submodules set $\{r\beta : r \in R\}$ is linearly ordered.
Conversely, assume that given any soft R-submodule β of the field of quotients K, the set of soft submodules $\{r\beta : r \in R\}$ is linearly ordered. For proving that a valuation ring is R. Suppose $r_1, r_2 \in R$, by supposition we mean, either $r_1\beta \widetilde{\subseteq} r_2\beta$ or $r_2\beta \widetilde{\subseteq} r_1\beta$. If $r_1\beta \widetilde{\subseteq} r_2\beta$, then clearly by Proposition 4.2, $r_2\beta_\alpha \widetilde{\subseteq} r_1\beta_\alpha$, for every $\alpha \in P(K)$. If β is chosen to be the characteristic function of R, then we state $\beta_* = \beta_\alpha = R$. It follows that $r_2R \widetilde{\subseteq} r_1R$. i.e., $(r_2) \widetilde{\subseteq} (r_1)$. If $r_2\beta \widetilde{\subseteq} r_1\beta$, then in a similar way we get $(r_1) \widetilde{\subseteq} (r_2)$. Hence, R is a valuation ring. □

Remark 4.4. Consider R an integral domain and K be the field of fractions of R. Assuming L be the subset of K whose characteristic function is symbolized by $\chi_{_L}$. Now for every $\alpha \in P(K)$, the soft subset of K is symbolized by $\chi_{_L}^\alpha$, which is outlined by $\chi_{_L}^\alpha(m) = Z$, if $m \in R$ and $\chi_{_L}^\alpha(m) = \alpha$, if $m \in K - R$.

We symbolized the soft subset of K by d_α outlined by $d_\alpha(m) = \alpha$ if $m = d$ and $d_\alpha(m) = 0$ otherwise, for every $\alpha \in P(K)$ and $d \in K$. The soft subset $d_\alpha(m)$ is called a soft singleton set. If soft sets δ and ω of K exists, then the definitions of soft sets $\delta \circ \omega$ and $\delta\omega$ of K are given by

$$(\delta \circ \omega)(m) = \cup\{\delta(p) \cap \omega(q) : m = pq \,;\, p, q \in K\}, \forall\, m \in K$$

$$(\delta\omega)(m) = \cup\{\bigcap_{i=1}^{n} (\delta(p_i) \cap \omega(q_i)) : p_i, q_i \in K, n \geq i \geq 1, n \in N, \quad m = \sum_{i=1}^{n} p_iq_i\}, \forall\, m \in K$$

Definition 4.5 ([14]). Let K be the field of fractions of an integral domain R. A soft R-submodule β of K is called a soft fractionary ideal of R, if there exists $d \in R \,;\, d \neq 0$ such that $d_{_R} \circ \beta \subseteq \chi_{_R}^\alpha$ for some $\alpha \in K - R$.

Theorem 4.6. *Consider R an integral domain and K be the field of fractions of R. Then the statements given below are satisfied :*

(i) R is a valuation ring.

(ii) For any soft R-submodule β of K, the set of soft submodules $\{r\beta : r \in R\}$ is linearly ordered.

Proof. The proof can be simply obtained by using Proposition 4.3, and thus omitted. □

Proposition 4.7. *Consider a prime ideal P of a valuation ring V. Then, the localization $V_{_P}$ of V at P is also a valuation ring.*

Proof. Consider a valuation ring V and K be the field of fractions of V. Then the fraction field of $V_{_P}$ is also K. Assume β be any soft $V_{_P}$-submodule of K. Consider the submodules set $\{a\beta : a \in V_{_P}\}$. We prove that this set of submodules is linearly ordered. Note that β is also a soft V-submodule of K. If $r \in V$ and $b \in K$, then $\beta(rb) = \beta((r/1)b) \supseteq \beta(b)$, since $r/1 \in V_{_P}$. Since V being a valuation ring, then clearly by Proposition 4.3, $\{r\beta : r \in R\}$ is linearly ordered. Let $a = r_1/s_1$, $c = r_2/s_2 \in V_{_P}$, then r_1s_2, $r_2s_1 \in V$. Thus either $(r_1s_2)\beta \widetilde{\subseteq} (r_2s_1)\beta$ or $(r_2s_1)\beta \widetilde{\subseteq} (r_1s_2)\beta$. But since $1/s_1s_2 \in V_{_P}$, clearly by Proposition 3.2, we have $(1/s_1s_2)(r_1s_2)\beta \widetilde{\subseteq} (1/s_1s_2)(r_2s_1)\beta$ or $(1/s_1s_2)(r_2s_1)\beta \widetilde{\subseteq} (1/s_1s_2)(r_1s_2)\beta$. i.e., $(r_1/s_1)\beta \widetilde{\subseteq} (r_2/s_2)\beta$ or $(r_2/s_2)\beta \widetilde{\subseteq} (r_1/s_1)\beta$. i.e., $a\beta \widetilde{\subseteq} c\beta$ or $c\beta \widetilde{\subseteq} a\beta$. Therefore the set of submodules $\{a\beta : a \in V_{_P}\}$ is linearly ordered. By Proposition 4.3, $V_{_P}$ is therefore a valuation ring. □

5 Soft fractional ideals and its R-multiple properties

Definition 5.1. Let R be an integral domain and β be a soft fractionary ideal of R, then β is soft invertible if there exists another soft fractionary ideal $\beta^{'}$ of R such that $\beta\beta^{'} = \chi_{_R}^{\alpha}$, for every $\alpha \in P(K)$.

The following theorem provides the necessary and sufficient condition for soft invertibility.

Theorem 5.2. *Let R be an integral domain and K be the fraction field of R. Let β be a soft fractionary ideal of R. If β is soft invertible, then $\cup\{\beta(m) : m \in K \setminus \beta_*\}$ exists iff β_* is an invertible fractionary ideal of R.*

Proof. Considering that if β is soft invertible, then $\cup\{\beta(m) : m \in K \setminus \beta_*\}$ exists. We must now demonstrate that β_* is an invertible fractionary ideal of R. First, we show that β_* is invertible fractionary ideal of R iff β_* is finitely generated principal ideal of R. Suppose that β_* is invertible fractionary ideal. i.e., $\beta_*\beta_*^{'} = R$. Then we have to demonstrate that β_* is finitely generated principal ideal of R . For this we claim that $1 = qi_jk_j$, $i_j \in \beta_*$ and $k_j, \beta_* \in \beta_*^{-1}$ be an ideal. Let $l \in \beta_*$, then $l = qi_jlk_j$. But $lk_j \in R$ as $l \in \beta_*$ and $n_{_l} \in \beta_*^{-1}$. Thus i_j generates β_*, so $\beta_* \neq 0$. Hence β_* is finitely generated. Moreover if $M \subset R$, then $(M^{-1}\beta_*)(M^{-1}\beta_*^{'}) = M^{-1}R$. Considering that i be a maximal ideal. Therefore β_{*_i} is an invertible over R_i. So β_{*_i} is principal ideal. More clearly, we take β_* is invertible, so $\beta_* \neq 0$. Say $1 = \sum i_jk_j$; $i_j \in \beta_*$ and $k_j \in \beta_*^{-1}$. Fix a nonzero $l \in \beta_*$, then $l = \sum i_jlk_j$. But $k_jl \in R$ as $l \in \beta_*$ and $k_l \in \beta_*^{-1}$. Consider the set $d = gcd\{k_jl\} \in R$ and $m = \sum(k_jl/d)i_j \in \beta_*$, then $l = dm$. Given $l^{'} \in \beta_*$, write $l^{'}/l = e/f$; e and f in R are prime. So $d^{'} = gcd\{k_jl^{'}\} = gcd\{k_jle/f\} = e(gcd\{k_jl\}/f) = ed/f$. So $l^{'} = (e/f)l = (ed/f)m = d^{'}m$, but $d^{'} \in R$. Thus $\beta_* = Rm$. Again let us consider that β_* is finitely generated principal ideal, then we need to demonstrate that β_* of R is invertible fractionary ideal. Consider the set $c = \beta_*(R : \beta_*) \subset R$. Since β_* be finitely generated, therefore $c_i = \beta_{*_i}(R_i : \beta_{*_i})$. Further assume that β_{*_i} is a nonzero principal ideal, so $c_i = R_i$. Therefore $c = R$ that implies $c = R = \beta_*\beta_*^{'}$. Thus β_* is an invertible fractionary ideal of R.
Conversely, consider the case when β_* is an invertible fractionary ideal of R. Then we demonstrate that if β is soft invertible, then $\cup\{\beta(m) : m \in K \setminus \beta_*\}$ exists. By Proposition 4.1, if β is any soft R-submodule of the fraction field K of R and $r \in R$, then we have $r(r\beta)_\alpha = \beta_\alpha$, for all $\alpha \in P(K)$. Particularly, $\beta_* = r(r\beta)_*$. Thus $(r\beta)_* = (1/r)\beta_*$. Let us consider that $\cup\{\beta(m) : m \in K \setminus \beta_*\}$ exists and $\cup\{\beta(m) : m \in K \setminus \beta_*\} = \alpha$. If $m \in K \setminus \beta_*$, then $m \notin \beta_*$. Therefore $\beta(m) \subset R$, thus $m \notin \beta_*$ or $m \in K \setminus \beta_*$. Therefore $\beta(m) \subseteq \alpha$. Hence $\cup\{\beta(m) : m \in K \setminus \beta_*\}$ exists. □

Proposition 5.3. *Consider an integral domain R and K be the fraction field of R. Let β be a soft R-submodule of K . Then*

(i) β is a soft fractionary ideal of $R \Longleftrightarrow r\beta$ is soft fractionary ideal of R, for any $r \in R$.

(ii) β is a soft invertible and $\cup\{\beta(x) : x \in K \setminus \beta_*\}$ exists $\Longleftrightarrow r\beta$ is soft invertible and $\cup\{(r\beta)(y) : y \in K \setminus \beta_*\}$ exists.

Proof. (i) Suppose β be a soft fractionary ideal of R, then $\exists\ d \neq 0$, $d \in R$ and $\alpha \in P(K)$ such that $d_{_R} \circ \beta \subseteq \chi_{_R}^{\alpha}$. Thus $\beta(x/d) \subseteq \alpha$, $\forall\ x \in K - R$. Since $(r\beta)(x/rd) = \beta(x/d)$, we get $(r\beta)(x/rd) \subseteq \alpha$, $\forall\ x \in K - R$. Hence $(rd)_{_R} \circ (r\beta) \subseteq \chi_{_R}^{\alpha}$. Therefore $r\beta$ is a soft fractionary ideal.
Conversely, suppose for any $r \in R$, $r\beta$ is soft fractionary ideal of R. Then $\exists\ h \in R$ and $\alpha \in P(K)$ such that $h_{_R} \circ (r\beta) \subseteq \chi_{_R}^{\alpha}$. Therefore $(r\beta)(x/h) \subseteq \alpha$, $\forall\ x \in K - R$. Thus $\beta(rx/h) \subseteq \alpha$, $\forall\ x \in K - R$. But since β is a soft R-submodule. It implies $\beta(rx/h) \supseteq \beta(x/h)$. Therefore $\beta(x/h) \subseteq \alpha$, $\forall\ x \in K - R$. It implies $h_{_R} \circ \beta \subseteq \chi_{_R}^{\alpha}$. Hence, β is a soft fractionary ideal of R.

(ii) By Proposition 4.1, we have $r(r\beta)_\alpha = \beta_\alpha$, for every $\alpha \in P(K)$. Particularly, $r(r\beta)_* = \beta_*$. Therefore $(r\beta)_* = (1/r)\beta_*$. Assume that $\cup\{\beta(x) : x \in K \setminus \beta_*\}$ exists and $\cup\{\beta(x) : x \in K \setminus \beta_*\} = \alpha$. If $y \in K \setminus (r\beta)_*$, then $y \notin (r\beta)_*$. Thus $\beta(ry) = (r\beta)(y) \subset R$, hence $ry \notin \beta_*$ or $ry \in K \setminus \beta_*$. Therefore $(r\beta)(y) = \beta(ry) \subseteq \alpha$. Thus $\cup\{(r\beta)(y) : y \in K \setminus (r\beta)_*\}$ exists. Moreover if β is soft invertible, then by Theorem 5.2, β_* is fractionary ideal of R which is invertible. It follows $(r\beta)_* = (1/r)\beta_*$ is invertible. By Theorem 5.2 again, we can conclude that $r\beta$ is soft

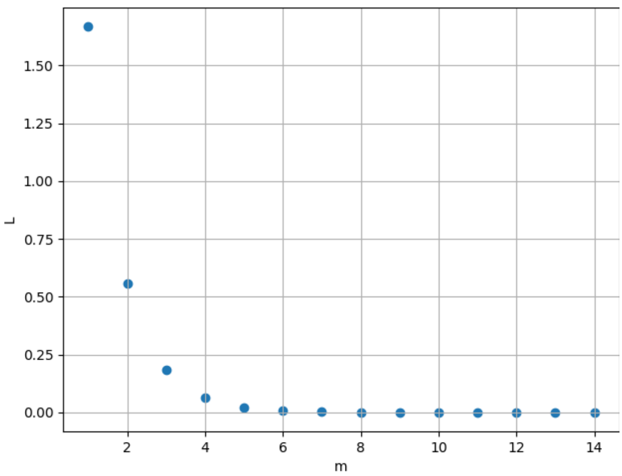

Figure 1. Graphical representation of L

invertible. By the same way, we may demonstrate that if $\cup\{(r\beta)(y) : y \in K \setminus (r\beta)_*\}$ exists, then $\cup\{\beta(x) : x \in K \setminus \beta_*\}$ exists and if in addition β is soft invertible, then $r\beta$ is also soft invertible. □

Proposition 5.4. *Let K be the fraction field of an integral domain R. Suppose β be a soft R-submodule of K. If $\beta \neq R_K$ and $\beta_* \neq (0)$, then $(r_1)\widetilde{\subsetneq}(r_2)$ implies $r_2\beta \widetilde{\subsetneq} r_1\beta$.*

Proof. We have proved that $(r_1) \widetilde{\subseteq} (r_2)$ implies $r_2\beta \widetilde{\subseteq} r_1\beta$ in Proposition 4.3. Let us assume that $\beta \neq R_K$ and $\beta_* \neq (0)$, and that $(r_1) \widetilde{\subsetneq} (r_2)$. Then $r_1 = r_2c$, where c is a non-unit of R. Choose $0 \neq x \in \beta_*$ but $x/c \notin \beta_*$. This is possible since $\beta \neq R_K$ and $\beta_* \neq 0$. For $\beta_* \neq 0$ implies that there exists $0 \neq x \in \beta_*$. If $x/c \in \beta_*$ for all $0 \neq x \in \beta_*$, then $\beta_*/c \widetilde{\subseteq} \beta_*$. At the same time $\beta_* \widetilde{\subseteq} \beta_*/c$, since $r/s \in \beta_* \Longrightarrow \beta(r/s) = R \Longrightarrow \beta(rc/s) \supseteq \beta(r/s) = R$, so that $\beta(rc/s) = R$ $\Longrightarrow r/s = rc/sc = (rc/s)/c \in \beta_*/c$. Either one suggests that $\beta_*/c = \beta_*$ or $\beta_* = c\beta_*$. But that is impossible when c becomes a unit. Now take $y = x/(r_2c)$, then $(r_1\beta)(y) = \beta(r_1y) = \beta(r_1x/r_2c) = \beta(x) = R$. Since $x \in \beta_*$, but $(r_2\beta)(y) = \beta(r_2y) = \beta(r_2x/r_2c) = \beta(x/c) \subset R$, since $x/c \notin \beta_*$. Thus $(r_2\beta)(y) \subset (r_1\beta)(y)$. Hence $(r_2\beta) \widetilde{\subsetneq} (r_1\beta) \Longrightarrow r_2\beta \widetilde{\subsetneq} r_1\beta$. □

Remark 5.5. The following example shows that the converse of the above Proposition is not true in general.

Example 5.6. Consider the integers ring Z and

$L = \widetilde{\bigcup}_{m=1}^{\infty}(5)/3^m = (5) \cup (5)/3 \cup (5)/3^2 \cup (5)/3^3 \cup ....$

Then L is a Z-submodule of the field of fractions Q. But L is not a fractional ideal. Let β be the characteristic function of Z-submodule L in Q, then β is a soft submodule of Q and $\beta_R = \beta_* = L$. We have $3L = 2L$, therefore $3\beta = 2\beta$. At the same time $4L \widetilde{\subsetneq} 2L$, thus $2\beta \widetilde{\subsetneq} 4\beta$. Therefore $3\beta \widetilde{\subsetneq} 4\beta$, but $(4) \nsubseteq (3)$.

Graphically Figure 1 shows that L is not a fractional ideal.

Definition 5.7. Let K be the field of fractions of an integral domain R. A soft fractional ideal β of K such that $\beta \neq R_K$ and $\beta_* \neq (0)$ is said to be minimal with respect to $r \in R$, if $r\beta \widetilde{\supsetneq} \beta$ and that $\exists$ no $s_1 \in R$ such that $r\beta \widetilde{\supsetneq} s_1\beta \widetilde{\supsetneq} \beta$. In other words, $r\beta \widetilde{\supseteq} s_1\beta \widetilde{\supseteq} \beta$, for $s_1 \in R$ implies that either $(r) = (s_1)$ or $(s_1) = R$.

Proposition 5.8. *Let K be the fraction field of a Dedekind domain R. Suppose β be a soft fractional ideal of K such that $\beta \neq R_K$ and $\beta_* \neq (0)$. Then β is minimal with respect to $r \in R$ if and only if for all proper principal ideals set of R, (r) is maximal.*

Proof. By assuming if (r) is not maximal in all proper principal ideals set of R, then $\exists\ s_1 \in r$ such that $(r) \widetilde{\subsetneq} (s_1) \widetilde{\subsetneq} (R)$. By Proposition 5.4, $\beta \widetilde{\subsetneq} s_1\beta \widetilde{\subsetneq} r\beta$. Hence β is not minimal with respect to $r \in R$.
Conversely, by assuming if β is not minimal in relation to r, then $\exists\ s_1 \in R$ such that $\beta \widetilde{\subsetneq} s_1\beta \widetilde{\subsetneq} r\beta$. Thus by Proposition 4.2, $r\beta_\alpha \widetilde{\subseteq} s_1\beta_\alpha \widetilde{\subseteq} \beta_\alpha$, for every $\alpha \in P(K)$. Since β is a soft fractionary ideal of a Dedekind domain R, β_α is a fractional ideal, for every α. Since in a Dedekind domain, fractional ideals are invertible and multiplying with its inverse, we get $(r) \widetilde{\subseteq} (s_1) \widetilde{\subseteq} (R)$. Now if $(r) = (s_1)$, then by Proposition 4.3, $s_1\beta = r\beta$ and if $(s_1) = (R)$, then $\beta = \beta_R$. Since $\beta \widetilde{\subsetneq} s_1\beta \widetilde{\subsetneq} r\beta$, this leads to a contradiction. Therefore $(r) \widetilde{\subsetneq} (s_1) \widetilde{\subsetneq} (R)$ and hence (r) is not maximal. □

Remark 5.9. Knowing this, (r) is maximal iff r is irreducible. If β is a soft fractionary ideal of a Dedekind domain R. Then the set of all minimal elements of β is equivalent to the set of all irreducible elements of R.

6 Conclusion

Motivated while studying and analysing the concepts in connection with fuzzification of fractional ideals proposed by Lee and Mordeson [18]. Properties of multiples play a dominant role in soft fractional ideals of an integral domain. We bring out the new concepts in relation to soft modules, soft submodules and equivalence of certain soft modules of the quotient field, called soft fractionary ideals. In this paper, using these concepts and freshly defined concepts, we have characterized valuation domains, integral domains especially that of Dedekind domains. The results carried out in this research article will create new ideas and provide a solid point of departure for studying and analysing soft fractional ideal theory.

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