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THE GEOMETRY OF INVARIANTS UNDER THE SL$(3,\mathbb{R})$ ACTION ON PROJECTIVE PLANE

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Abstract In this paper, we have investigated the projective action of the Lie group SL$(3,\mathbb{R})$ on the homogeneous space $\mathbb{RP}^2$. Following the Erlangen approach, the action of the subgroups of SL$(3,\mathbb{R})$ on the projective lines has been examined. Projective properties, in particular, plane duality and cross-ratio, are discussed for the SL$(3,\mathbb{R})$ action.

1 Introduction

The study of geometry reveals the essence of any subject. The Erlangen programme, proposed by Felix Klein in 1872, is a framework for classifying and understanding geometries based on the types of transformations that are allowed in the system. The program defines geometry as the study of properties that are invariant under certain groups of transformations [8, 19]. In accordance with this line of thought, we propose to investigate the geometric properties of the action of the Lie group SL$(3,\mathbb{R})$ on the two-dimensional homogeneous space $\mathbb{RP}^2$. Thus, by studying invariants, we propose to construct geometry of the homogeneous space $\mathbb{RP}^2$ in a systematic way. In this trend, numerous authors have conducted substantial research on geometry in terms of invariants and used the Erlangan approach to construct the geometry of one and two-dimensional spaces [7, 16, 18]. The action of $G = \mathrm{SL}(2,\mathbb{R})$ on the one-dimensional homogeneous space G/H represented by Möbius transformation was extensively studied and investigated by several authors, for example, in [1, 5, 17]. In the series of works [2, 7, 15], the authors proposed to develop an analytic function theory in the spirit of the Erlangen programme. A similar approach was taken in [3, 4, 9] to investigate the SL$(2,\mathbb{R})$ invariant geodesic curves and metrics. These series of works result in a number of natural and effective generalizations, making it imperative to investigate higher-dimensional instances. In the parallel line, there is a wide range of literature devoted to the representation theory of Lie groups and symmetric spaces, where the classical cases such as the Lie groups SL$(2,\mathbb{R})$, SL$(2,\mathbb{C})$, SL$(3,\mathbb{R})$, SU(2) etc. have provided powerful tools for studying these groups and other related results, such references include [10, 12, 20, 22, 23, 24].

Since projective geometry is a rich and intriguing field that provides a geometric foundation for numerous disciplines, we are trying to study the fundamental objects and their interactions, such as points, lines, and their connections. The axiomatic definition of the projective plane is revisited in an effort to systematically develop the geometry on it. In this direction, we consider the two-dimensional space $\mathbb{RP}^2$ which is obtained as the homogeneous space G/H, where $G = \mathrm{SL}(3,\mathbb{R})$ is the transformation group and H is a six-dimensional closed subgroup of G. We layout details for $\mathbb{RP}^2$ and show that plane duality, as well as cross-ratio of four collinear points, are invariants under the SL$(3,\mathbb{R})$ action.

2 Preliminaries

In this section, we review some basic definitions and theorems that are pertinent to this work.

Definition 2.1. A transformation group M can be defined as a non-empty collection of mappings from a set P to itself, adhering to the conditions

(i) the identity map is an element of M,

(ii) if $m_1 \in M$ and $m_2 \in M$, then $m_1 m_2 \in M$,

(iii) if $m \in M$, then the inverse m^{-1} exists and is a member of M.

Definition 2.2. A group action $\varphi : M \times P \to P$ is called transitive if for every $p,\ q \in P$, there exists $m \in M$ satisfying $m \cdot p = q$. Furthermore, a homogeneous space is a pair (M, P), where the action of the group M on P is transitive and P is a topological space.

Definition 2.3. [13] A subgroup H of GL$(n, \mathbb{C})$ is referred to as a matrix Lie group if it exhibits the subsequent characteristic – when considering a sequence of matrices C_k within the group H that converges to a matrix C, then either C belongs to H, or C is not invertible.

2.1 Group action on coset spaces

Let G be a Lie group and H be a closed subgroup of G. Then H is a Lie group by Cartan's Theorem, [21]. Let $G/H = \{gH : g \in G\}$ denote the space of left cosets of H.

Theorem 2.4. *[14] Let G be a Lie group and H a closed subgroup of G. Let G/H have the quotient topology. Then G/H has a unique smooth manifold structure such that the projection map $p : G \to G/H$ is a smooth submersion and G acts smoothly on G/H.*

Definition 2.5. The projection map $p : G \to G/H$ is given by sending $g \in G$ to its equivalence class $[g]$ and is defined as $p(g) = gH = [g]$.

Definition 2.6. A section s of a projection map p is defined as the right inverse of p such that $s : G/H \to G$ and $p(s(x)) = x, \forall\, x \in G/H$.

Remark 2.7. We consider the action $G \times G/H \to G/H$ defined by $(a, gH) \mapsto agH$. The action can be viewed as a composition of smooth maps as follows:

$$\phi : G \times G/H \to G/H$$
$$\phi(g, x) = g \cdot x = p(g * s(x)),$$

where $*$ denotes the group operation on G (see Lemma 3.3 for the details).

2.2 Real projective space $\mathbb{RP}^n$

Let $\mathbb{R}^{n+1} = \{(x_1, x_2, \cdots, x_{n+1}) : x_i \in \mathbb{R}\}$. The real projective space $\mathbb{RP}^n$ consists of points which are equivalence classes of the set $\mathbb{R}^{n+1} \setminus \{0\}$ modulo the equivalence relation $x \sim \lambda x$, for all λ in $\mathbb{R} \setminus \{0\}$.

In particular, for $n = 1$, the space $\mathbb{RP}^1$ is called the real projective line. For $n = 2$, the space $\mathbb{RP}^2$ is called the real projective plane. Any point in the projective plane is represented by a triple $(X, Y, Z)^t$, called homogeneous coordinates or projective coordinates of the point, where X, Y, and Z are not all 0. Since points in $\mathbb{RP}^2$ are equivalence classes, in the homogeneous coordinated setup, the coordinates $(X, Y, Z)^t$ and $(\lambda X, \lambda Y, \lambda Z)^t$ are considered to represent the same point, for all $\lambda \neq 0$ in $\mathbb{R}$, [11].

2.3 Lines and Conics in $\mathbb{RP}^2$

Line L in $\mathbb{RP}^2$ is represented by the homogeneous coordinates $L = (a, b, c)^t$ and is defined as $ax + by + cz = 0$, or in matrix notation $q^t L = 0$, where $q = (x, y, z)^t$ is any point in L.

Conic in $\mathbb{RP}^2$ is the set of points for which a quadratic form on $\mathbb{R}^3$ equals zero. Thus, the conic associated with a quadratic form (or matrix) is

$$\mathbf{C}_{\mathcal{A}} = \{[q] \in \mathbb{RP}^2 : q^t \mathcal{A} q = 0\}, [11].$$

The next result gives us an explicit way of describing points on a line L in $\mathbb{RP}^2$, provided two distinct points on L are given.

Lemma 2.8. *Consider two distinct points p and q in $\mathbb{RP}^1$. The collection of all points lying on the line passing through these two points can be expressed as*

$$\{\mu p + \lambda q \mid \mu,\ \lambda \in \mathbb{R} \text{ with } \mu \text{ or } \lambda \text{ non-zero}\},\ [11].$$

Remark 2.9. In Lemma 2.8, when L is defined as the linear span of two distinct points, it is said that the points p, q constitute the basis of coordinate vectors for the line L. In a projective context, the pair (μ, λ) is only defined up to scale factor, signifying homogeneous coordinates along the projective line $\mathbb{RP}^1$ from p to q, expressed with respect to the basis of coordinate vectors p, q.

Notation 2.1. If $P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ are two-dimensional vectors, we use the notation $[P, Q]$ as $[P, Q] = \det \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$.
Similarly, if $P = (p_1, p_2, p_3)^t,\ Q = (q_1, q_2, q_3)^t,\ R = (r_1, r_2, r_3)^t$ are three-dimensional vectors, then $[P, Q, R] = \det \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}$.

2.4 Cross-ratio in $\mathbb{RP}^2$

Definition 2.10. Consider any line L in $\mathbb{RP}^2$ that contains two points, p and q, represented by $\mu p + \lambda q$. Let $R_i = \mu_i p + \lambda_i q$ be any four points on this line, where μ_i and λ_i $(i = 1, 2, 3, 4)$ are real numbers. Then, the cross-ratio is defined as

$$(R_1, R_2; R_3, R_4) = \frac{[r_1, r_3][r_2, r_4]}{[r_2, r_3][r_1, r_4]}, \tag{2.1}$$

where $r_i = \begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix}$ for $i = 1,\ 2,\ 3,\ 4$, as introduced in [11].

Remark 2.11. The expression on the right-hand side of the equation (2.1) represents a projective quantity. This can be observed by considering non-zero scalars λ_x and λ_y in $\mathbb{R}$ and a 2×2 matrix M with a non-vanishing determinant. We see that the relationships $[\lambda_x x, \lambda_y y] = \lambda_x \lambda_y [x, y]$ (since $[x, y]$ represents a determinant with columns x and y), and $[M \cdot x, M \cdot y] = \det(M) \cdot [x, y]$ hold. Hence,

$$\begin{aligned}(\lambda_1 R_1, \lambda_2 R_2; \lambda_3 R_3, \lambda_4 R_4) &= \frac{[\lambda_1 r_1, \lambda_3 r_3][\lambda_2 r_2, \lambda_4 r_4]}{[\lambda_2 r_2, \lambda_3 r_3][\lambda_1 r_1, \lambda_4 r_4]} \\ &= \frac{\lambda_1 \lambda_2 \lambda_3 \lambda_4 [r_1, r_3][r_2, r_4]}{\lambda_1 \lambda_2 \lambda_3 \lambda_4 [r_2, r_3][r_1, r_4]} \\ &= (R_1, R_2; R_3, R_4),\end{aligned}$$

and

$$\begin{aligned}(M \cdot R_1, M \cdot R_2; M \cdot R_3, M \cdot R_4) &= \frac{[M \cdot r_1, M \cdot r_3][M \cdot r_2, M \cdot r_4]}{[M \cdot r_2, M \cdot r_3][M \cdot r_1, M \cdot r_4]} \\ &= \frac{(\det(M))^2 [r_1, r_3][r_2, r_4]}{(\det(M))^2 [r_2, r_3][r_1, r_4]} \\ &= (R_1, R_2; R_3, R_4),\end{aligned}$$

showing that the expression $\dfrac{[r_1,r_3][r_2,r_4]}{[r_2,r_3][r_1,r_4]}$ remains invariant under projective transformations.

3 Two-dimensional Homogeneous space $\mathbb{RP}^2$

We consider the matrix Lie group $G = \mathrm{SL}(3,\mathbb{R}) = \{A \in M(3,\mathbb{R}) : \det(A) = 1\}$ and the action of it on the space of left cosets $X = G/H$, where H is a closed subgroup of SL$(3,\mathbb{R})$ and hence a matrix Lie group, [21]. The subgroup H is defined as

$$H = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \middle| a_{22}a_{33} - a_{23}a_{32} = \frac{1}{a_{11}}, a_{11} \neq 0 \right\}.$$

The space X becomes a homogeneous space with the left G-action $g : z \mapsto g \cdot z$. For a more intuitive geometric understanding, it is advantageous to parametrize X and represent the G-action using these parameters, as illustrated below.

By the Dimension theorem of the quotient space, dim $X = \dim(G) - \dim(H) = 2$. X being a two-dimensional space, we are parametrizing every element of X by a pair (x, y). With this parametrization, we can express the set-theoretic action of SL$(3,\mathbb{R})$ on SL$(3,\mathbb{R})/H$ as a composition of smooth mappings in the following manner

$$g : x \mapsto g \cdot x = p(g * s(x)).$$

We can define another map $r : G \to H$ such that $r(g) = h$, where $h = s(p(g))^{-1}g$. Hence g can be uniquely written as $g = s(p(g))r(g)$.
Consequently, we have a decomposition $g = s(p(g))r(g)$ of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & \dfrac{a_{21}}{a_{31}}a_{32} - a_{22} & \dfrac{a_{21}}{a_{31}}a_{33} - a_{23} \\ 0 & a_{12} - \dfrac{a_{11}}{a_{31}}a_{32} & a_{13} - \dfrac{a_{11}}{a_{31}}a_{33} \end{pmatrix},$$

provided $a_{31} \neq 0$.

In this set up, we define $p(g) = \left(\dfrac{a_{11}}{a_{31}}, \dfrac{a_{21}}{a_{31}}\right)$, where $\begin{pmatrix} a_{11}/a_{31} & 0 & 1 \\ a_{21}/a_{31} & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is the matrix representation of the equivalence class of $[g]$.
We define section, $s : X \to \mathrm{SL}(3,\mathbb{R})$ as

$$s(x, y) = \begin{pmatrix} x & 0 & 1 \\ y & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

such that $p(s(x,y)) = p \begin{pmatrix} x & 0 & 1 \\ y & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (x, y)$.
Then the SL$(3,\mathbb{R})$ action takes the form

$$(x, y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \right),$$

provided $a_{31}x + a_{32}y + a_{33} \neq 0$, [6].

Remark 3.1. If we allow $a_{31}x + a_{32}y + a_{33} = 0$, then this action indeed gives us a projective transformation of the space $\mathbb{RP}^2$ as follows:

Let any point $[p] \in \mathbb{RP}^2$ be represented by three dimensional column vector $(X, Y, Z)^t$ in homogeneous coordinates and let the action of $g = (a_{ij})$ be defined by matrix multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a_{11}X + a_{12}Y + a_{13}Z \\ a_{21}X + a_{22}Y + a_{23}Z \\ a_{31}X + a_{32}Y + a_{33}Z \end{pmatrix}$$

This $\mathrm{SL}(3, \mathbb{R})$ action on $\mathbb{RP}^2$ is denoted as $g : [p] \mapsto [g \cdot p]$. Let ϕ be the action.

$$\phi : \mathrm{SL}(3, \mathbb{R}) \times \mathbb{RP}^2 \to \mathbb{RP}^2$$
$$\phi(g, [p]) = [g \cdot p]$$

We consider the projective transformation $\phi_g : \mathbb{RP}^2 \to \mathbb{RP}^2$ such that $\phi_g([p]) = [g \cdot p]$ for all $g \in \mathrm{SL}(3, \mathbb{R})$, [11].

Lemma 3.2. *For any element $g \in G$, we have $s(p(g)) = gh$, where h is a member of the subgroup H and depends on the specific choice of g.*

Proof. We consider the projection map $p : G \to G/H$, the section map $s : G/H \to G$, and the composition map $s \circ p : G \to G$ such that $p(s(x)) = x$ for all $x \in G/H$. Let $(s \circ p)(g) = s(p(g)) = g'$. Then,

$$\begin{aligned} & p(s(p(g))) = p(g') \Rightarrow p(g) = p(g') \\ & \Rightarrow gH = g'H \text{ (cf. Definition 2.5)} \Rightarrow g^{-1}g' \in H \\ & \Rightarrow g^{-1}g' = h \text{ for some } h \in H \Rightarrow g' = gh. \end{aligned}$$

Therefore, $s(p(g)) = gh$, for some $h \in H$ that depends on g. □

Lemma 3.3. *The action $\phi : G \times G/H \to G/H$, defined as $\phi(g, x) = p(g * s(x))$, is a transitive group action.*

Proof. To verify the group action, we need to confirm two conditions:

i) For the identity element e of the group G

$$\phi(e, y) = p(e * s(y)) = p(s(y)) = y \ (\because s \text{ is the right inverse of } p)$$

$\therefore\ \phi(e, y) = y, \forall\, y \in G/H.$

ii) Let $g_1, g_2 \in G$ and $x \in G/H$.

$$\begin{aligned} \phi(g_1, \phi(g_2, x)) &= \phi(g_1, y) \ \text{ where } \ y = \phi(g_2, x) = p(g_2 * s(x)) \\ &= p(g_1 * s(y)) = p(g_1, s(p(g_2 * s(x))) \\ &= p(g_1 * (g_2 * s(x))h) \ \text{ for some } h \in H \text{ depending on } g_2 * s(x) \\ &= g_1 g_2 s(x) H \end{aligned}$$

Again, $\phi(g_1 g_2, x) = p(g_1 g_2 * s(x)) = g_1 g_2 s(x) H.$
$\therefore\ \phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x),\ \forall\, g_1,\ g_2 \in G$ and $\forall\, x \in G/H.$

For transitivity, we need to check that for any pair of $aH,\ bH \in G/H$, $\exists\, g \in G$ such that $\phi(g, aH) = bH$. We take two elements $aH,\ bH \in G/H$ and consider $g = bs(aH)^{-1}$, then,

$$\begin{aligned} \phi(g, aH) &= \phi(bs(aH)^{-1}, aH) \\ &= p(bs(aH)^{-1} * s(aH)) \ (\because\ \phi(g, x) = p(g * s(x))) \\ &= p(b) = bH \end{aligned}$$

Therefore, for any pair $aH,\ bH \in G/H$, $\exists\, g \in G$ such that $\phi(g, aH) = bH.$
Hence ϕ is a transitive group action. □

Lemma 3.4. *Consider two sections, denoted as s_1 and s_2, such that $p(s_1(y)) = p(s_2(y)) = y$ for all $y \in X$. It follows that $p(g * s_1(y)) = p(g * s_2(y))$ for all $g \in G$.*

Proof. We have

$$\begin{aligned} & p(s_1(y)) = p(s_2(y)) \\ \Rightarrow\ & s_1(p(s_1(y))) = s_1(p(s_2(y))) \\ \Rightarrow\ & s_1(y)h_1 = s_2(y)h_2 \text{ for some } h_1,\ h_2 \in H \text{ (using the Lemma 3.2).} \end{aligned}$$

Let $x \in p(g * s_1(y))$. Then,

$$\begin{aligned} p(g * s_1(y)) &= p(g * s_2(y)h_2h_1^{-1}) \ (\because s_1(y) = s_2(y)h_2h_1^{-1}) \\ &= (g * s_2(y))H \text{ (cf. Definition 2.5)} \\ &= p(g * s_2(y)) \end{aligned}$$

Thus,

$$\begin{aligned} x \in p(g * s_1(y)) &\Rightarrow x \in p(g * s_2(y)) \\ \therefore\ p(g * s_1(y)) &\subseteq p(g * s_2(y)) \end{aligned} \tag{3.1}$$

Again, let $z \in p(g * s_2(y))$. Then

$$\begin{aligned} p(g * s_2(y)) &= p(g * s_1(y)h_1h_2^{-1}) \ (\because s_2(x) = s_1(x)h_1h_2^{-1}) \\ &= (g * s_1(y))H \\ &= p(g * s_1(y)) \end{aligned}$$

Thus,

$$\begin{aligned} z \in p(g * s_2(y)) &\Rightarrow z \in p(g * s_1(y)) \\ \therefore\ p(g * s_2(y)) &\subseteq p(g * s_1(y)) \end{aligned} \tag{3.2}$$

From equations (3.1) and (3.2), we have $p(g * s_1(y)) = p(g * s_2(y)), \forall\, g \in G$.
Hence the result follows. □

4 Plane Duality and SL$(3,\mathbb{R})$ action on $\mathbb{RP}^2$

The principle of duality states that the axioms for a projective plane hold true when the terms "line" and "point" are used interchangeably. This principle applies to the real projective plane $\mathbb{RP}^2$. In this regard, by considering the axiomatic definition of $\mathbb{RP}^2$, we demonstrate that these axioms remain unaltered even under the action of SL$(3,\mathbb{R})$ i.e., the axioms

(i) Any two distinct points X, and Y are incidence with exactly one line L.

(ii) Any two distinct lines L_1, L_2 are incidence with exactly one point X,

will be converted under the action of SL$(3,\mathbb{R})$ to the following axioms:

(i) Image of any two distinct points i.e., PX, PY are incidence with exactly one line L', where $L' = (P^{-1})^t L$.

(ii) When $P \in$ SL$(3,\mathbb{R})$ acts, the transformed lines $(P^{-1})^t L_1$, $(P^{-1})^t L_2$ are incidence with exactly one point X', where $X' = PX$, respectively.

Here, PX denotes the action of $P \in$ SL$(3,\mathbb{R})$ on X.

Theorem 4.1. *If two distinct points $X = (x_1, y_1, z_1)^t$, $Y = (x_2, y_2, z_2)^t$ are incidence (lies on) with exactly one line $L = (l_1, m_1, n_1)^t$, then image of two distinct points i.e., PX, PY are incidence with exactly one line L', where $L' = (P^{-1})^t L$.*

Proof. It is clear that under the action of SL$(3,\mathbb{R})$, distinct points maps to distinct points, otherwise, if

$$PX = \lambda PY \Rightarrow PX = P(\lambda Y) \Rightarrow X = (\lambda Y) \ \text{(since } P \text{ is invertible)},$$

which contradicts that X and Y are distinct.
Let $X = (x_1, x_2, x_3)^t$, $Y = (y_1, y_2, y_3)^t$ are incidence with exactly one line $L = (l_1, m_1, n_1)^t$. Then, the line is uniquely determined by cross product $X \times Y$ and is given by

$$(x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - y_1x_2)^t .$$

Therefore, in homogeneous coordinates,

$$L = \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - y_1x_2 \end{pmatrix} . \tag{4.1}$$

Now, under the action of $P = (a_{ij})_{3\times 3} \in \mathrm{SL}(3, \mathbb{R})$, the points X and Y map to

$$PX = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} ,$$

and

$$PY = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_2 \end{pmatrix} ,$$

respectively.
Again, if we explicitly calculate the unique line L' joining these two points PX and PY, then it will be determined by $PX \times PY$, where $\times$ is the cross product in $\mathbb{R}^3$. Hence, computation on this cross product gives

$$\begin{aligned} L' &= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} \times \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{pmatrix} \\ &= \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{23}a_{31} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\ a_{13}a_{32} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\ a_{12}a_{23} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix} \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - y_1x_2 \end{pmatrix} . \end{aligned} \tag{4.2}$$

Thus, from equations (4.1) and (4.2), we obtain $L' = (P^{-1})^t L$.
Hence, the theorem follows. □

Corollary 4.2. *Collinearity is preserved under the action of SL$(3, \mathbb{R})$.*

Proof. Collinearity of a set of points is the property of their lying on the same line. Consider the line L given by $ax + by + cz = 0$ or, in matrix notation $X^tL = 0$ where $L = (a, b, c)$ and $X = (x, y, z)$ are the homogeneous coordinates of the line L and point X, respectively.
Now, under the action of $P \in \mathrm{SL}(3, \mathbb{R})$, the point X maps to $X' = PX$ and X' lies on $L' = (P^{-1})^t L$. Hence,

$$X^tL = 0 \Leftrightarrow (P^{-1}X')^t L = 0 \Leftrightarrow (X')^t(P^{-1})^tL = 0.$$

Therefore, any point X lies on L, the corresponding point X' lies on the corresponding line L', where $L' = (P^{-1})^tL$, which implies that collinearity is preserved. □

Remark 4.3. Under SL$(3, \mathbb{R})$ action, projective line maps to projective line.

Theorem 4.4. *If two distinct lines $L_1 = (l_1, m_1, n_1)^t$, $L_2 = (l_2, m_2, n_2)^t$ are incidence (passes through) with exactly one point $X = (x_1, y_1, z_1)^t$, then the transformed distinct lines $(P^{-1})^t L_1$, $(P^{-1})^t L_2$ are incidence with exactly one point $X' = PX$.*

Proof. From the collinearity property, it is clear that under the action of $P \in \mathrm{SL}(3,\mathbb{R})$, a transformation of two distinct lines will be again distinct lines.
Let two distinct lines $L_1 = (l_1, m_1, n_1)^t$, $L_2 = (l_2, m_2, n_2)^t$ are both incidence with one point $X = (x_1, y_1, z_1)^t$. Thus, $X^t L_1 = X^t L_2 = 0$ and this X is given by the homogeneous coordinates of $L_1 \times L_2$. Hence, performing the cross product $\times$ in $\mathbb{R}^3$, we get the homogeneous coordinates

$$X = \begin{pmatrix} m_1n_2 - m_2n_1 \\ l_2n_1 - l_1n_2 \\ l_1m_2 - l_2m_1 \end{pmatrix}. \tag{4.3}$$

Again, let under the action of $P = (a_{ij})_{3\times 3} \in \mathrm{SL}(3,\mathbb{R})$, the transformed lines $(P^{-1})^t L_1$, $(P^{-1})^t L_2$ meet at X' (cf. Theorem 4.1). Therefore, the homogeneous coordinates of X' are given by the cross product $(P^{-1})^t L_1 \times (P^{-1})^t L_2$. Thus, after explicit calculation, we get

$$X' = \begin{pmatrix} (a_{22}a_{33} - a_{23}a_{32})l_1 + (a_{23}a_{31} - a_{21}a_{33})m_1 + (a_{21}a_{32} - a_{22}a_{31})n_1 \\ (a_{13}a_{32} - a_{12}a_{33})l_1 + (a_{11}a_{33} - a_{13}a_{31})m_1 + (a_{12}a_{31} - a_{11}a_{32})n_1 \\ (a_{12}a_{23} - a_{13}a_{22})l_1 + (a_{13}a_{21} - a_{11}a_{23})m_1 + (a_{11}a_{22} - a_{12}a_{21})n_1 \end{pmatrix} \times$$

$$\begin{pmatrix} (a_{22}a_{33} - a_{23}a_{32})l_2 + (a_{23}a_{31} - a_{21}a_{33})m_2 + (a_{21}a_{32} - a_{22}a_{31})n_2 \\ (a_{13}a_{32} - a_{12}a_{33})l_2 + (a_{11}a_{33} - a_{13}a_{31})m_2 + (a_{12}a_{31} - a_{11}a_{32})n_2 \\ (a_{12}a_{23} - a_{13}a_{22})l_2 + (a_{13}a_{21} - a_{11}a_{23})m_2 + (a_{11}a_{22} - a_{12}a_{21})n_2 \end{pmatrix}$$

$$= \begin{pmatrix} (m_1n_2 - m_2n_1)a_{11} + (l_2n_1 - l_1n_2)a_{12} + (l_1m_2 - l_2m_1)a_{13} \\ (m_1n_2 - m_2n_1)a_{21} + (l_2n_1 - l_1n_2)a_{22} + (l_1m_2 - l_2m_1)a_{23} \\ (m_1n_2 - m_2n_1)a_{31} + (l_2n_1 - l_1n_2)a_{32} + (l_1m_2 - l_2m_1)a_{33} \end{pmatrix}. \tag{4.4}$$

Therefore, comparing equations (4.3) and (4.4), we have $X' = PX$. □

Corollary 4.5. *Concurrence of lines is preserved under the action of SL$(3,\mathbb{R})$.*

Proof. A set of lines is said to be concurrent if they all intersect at the same point.
Suppose the lines L_1 and L_2 meet at $X = L_1 \times L_2$. Here, under $P \in \mathrm{SL}(3,\mathbb{R})$ action, the lines L_1 and L_2 are transformed to lines $(P^{-1})^t L_1$ and $(P^{-1})^t L_2$, respectively. The intersecting point X' of these transformed lines is given by $X' = (P^{-1})^t L_1 \times (P^{-1})^t L_2$, with $\times$ representing the cross product in $\mathbb{R}^3$. After performing explicit calculations on the coordinates of $(P^{-1})^t L_1 \times (P^{-1})^t L_2$, we obtain $X' = PX$. □

Therefore, from Corollary 4.2 and Corollary 4.5, it follows that the incidence relation of points and lines is preserved under the action of $\mathrm{SL}(3,\mathbb{R})$. Hence, we can conclude that duality between points and lines in the projective plane is preserved under $\mathrm{SL}(3,\mathbb{R})$ action. In particular, in the space $\mathbb{RP}^2$, the projective duality intertwines the action by $g \in \mathrm{SL}(3,\mathbb{R})$ to the dual action by $(g^{-1})^t \in \mathrm{SL}(3,\mathbb{R})$.

5 Cross-ratio and SL$(3,\mathbb{R})$ action

In addition to plane duality, one of the fundamental concepts in the projective plane is the cross-ratio of four collinear points. In this section, we present two proofs to establish that the cross-ratio of four collinear points is preserved under the $\mathrm{SL}(3,\mathbb{R})$ action.

Theorem 5.1. *Let K_X, under the action of SL$(3,\mathbb{R})$ on $\mathbb{RP}^2$, be the isotropy subgroup of the line $X = 0$. Then, the action of K_X on the line $X = 0$ coincides with the action of SL$(2,\mathbb{R})$ on the real line.*

Proof. Any line $aX + bY + cZ = 0$ in $\mathbb{RP}^2$ is determined by the homogeneous coordinates $L = (a,b,c)^t$. Thus, the line $L_\circ$ defined by $X = 0$ can be written as $1 \cdot X + 0 \cdot Y + 0 \cdot Z = 0$ or, in matrix notation $p^t L_\circ = 0$, where $L_\circ = (1,0,0)^t$ and $p = (0,Y,Z)^t$ are the homogeneous coordinates of the line $L_\circ$ and the point p, respectively.

Now, under the action of $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SL}(3,\mathbb{R})$, any point p in the line $L_\circ$ maps to $g \cdot p$, whereas the line $L_\circ$ maps to $(g^{-1})^t L_\circ$ (see Corollary 4.2).
Therefore, the isotropy subgroup of the line $X = 0$ is determined by

$$K_x = \left\{ g \in \mathrm{SL}(3,\mathbb{R}) \middle| (g^{-1})^t L_\circ = L_\circ \right\}.$$

Hence, $g \in K_x \Leftrightarrow g \cdot p = q,$ where $p,\ q \in L_\circ$. Thus,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ Y' \\ Z' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_{12}Y + a_{13}Z \\ a_{22}Y + a_{23}Z \\ a_{32}Y + a_{33}Z \end{pmatrix} = \begin{pmatrix} 0 \\ Y' \\ Z' \end{pmatrix}$$

$$\Rightarrow a_{12}Y + a_{13}Z = 0 \text{ for all Y, Z.}$$

$$\Rightarrow a_{12} = 0 \text{ and } a_{13} = 0.$$

Therefore, the subgroup of $\mathrm{SL}(3,\mathbb{R})$ which fixes the line $X = 0$, is given by

$$K_x = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \middle| a_{22}a_{33} - a_{32}a_{23} = \tfrac{1}{a_{11}},\ a_{11} \neq 0 \right\}.$$

Let us now consider the action of K_x on the line $X = 0$. For any point p in $L_\circ$, we have

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ a_{22}Y + a_{23}Z \\ a_{32}Y + a_{33}Z \end{pmatrix}.$$

Given that each point on the line $X = 0$ has the form $(0, Y, Z)^t$, each point on that line can be uniquely interpreted as $(Y, Z)^t$, [11]. Therefore the action of K_x on the line $X = 0$ is given by

$$\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \cdot w = \frac{a_{22}w + a_{23}}{a_{32}w + a_{33}}, \text{ for } w = (Y, Z)^t \in L_\circ \text{ and } a_{22}a_{33} - a_{32}a_{23} \neq 0,$$

which is same as the $\mathrm{SL}(2,\mathbb{R})$ action on the real line. □

Corollary 5.2. *Under the projective transformation, any three distinct points x, y, z of $X = 0$ can be uniquely mapped to the points* $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}$.

Proof. Let $(0, X_1, X_2)^t$, $(0, Y_1, Y_2)^t$ and $(0, Z_1, Z_2)^t$ be the homogeneous coordinates of three distinct points x, y, z of $X = 0$, respectively. Since, $(0, X_1, X_2)^t$, $(0, Y_1, Y_2)^t$ and $(0, Z_1, Z_2)^t$ are distinct points, we have $Y_1Z_2 - Y_2Z_1 \neq 0$, $X_1Z_2 - X_2Z_1 \neq 0$, and $X_1Y_2 - X_2Y_1 \neq 0$.

Now, there is a 3×3 non singular matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\dfrac{c_1Y_2}{Y_1Z_2 - Y_2Z_1} & \dfrac{c_1Y_1}{Y_1Z_2 - Y_2Z_1} \\ 0 & -\dfrac{c_2X_2}{X_1Z_2 - X_2Z_1} & \dfrac{c_2X_1}{X_1Z_2 - X_2Z_1} \end{pmatrix}$ such that

$$M \cdot \begin{pmatrix} 0 \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; M \cdot \begin{pmatrix} 0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } M \cdot \begin{pmatrix} 0 \\ Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}.$$

Hence, the result follows. □

Lemma 5.3. *Under the $SL(3,\mathbb{R})$ action, the case of any line L mapping to line L' can be reduced to the case of line $L_\circ$ mapping to line L'' by conjugation, where $L_\circ$ is the line $X = 0$ and L'' is some line depending on L and L'.*

Proof. Let under the action of $g \in \mathrm{SL}(3,\mathbb{R})$, the line L maps to L'. Therefore $L' = (g^{-1})^t L$ (see Corollary 4.2). Now, we consider the line $L_\circ$, where $L_\circ$ is the line $X = 0$. Since $\mathrm{SL}(3,\mathbb{R})$ acts transitively on $\mathbb{RP}^2$, there is some $h \in \mathrm{SL}(3,\mathbb{R})$ such that $L_\circ$ maps to L, which means $L = (h^{-1})^t L_0$.
We take $L'' = h^t L' = (g^{-1}h)^t L$, then it follows that $L_\circ$ maps to L'' and $L'' = (k^{-1})^t L_\circ$, where $k = h^{-1}gh \in \mathrm{SL}(3,\mathbb{R})$ is conjugate to g and L'' is some line depending on L and L'.
Therefore, we can reduce any line L mapping to L' to the case of line $L_\circ$ mapping to L'' by conjugation. □

Lemma 5.4. *Let $L_\circ$ be the line $X = 0$. If the line $L_\circ$ maps to line L'' under the action of $g_\circ \in SL(3,\mathbb{R})$, then the action can be factored as $g_\circ h$ action, where $h \in K_X$, K_X being the isotropy subgroup of the line $L_\circ$.*

Proof. This follows from Lemma 5.3 and the observation that

$$\begin{aligned} L'' &= ({g_\circ}^{-1})^t L_\circ \text{ (see Corollary 4.2)} \\ &= ({g_\circ}^{-1})^t (h^{-1})^t L_\circ \; (\because h \in K_X \text{ implies } (h^{-1})^t L_\circ = L_\circ) \\ &= ((g_\circ h)^{-1})^t L_\circ. \end{aligned}$$

□

Theorem 5.5. *Cross-ratio of four collinear points is invariant under SL$(3,\mathbb{R})$ action.*

Proof. We need to show that for any $g \in \mathrm{SL}(3,\mathbb{R})$ and for any four collinear points a, b, c, d in a line L, $(ga, gb; gc, gd) = (a, b; c, d)$.
Recalling the definition of the cross-ratio (see Definition 2.10), we have that if L is any line in $\mathbb{RP}^2$ with two points p and q such that the points on the line are represented by $\mu p + \lambda q$, then the cross-ratio of four points $K_i = \mu_i p + \lambda_i q$, $(i = 1,\ 2,\ 3,\ 4)$ on this line is defined as

$$(K_1, K_2; K_3, K_4) = \frac{[k_1, k_3][k_2, k_4]}{[k_2, k_3][k_1, k_4]}, \tag{5.1}$$

where $k_i = \begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix}$ for $i = 1,\ 2,\ 3,\ 4$. Now, by the Remark 2.11, we see that the value of the expression $\frac{[k_1, k_3][k_2, k_4]}{[k_2, k_3][k_1, k_4]}$ is invariant under projective transformations, i.e.,

$$(M \cdot K_1,\ M \cdot K_2;\ M \cdot K_3,\ M \cdot K_4) = (K_1,\ K_2;\ K_3,\ K_4)\,, \text{ for non singular matrix } M.$$

Therefore, for the proof, instead of any line L, we can take the line as $X = 0$ (using the Lemma 5.4) and choose the points on it as $a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ c_1 \\ c_2 \end{pmatrix}$ by using the Corollary 5.2.

We also take the point $d = \begin{pmatrix} 0 \\ d_1 \\ d_2 \end{pmatrix}$ on the line $X = 0$.

As $a = (0,1,0)^t$ and $b = (0,0,1)^t$ are two distinct points on the line $X = 0$, we now describe all the points on this line through these points, which are given by

$$\{\mu a + \lambda b \mid \mu,\ \lambda \in \mathbb{R} \text{ with } \mu \text{ or } \lambda \text{ non zero}\},\ \text{(cf. Lemma 2.8)}.$$

Hence, we get

$$a = 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad b = 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$c = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad d = d_1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + d_2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, by using the definition (see equation (5.1)),

$$(a,\ b;\ c,\ d) = \frac{\det\begin{pmatrix} 1 & c_1 \\ 0 & c_2 \end{pmatrix} \det\begin{pmatrix} 0 & d_1 \\ 1 & d_2 \end{pmatrix}}{\det\begin{pmatrix} 0 & c_1 \\ 1 & c_2 \end{pmatrix} \det\begin{pmatrix} 1 & d_1 \\ 0 & d_2 \end{pmatrix}} = \frac{c_2(-d_1)}{(-c_1)d_2} = \frac{c_2 d_1}{c_1 d_2}. \tag{5.2}$$

Now, under the action of $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SL}(3,\mathbb{R})$, the points $a,\ b,\ c,\ d$ in L map to

$$g \cdot a = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix},\ g \cdot b = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},\ g \cdot c = \begin{pmatrix} c_1 a_{12} + c_2 a_{13} \\ c_1 a_{22} + c_2 a_{23} \\ c_1 a_{32} + c_2 a_{33} \end{pmatrix},\ g \cdot d = \begin{pmatrix} d_1 a_{12} + d_2 a_{13} \\ d_1 a_{22} + d_2 a_{23} \\ d_1 a_{32} + d_2 a_{33} \end{pmatrix},$$

respectively in L''. In this case, the line L'' can be expressed as the linear span of two points $g \cdot a$ and $g \cdot b$ on it, see Lemma 2.8. Thus,

$$g \cdot a = 1 \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + 0 \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},\quad g \cdot b = 0 \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + 1 \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},$$
$$g \cdot c = c_1 \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + c_2 \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix},\quad g \cdot d = d_1 \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + d_2 \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}.$$

Therefore, calculating cross-ratio with respect to the coordinate points $g \cdot a$ and $g \cdot b$ of the line L'', we get

$$(g \cdot a,\ g \cdot b;\ g \cdot c,\ g \cdot d) = \frac{\det\begin{pmatrix} 1 & c_1 \\ 0 & c_2 \end{pmatrix} \det\begin{pmatrix} 0 & d_1 \\ 1 & d_2 \end{pmatrix}}{\det\begin{pmatrix} 0 & c_1 \\ 1 & c_2 \end{pmatrix} \det\begin{pmatrix} 1 & d_1 \\ 0 & d_2 \end{pmatrix}} = \frac{c_2(-d_1)}{(-c_1)d_2} = \frac{c_2 d_1}{c_1 d_2}. \tag{5.3}$$

Therefore, from equations (5.2) and (5.3), we obtain

$$(a,\ b;\ c,\ d) = (ga,\ gb;\ gc,\ gd).$$

Thus, cross-ratio is invariant under $\mathrm{SL}(3,\mathbb{R})$ action. □

Alternative proof of the Theorem 5.5

Here we try to give an alternative proof of the result that cross-ratio is invariant under SL$(3,\mathbb{R})$ action by using quotients of 3×3 determinants.

Lemma 5.6. *Let $[a']$, $[b']$, $[c']$, and $[d']$ be four distinct points in $\mathbb{RP}^2$, with no three of them lying on the same line. Similarly, let $[a'']$, $[b'']$, $[c'']$, and $[d'']$ be another set of four distinct points in $\mathbb{RP}^2$, with no three of them collinear as well. Then, there exists matrix $T\in SL(3,\mathbb{R})$ such that $T\cdot[a']=[a'']$, $T\cdot[b']=[b'']$, $T\cdot[c']=[c'']$, and $T\cdot[d']=[d'']$, (see [11]).*

Proof. Let $a',\ b',\ c',\ d',\ a'',\ b'',\ c'',\ d''\in\mathbb{R}^3$ be the representatives of the corresponding equivalence classes. Now, we first consider the case where $a'=(1,0,0),\ b'=(0,1,0),\ c'=(0,0,1)$, and $d'=(1,1,1)$. Also, we take $a''=(a_1,a_2,a_3),\ b''=(b_1,b_2,b_3),\ c''=(c_1,c_2,c_3)$, and $d''=(d_1,d_2,d_3)$. Thus, the matrix must have the form $T=\begin{pmatrix}\lambda a_1 & \mu b_1 & \tau c_1\\ \lambda a_2 & \mu b_2 & \tau c_2\\ \lambda a_3 & \mu b_3 & \tau c_3\end{pmatrix}$, since the columns are the images of the unit vectors.
Therefore, the image of d' is $(\lambda a_1+\mu b_1+\tau c_1,\ \lambda a_2+\mu b_2+\tau c_2,\ \lambda a_3+\mu b_3+\tau c_3)$ and hence, this must be a multiple of d''. Now, to find the appropriate parameters λ, μ, τ, we solve the system of linear equations

$$\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\cdot\begin{pmatrix}\lambda\\ \mu\\ \tau\end{pmatrix}=\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}.$$

As $a'',\ b'',\ c''$ are non-collinear, the matrix with column $a'',\ b'',\ c''$ has non-zero determinant. Therefore, this system has a solution, namely $\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}^{-1}\cdot\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}$. Further, none of the parameters is zero as no three of $a'',\ b'',\ c''$, and d'' are collinear and hence the determinants of the 3×3 matrices formed by these vectors become non-zero.

We next use this part to find a matrix T_1, which transforms $(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1)$ into the points $a',\ b',\ c',\ d'$, respectively. Additionally, we calculate matrix T_2, which maps $(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,1,1)$ to $a'',\ b'',\ c'',\ d''$. Therefore, the required matrix is $T_2T_1^{-1}$. □

Lemma 5.7. *Consider four collinear points $a',\ b',\ c',\ d'$ in $\mathbb{RP}^2$ and let p be a point not on the line. Then the cross-ratio can be calculated as*

$$(a',b';c',d')=\frac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']}. \tag{5.4}$$

Proof. We can see that the expression on the right-hand side of equation (5.4), specifically $\dfrac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']}$, is invariant under projective transformations, as the Remark 2.11 suggests.

Hence, by the Lemma 5.6, we may consider four points $a'=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $b'=\begin{pmatrix}0\\0\\1\end{pmatrix}$, $p=\begin{pmatrix}1\\0\\0\end{pmatrix}$, and $q=\begin{pmatrix}1\\1\\1\end{pmatrix}$ such that no three are collinear.

Next, we take the four collinear points a', b', c', and d'. Thus, under the collinearity assumption, the points c' and d' have coordinates as $c'=\begin{pmatrix}0\\c_1\\c_2\end{pmatrix}$, $d'=\begin{pmatrix}0\\d_1\\d_2\end{pmatrix}$.

Therefore, as required, we have four collinear points a', b', c', d' in $\mathbb{RP}^2$ and p be a point not on this line. Now, explicitly calculating the values of both sides of the equation (5.4), we get

$$(a',b';c',d') = \frac{\det\begin{pmatrix}1 & c_1\\ 0 & c_2\end{pmatrix}\det\begin{pmatrix}0 & d_1\\ 1 & d_2\end{pmatrix}}{\det\begin{pmatrix}0 & c_1\\ 1 & c_2\end{pmatrix}\det\begin{pmatrix}1 & d_1\\ 0 & d_2\end{pmatrix}} \text{ (same as in Theorem 5.5)}$$
$$= \frac{c_2(-d_1)}{(-c_1)d_2} = \frac{c_2 d_1}{c_1 d_2}.$$

Again,

$$\frac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']} = \frac{\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & c_1\\ 0 & 0 & c_2\end{pmatrix}\det\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & d_1\\ 0 & 1 & d_2\end{pmatrix}}{\det\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & c_1\\ 0 & 1 & c_2\end{pmatrix}\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & d_1\\ 0 & 0 & d_2\end{pmatrix}} \text{ (see the Notation 2.1)}$$
$$= \frac{\begin{vmatrix}1 & c_1\\ 0 & c_2\end{vmatrix}\begin{vmatrix}0 & d_1\\ 1 & d_2\end{vmatrix}}{\begin{vmatrix}0 & c_1\\ 1 & c_2\end{vmatrix}\begin{vmatrix}1 & d_1\\ 0 & d_2\end{vmatrix}} = \frac{c_2(-d_1)}{(-c_1)d_2} = \frac{c_2 d_1}{c_1 d_2}.$$

Therefore, $(a',b';c',d') = \dfrac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']}$. □

Theorem 5.8. *The cross-ratio of four collinear points remains invariant under* $SL(3,\mathbb{R})$ *action.*

Proof. Let a', b', c', d' are the collinear points on a line and p be a point not on the same line. Then by the Lemma 5.7, $(a',b';c',d') = \dfrac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']}$.
Now under $M \in \mathrm{SL}(3,\mathbb{R})$ action, the cross-ratio becomes $(M\cdot a', M\cdot b'; M\cdot c', M\cdot d')$ (as $\mathrm{SL}(3,\mathbb{R})$ preserves collinearity).
Thus,

$$\begin{aligned}
&(M\cdot a', M\cdot b'; M\cdot c', M\cdot d')\\
&= \frac{[M\cdot p, M\cdot a', M\cdot c'][M\cdot p, M\cdot b', M\cdot d']}{[M\cdot p, M\cdot b', M\cdot c'][M\cdot p, M\cdot a', M\cdot d']} \text{ (by the Lemma 5.7)}\\
&= \frac{\det(M)^2[p,a',c'][p,b',d']}{\det(M)^2[p,b',c'][p,a',d']} \text{ (see the Remark 2.11)}\\
&= \frac{[p,a',c'][p,b',d']}{[p,b',c'][p,a',d']} \ (\because \det(M) = 1)\\
&= (a',b';c',d').
\end{aligned}$$

Hence, cross-ratio remains invariant under $\mathrm{SL}(3,\mathbb{R})$ action. □

Therefore, we obtain the same value of cross-ratio using two different approaches, demonstrating that cross-ratio is indeed invariant under the $\mathrm{SL}(3,\mathbb{R})$ action.

6 Conclusion

We have discussed the action of the transformation group $\mathrm{SL}(3,\mathbb{R})$ on the two-dimensional homogeneous space $\mathbb{RP}^2$. Here, we have focused on the projective invariant and obtained that both the plane duality and the cross-ratio of four collinear points remain invariant under the group action.

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