**Palestine Journal of Mathematics**

**Vol 13(3)(2024)** , 415–427 

# SUMS OF THE POWERS OF BI-PERIODIC FIBONACCI AND LUCAS NUMBERS

N. Belaggoun and H. Belbachir

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11B39; Secondary 05A15; 11B65.

Keywords and phrases: Bi-periodic Fibonacci and Lucas numbers, Binet's formula, Melham's sums, generating function.

**Abstract** In this paper, we establish various formulas for the sums of powers of bi-periodic Fibonacci and Lucas numbers, along with their associated generating functions.

## 1 Introduction

For any integer $n \geq 2$, the well-known Fibonacci and Lucas numbers are defined by the following recurrence relations

$$F_n = F_{n-1} + F_{n-2} \quad \text{and} \quad L_n = L_{n-1} + L_{n-2},$$

with initial conditions $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$, respectively. Several recent studies have explored generalizations of Fibonacci and Lucas numbers along with their properties [2, 3, 4, 7, 9, 10, 14, 19]. Specifically, Edson and Yayenie [7] defined the bi-periodic Fibonacci numbers for any integer $n \geq 2$ as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even,} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \tag{1.1}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where $a$ and $b$ are nonzero reals numbers. Note that if $a = b = 1$, then $q_n$ is the $n$th Fibonacci number. Afterward, Bilgici [4] defined the bi-periodic Lucas numbers for any integer $n \geq 2$ as follows

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even,} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \tag{1.2}$$

with initial values $l_0 = 2$ and $l_1 = a$. When $a = b = 1$, we obtain the classic Lucas numbers. The properties of bi-periodic Fibonacci and Lucas numbers have been investigated through diverse approaches in various studies [1, 11, 16].

Their Binet formulas are expressed as follows

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor n/2 \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

and

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor (n+1)/2 \rfloor}} \left( \alpha^n + \beta^n \right),$$

where $\lfloor x \rfloor$ denotes the floor function of $x$, $\xi(n) = n - 2\lfloor n/2 \rfloor$ is the parity function (i.e., $\xi(n) = 0$ when $n$ is even and $\xi(n) = 1$ when $n$ is odd), and $\alpha$ and $\beta$ denote the roots of the characteristic equation $x^2 - abx - ab = 0$, given by

$$\alpha = \frac{ab + \sqrt{ab(ab+4)}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{ab(ab+4)}}{2}.$$

The evaluation of sums involving powers of Fibonacci and Lucas numbers poses a challenging problem that has interested mathematicians for many years. Wiemann and Cooper [18] reported certain conjectures proposed by Melham related to the sum $\sum_{k=1}^{n} F_{2k}^{2m+1}$ as given in [12], where $m$ is a positive integer. Subsequently, Ozeki [13] provided an explicit expansion for the sum $\sum_{k=1}^{n} F_{2k}^{2m+1}$ as a polynomial in powers of $F_{2n+1}$. Prodinger [15] independently obtained similar result and derived general expansion formulas for sums

$$\sum_{k=0}^{n} F_{2k+\delta}^{2m+\varepsilon} \quad \text{and} \quad \sum_{k=0}^{n} L_{2k+\delta}^{2m+\varepsilon}, \quad \text{where} \quad \varepsilon, \delta \in \{0,1\}.$$

In this paper, we initially derive explicit formulas for the following sums

$$\sum_{k=0}^{n} q_{2k+\delta}^{2m+\varepsilon} \quad \text{and} \quad \sum_{k=0}^{n} l_{2k+\delta}^{2m+\varepsilon}, \text{ where } m \text{ is a positive integer and } \varepsilon, \delta \in \{0,1\}.$$

These formulas extend Melham's sums to the powers of the bi-periodic Fibonacci and Lucas numbers while also introducing new expressions for the classical Fibonacci and Lucas numbers. Subsequently, we establish the generating functions for the powers of the bi-periodic Fibonacci and Lucas sequences. To prove the main results, we require the following lemmas.

**Lemma 1.1.** *For any real numbers $x$ and $y$, we have*

$$x^{2k} - y^{2k} = (x-y) \sum_{j=0}^{k-1} \binom{k+j}{2j+1} (-xy)^{k-j-1} (x+y)^{2j+1}, \tag{1.3}$$

$$x^{2k+1} - y^{2k+1} = (x-y) \sum_{j=0}^{k} \binom{k+j}{2j} (-xy)^{k-j} (x+y)^{2j}. \tag{1.4}$$

*Proof.* For any integer $n \geq 1$ and real numbers $x$ and $y$, we have the following identity (see [5, page 23]),

$$\frac{x^n - y^n}{x-y} = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (-1)^j \binom{n-j-1}{j} (x+y)^{n-2j-1} (xy)^j.$$

We obtain the results by considering even and odd values of $n$ separately. □

**Lemma 1.2.** *For any real numbers $x$ and $y$, we have*

$$x^{2k+1} + y^{2k+1} = \sum_{j=0}^{k} \frac{2k+1}{2j+1} \binom{k+j}{2j} (-xy)^{k-j} (x+y)^{2j+1}. \tag{1.5}$$

*Proof.* For any integer $n \geq 1$ and real numbers $x$ and $y$, we have the following identity (see [6, Section 4.9], [8, 17])

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (x+y)^{n-2j} (xy)^j.$$

We achieve the result by specifically considering the odd values of $n$. □

## 2 Melham's sums for bi-periodic Fibonacci numbers

In this section, we will provide the sums of the powers of the bi-periodic Fibonacci numbers.

**Theorem 2.1.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=1}^{n} q_{2k}^{2m+1} = \frac{a^{m+1}}{b^{m+1}(ab+4)^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{q_{(2n+1)(2m+1-2j)} - q_{2m+1-2j}}{l_{2m+1-2j}}, \tag{2.1}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m+1} = \frac{1}{(ab+4)^m} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{q_{2(n+1)(2m+1-2j)}}{l_{2m+1-2j}}, \tag{2.2}$$

$$\sum_{k=1}^{n} q_{2k}^{2m} = \frac{a^m}{b^m(ab+4)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} + \frac{(-1)^m a^m}{b^m(ab+4)^m} \binom{2m}{m} (n+\frac{1}{2}), \tag{2.3}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m} = \frac{1}{(ab+4)^m} \sum_{j=0}^{m-1} \binom{2m}{j} \frac{q_{2(n+1)(2m-2j)}}{q_{2m-2j}} + \frac{n+1}{(ab+4)^m} \binom{2m}{m}. \tag{2.4}$$

*Proof.* Using Binet's formulas of the bi-periodic Fibonacci and Lucas numbers and since $\alpha\beta = -ab$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} q_{2k}^{2m+1} \\
&= \left(\frac{a}{\alpha-\beta}\right)^{2m+1} \sum_{k=1}^{n} \left(\frac{\alpha^{2k}-\beta^{2k}}{(ab)^k}\right)^{2m+1} \\
&= \frac{a^{2m+1}}{(\alpha-\beta)^{2m+1}} \sum_{k=1}^{n} \sum_{j=0}^{2m+1} (-1)^{j+1} \binom{2m+1}{j} \frac{\alpha^{2kj}\beta^{2k(2m+1-j)}}{(ab)^{k(2m+1)}} \\
&= \frac{a^{2m+1}}{(\alpha-\beta)^{2m+1}} \sum_{k=1}^{n} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{(\alpha\beta)^{2kj}}{(ab)^{k(2m+1)}} \left(\alpha^{2k(2m+1-2j)} - \beta^{2k(2m+1-2j)}\right) \\
&= \frac{a^{2m+1}}{(\alpha-\beta)^{2m+1}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \sum_{k=1}^{n} \left(\left(\frac{\alpha^2}{ab}\right)^{k(2m+1-2j)} - \left(\frac{\beta^2}{ab}\right)^{k(2m+1-2j)}\right) \\
&= \frac{a^{2m+1}}{(\alpha-\beta)^{2m+1}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \left(\frac{\left(\frac{\alpha^2}{ab}\right)^{(n+1)(2m+1-2j)} - 1}{\left(\frac{\alpha^2}{ab}\right)^{(2m+1-2j)} - 1} - \frac{\left(\frac{\beta^2}{ab}\right)^{(n+1)(2m+1-2j)} - 1}{\left(\frac{\beta^2}{ab}\right)^{(2m+1-2j)} - 1}\right) \\
&= \frac{a^{2m+1}}{(\alpha-\beta)^{2m+1}} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{\frac{\alpha^{(2n+1)(2m+1-2j)} - \beta^{(2n+1)(2m+1-2j)}}{(ab)^{n(2m+1-2j)}} + \beta^{2m+1-2j} - \alpha^{2m+1-2j}}{\alpha^{2m+1-2j} + \beta^{2m+1-2j}} \\
&= \frac{a^{m+1}}{b^{m+1}(ab+4)^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{q_{(2n+1)(2m+1-2j)} - q_{2m+1-2j}}{l_{2m+1-2j}}.
\end{aligned}$$

We perform a similar operation on the odd elements, yielding

$$\sum_{k=0}^{n} q_{2k+1}^{2m+1} = \frac{1}{(ab+4)^m} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{q_{2(n+1)(2m+1-2j)}}{l_{2m+1-2j}}.$$

Now, considering even power and using Binet's formula for the number $q_n$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} q_{2k}^{2m} &= \sum_{k=1}^{n} \frac{a^{2m}}{(ab)^{2km}} \left(\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta}\right)^{2m} \\
&= \left(\frac{a}{\alpha-\beta}\right)^{2m} \sum_{k=1}^{n} \frac{1}{(ab)^{2km}} \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \alpha^{2kj} \beta^{2k(2m-j)} \\
&= \left(\frac{a}{\alpha-\beta}\right)^{2m} \sum_{k=1}^{n} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{(\alpha\beta)^{2kj}}{(ab)^{2km}} \left(\alpha^{2k(2m-2j)} + \beta^{2k(2m-2j)}\right) \\
&\quad + (-1)^m n \left(\frac{a}{\alpha-\beta}\right)^{2m} \binom{2m}{m} \\
&= \frac{a^m}{b^m (a^2b^2+4ab)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \sum_{k=1}^{n} \left( \left(\frac{\alpha^2}{ab}\right)^{k(2m-2j)} + \left(\frac{\beta^2}{ab}\right)^{k(2m-2j)} \right) \\
&\quad + \frac{(-1)^m a^m n}{b^m (a+4)^m} \binom{2m}{m} \\
&= \left(\frac{a}{b}\right)^m \frac{1}{(ab+4)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \left( \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} - 1 \right) + \frac{(-1)^m a^m n}{b^m (a+4)^m} \binom{2m}{m} \\
&= \frac{a^m}{b^m (ab+4)^m} \left( \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} + (-1)^m \binom{2m-1}{m-1} \right) \\
&\quad + \frac{(-1)^m a^m n}{b^m (a+4)^m} \binom{2m}{m} \\
&= \frac{a^m}{b^m (ab+4)^m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} + \frac{(-1)^m a^m}{b^m (ab+4)^m} \binom{2m}{m} (n + \frac{1}{2}).
\end{aligned}$$

We perform a similar operation on the odd elements, yielding

$$\sum_{k=0}^{n} q_{2k+1}^{2m} = \frac{1}{(ab+4)^m} \sum_{j=0}^{m-1} \binom{2m}{j} \frac{q_{2(n+1)(2m-2j)}}{q_{2m-2j}} + \frac{n+1}{(ab+4)^m} \binom{2m}{m}.$$

□

Note that, for $a = b = 1$, we obtain the sums of the powers of the classical Fibonacci numbers (see [15]).

The sums of the powers of the bi-periodic Fibonacci numbers can be expressed in terms of the powers of the numbers themselves.

**Theorem 2.2.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\begin{aligned}
\sum_{k=1}^{n} q_{2k}^{2m+1} &= \left(\frac{a}{b}\right)^{m+1} \sum_{k=0}^{m} q_{2n+1}^{2k+1} \sum_{j=0}^{m-k} (-1)^{m-k} \frac{(ab+4)^{k-m}}{l_{2m+1-2j}} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{2m+1-2j}{2k+1} \\
&\quad - \left(\frac{a}{b}\right)^{m+1} \frac{1}{(ab+4)^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{q_{2m+1-2j}}{l_{2m+1-2j}},
\end{aligned} \tag{2.5}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m+1} = \sum_{k=0}^{m} q_{2(n+1)}^{2k+1} \sum_{j=0}^{m-k} \left(\frac{b}{a}\right)^{k} \frac{(ab+4)^{k-m}}{l_{2m+1-2j}} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{2m+1-2j}{2k+1}, \tag{2.6}$$

$$\sum_{k=1}^{n} q_{2k}^{2m} = \left(\frac{a}{b}\right)^m l_{2n+1} \sum_{k=0}^{m-1} q_{2n+1}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^{m-k-1} \binom{2m}{j} \binom{m-j+k}{2k+1} (ab+4)^{k-m} + \frac{(-1)^m a^m}{b^m (ab+4)^m} \binom{2m}{m} \left(n+\frac{1}{2}\right), \tag{2.7}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m} = l_{2(n+1)} \sum_{k=0}^{m-1} q_{2(n+1)}^{2k+1} \sum_{j=0}^{m-k-1} \left(\frac{b}{a}\right)^k \frac{(ab+4)^{k-m}}{q_{2m-2j}} \binom{2m}{j} \binom{m-j+k}{2k+1} + \left(\frac{a}{b}\right)^{m+1} \frac{n+1}{(ab+4)^m} \binom{2m}{m}. \tag{2.8}$$

*Proof.* Taking $x = \frac{\alpha^{2n+1}}{\sqrt{ab}(ab)^n}$ and $y = -\frac{\beta^{2n+1}}{\sqrt{ab}(ab)^n}$ in (1.5), we derive the following formula

$$q_{(2n+1)(2m-2j+1)} = \sum_{k=0}^{m-j} (-1)^{m-j-k} \frac{2m+1-2j}{2k+1} \binom{m-j+k}{2k} (ab+4)^k \, q_{2n+1}^{2k+1}. \tag{2.9}$$

Substituting (2.9) into (2.1), we get

$$\begin{aligned}\sum_{k=1}^{n} q_{2k}^{2m+1} &= \frac{a^{m+1}}{b^{m+1}(ab+4)^m} \sum_{j=0}^{m} \binom{2m+1}{j} \sum_{k=0}^{m-j} (-1)^{m-k} \frac{2m+1-2j}{(2k+1) l_{2m+1-2j}} \binom{m-j+k}{2k} \\ &\quad \times (ab+4)^k \, q_{2n+1}^{2k+1} - \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{q_{2m+1-2j}}{l_{2m+1-2j}} \\ &= \left(\frac{a}{b}\right)^{m+1} \sum_{k=0}^{m} q_{2n+1}^{2k+1} \sum_{j=0}^{m-k} (-1)^{m-k} \frac{(ab+4)^{k-m}}{l_{2m+1-2j}} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{2m+1-2j}{2k+1} \\ &\quad - \left(\frac{a}{b}\right)^{m+1} \frac{1}{(ab+4)^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{q_{2m+1-2j}}{l_{2m+1-2j}}.\end{aligned}$$

Taking $x = \frac{\alpha^{2(n+1)}}{(ab)^{n+1}}$ and $y = -\frac{\beta^{2(n+1)}}{(ab)^{n+1}}$ in (1.5), we obtain

$$q_{2(n+1)(2m+1-2j)} = \sum_{k=0}^{m-j} \frac{2m+1-2j}{2k+1} \binom{m-j+k}{2k} \left(\frac{b}{a}\right)^k (ab+4)^k \, q_{2(n+1)}^{2k+1}. \tag{2.10}$$

Substituting (2.10) into (2.2), we get

$$\begin{aligned}\sum_{k=0}^{n} q_{2k+1}^{2m+1} &= \frac{1}{(ab+4)^m} \sum_{j=0}^{m} \binom{2m+1}{j} \sum_{k=0}^{m-j} \frac{2m+1-2j}{2k+1} \binom{m-j+k}{2k} \left(\frac{b}{a}\right)^k \frac{(ab+4)^k}{l_{2m+1-2j}} q_{2(n+1)}^{2k+1} \\ &= \sum_{k=0}^{m} q_{2(n+1)}^{2k+1} \sum_{j=0}^{m-k} \left(\frac{b}{a}\right)^k \frac{(ab+4)^{k-m}}{l_{2m+1-2j}} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{2m+1-2j}{2k+1}.\end{aligned}$$

Taking $x = \frac{\alpha^{2n+1}}{\sqrt{ab}(ab)^n}$ and $y = -\frac{\beta^{2n+1}}{\sqrt{ab}(ab)^n}$ in (1.3), we obtain

$$q_{(2n+1)(2m-2j)} = l_{2n+1} \sum_{k=0}^{m-j-1} (-1)^{m-j-k-1} \binom{m-j+k}{2k+1} (ab+4)^k q_{2n+1}^{2k+1}. \tag{2.11}$$

Substituting (2.11) into (2.3), we get

$$\sum_{k=1}^{n} q_{2k}^{2m} = \frac{a^m}{b^m(ab+4)^m}\sum_{j=0}^{m-1}\binom{2m}{j}\frac{l_{2n+1}}{q_{2m-2j}}\sum_{k=0}^{m-j-1}(-1)^{m-k-1}\binom{m-j+k}{2k+1}(ab+4)^k q_{2n+1}^{2k+1}$$
$$+\frac{(-1)^m a^m}{b^m(ab+4)^m}\binom{2m}{m}(n+\frac{1}{2})$$
$$=\left(\frac{a}{b}\right)^m l_{2n+1}\sum_{k=0}^{m-1} q_{2n+1}^{2k+1}\sum_{j=0}^{m-k-1}(-1)^{m-k-1}\binom{2m}{j}\binom{m-j+k}{2k+1}(ab+4)^{k-m}$$
$$+\frac{(-1)^m a^m}{b^m(ab+4)^m}\binom{2m}{m}\left(n+\frac{1}{2}\right).$$

Taking $x = \frac{\alpha^{2(n+1)}}{(ab)^{n+1}}$ and $y = -\frac{\beta^{2(n+1)}}{(ab)^{n+1}}$ in (1.3), we obtain

$$q_{2(n+1)(2m-2j)} = l_{2(n+1)}\sum_{k=0}^{m-j-1}\left(\frac{b}{a}\right)^k\binom{m-j+k}{2k+1}(ab+4)^k q_{2(n+1)}^{2k+1}. \tag{2.12}$$

Substituting (2.12) into (2.4), we get

$$\sum_{j=0}^{n} q_{2j+1}^{2m} = \frac{1}{(ab+4)^m}\sum_{k=0}^{m-1}\binom{2m}{k}\frac{l_{2(n+1)}}{q_{2m-2k}}\sum_{k=0}^{m-j-1}\left(\frac{b}{a}\right)^k\binom{m-j+k}{2k+1}(ab+4)^k q_{2(n+1)}^{2k+1}$$
$$+\frac{n+1}{(ab+4)^m}\binom{2m}{m}$$
$$= l_{2(n+1)}\sum_{k=0}^{m-1} q_{2(n+1)}^{2k+1}\sum_{j=0}^{m-k-1}\left(\frac{b}{a}\right)^k\frac{(ab+4)^{k-m}}{q_{2m-2j}}\binom{2m}{j}\binom{m-j+k}{2k+1}$$
$$+\left(\frac{a}{b}\right)^{m+1}\frac{n+1}{(ab+4)^m}\binom{2m}{m}.$$

□

The sums of the powers of bi-periodic Fibonacci numbers can be expressed in terms of the powers of bi-periodic Lucas numbers.

**Theorem 2.3.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=1}^{n} q_{2k}^{2m+1} = \frac{q_{2n+1}}{(ab+4)^m}\sum_{k=0}^{m} l_{2n+1}^{2k}\sum_{j=0}^{m-k}\frac{(-1)^j}{l_{2m+1-2j}}\left(\frac{a}{b}\right)^{m+1-k}\binom{2m+1}{j}\binom{m-j+k}{2k}$$
$$-\left(\frac{a}{b}\right)^{m+1}\frac{1}{(ab+4)^m}\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}\frac{q_{2m+1-2j}}{l_{2m+1-2j}}, \tag{2.13}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m+1} = \frac{q_{2n+2}}{(ab+4)^m}\sum_{k=0}^{m} l_{2n+2}^{2k}\sum_{j=0}^{m-k}\frac{(-1)^{m-j-k}}{l_{2m+1-2j}}\binom{2m+1}{j}\binom{m-j+k}{2k}, \tag{2.14}$$

$$\sum_{k=1}^{n} q_{2k}^{2m} = \frac{q_{2n+1}}{(ab+4)^m}\sum_{k=0}^{m-1} l_{2n+1}^{2k+1}\sum_{j=0}^{m-k-1}\frac{(-1)^j}{q_{2m-2j}}\left(\frac{a}{b}\right)^{m-k}\binom{2m}{j}\binom{m-j+k}{2k+1}$$
$$+\frac{a^{2m}(-1)^m}{(a^2b^2+4ab)^m}\binom{2m}{m}\left(n+\frac{1}{2}\right), \tag{2.15}$$

$$\sum_{k=0}^{n} q_{2k+1}^{2m} = \frac{q_{2n+2}}{(ab+4)^m}\sum_{k=0}^{m-1} l_{2n+2}^{2k+1}\sum_{k=0}^{m-k-1}\frac{(-1)^{m-j-k+1}}{q_{2m-2j}}\binom{2m}{j}\binom{m-j+k}{2k+1}$$
$$+\frac{n+1}{(ab+4)^m}\binom{2m}{m}. \tag{2.16}$$

*Proof.* Substituting $x = \frac{\alpha^{2n+1}}{\sqrt{ab}(ab)^n}$, $y = \frac{\beta^{2n+1}}{\sqrt{ab}(ab)^n}$, and $x = \frac{\alpha^{2(n+1)}}{(ab)^{n+1}}$, $y = \frac{\beta^{2(n+1)}}{(ab)^{n+1}}$ into equations (1.4) and (1.3) leads to the following respective formulas:

$$q_{(2n+1)(2m+1-2j)} = q_{2n+1} \sum_{k=0}^{m-j} \left(\frac{b}{a}\right)^k \binom{m-j+k}{2k} l_{2n+1}^{2k}, \tag{2.17}$$

$$q_{2(n+1)(2m+1-2j)} = q_{2(n+1)} \sum_{k=0}^{m-j} (-1)^{m-j-k} \binom{m-j+k}{2k} l_{2n+2}^{2k}, \tag{2.18}$$

$$q_{(2n+1)(2m-2j)} = q_{2n+1} \sum_{k=0}^{m-j-1} \left(\frac{b}{a}\right)^k \binom{m-j+k}{2k+1} l_{2n+1}^{2k+1}, \tag{2.19}$$

$$q_{2(n+1)(2m-2j)} = q_{2(n+1)} \sum_{k=0}^{m-j-1} (-1)^{m-j-k+1} \binom{m-j+k}{2k+1} l_{2n+2}^{2k+1}. \tag{2.20}$$

Substituting (2.17), (2.18), (2.19), and (2.20) into (2.1), (2.2), (2.3), and (2.4), respectively, yields the results. □

Note that if we substitute $a = b = 1$ into equations (2.7), (2.13), (2.14), and (2.16), we obtain the following results.

**Corollary 2.4.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=1}^{n} F_{2k}^{2m} = L_{2n+1} \sum_{k=0}^{m-1} F_{2n+1}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^{m-k-1} \binom{2m}{j} \binom{m-j+k}{2k+1} 5^{k-m} + \frac{(-1)^m}{5^m} \binom{2m}{m} \left(n + \frac{1}{2}\right),$$

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m} \sum_{k=0}^{m} L_{2n+1}^{2k} \sum_{j=0}^{m-k} (-1)^j \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{F_{2n+1}}{L_{2m+1-2j}} - \frac{1}{5^m} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{F_{2m+1-2j}}{L_{2m+1-2j}},$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m+1} = \frac{F_{2n+2}}{5^m} \sum_{k=0}^{m} L_{2n+2}^{2k} \sum_{j=0}^{m-k} \frac{(-1)^{m-j-k}}{L_{2m+1-2j}} \binom{2m+1}{j} \binom{m-j+k}{2k},$$

$$\sum_{k=0}^{n} F_{2k+1}^{2m} = \frac{F_{2n+2}}{5^m} \sum_{k=0}^{m-1} L_{2n+2}^{2k+1} \sum_{k=0}^{m-k-1} \frac{(-1)^{m-j-k+1}}{F_{2m-2j}} \binom{2m}{j} \binom{m-j+k}{2k+1} + \frac{n+1}{5^m} \binom{2m}{m}.$$

**Remark 2.5.** Note that if we take $a = b = 1$ in (2.5), (2.6), (2.8), and (2.15), we obtain closed formulas for the sums of the powers of the classical Fibonacci numbers (see [15]).

## 3 Melham's sums for bi-periodic Lucas numbers

In the following theorem, we provide closed formulas for the sums of the powers of bi-periodic Lucas numbers.

**Theorem 3.1.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=0}^{n} l_{2k}^{2m+1} = \sum_{j=0}^{m} \binom{2m+1}{j} \frac{l_{(2n+1)(2m+1-2j)}}{l_{2m+1-2j}} + 4^m, \tag{3.1}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m+1} = \left(\frac{a}{b}\right)^{m+1} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{l_{2(n+1)(2m+1-2j)} - 2}{l_{2m+1-2j}}, \tag{3.2}$$

$$\sum_{k=0}^{n} l_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} + 2^{2m-1} + \binom{2m}{m}(n + \frac{1}{2}), \tag{3.3}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m} = \left(\frac{a}{b}\right)^{m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{q_{2(n+1)(2m-2j)}}{q_{2m-2j}} + (-1)^m \left(\frac{a}{b}\right)^{m} \binom{2m}{m}(n+1). \tag{3.4}$$

*Proof.* From Binet's formula for the bi-periodic Lucas numbers, we write

$$\begin{aligned}
\sum_{k=0}^{n} l_{2k}^{2m+1} &= \sum_{k=0}^{n} \left(\frac{\alpha^{2k} + \beta^{2k}}{(ab)^k}\right)^{2m+1} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{2m+1}{j} \frac{(\alpha\beta)^{2kj}}{(ab)^{k(2m+1)}} \left(\alpha^{2k(2m+1-2j)} + \beta^{2k(2m+1-2j)}\right) \\
&= \sum_{j=0}^{m} \binom{2m+1}{j} \sum_{k=0}^{n} \left(\left(\frac{\alpha^2}{ab}\right)^{k(2m+1-2j)} + \left(\frac{\beta^2}{ab}\right)^{k(2m+1-2j)}\right) \\
&= \sum_{j=0}^{m} \binom{2m+1}{j} \left(\frac{\alpha^{(2n+1)(2m+1-2j)} + \beta^{(2n+1)(2m+1-2j)}}{(ab)^{n(2m+1-2j)} \left(\alpha^{2m+1-2j} + \beta^{2m+1-2j}\right)}\right) + \sum_{j=0}^{m} \binom{2m+1}{j} \\
&= \sum_{j=0}^{m} \binom{2m+1}{j} \frac{l_{(2n+1)(2m+1-2j)}}{l_{2m+1-2j}} + 4^m.
\end{aligned}$$

We perform a similar operation on the odd elements, resulting in

$$\sum_{k=0}^{n} l_{2k+1}^{2m+1} = \left(\frac{a}{b}\right)^{m+1} \sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} \frac{l_{2(n+1)(2m+1-2j)} - 2}{l_{2m+1-2j}}.$$

Similarly, using Binet's formulas for numbers $q_n$ and $l_n$, we obtain the following equations for the even power

$$\sum_{k=0}^{n} l_{2k}^{2m} = \sum_{j=0}^{m-1} \binom{2m}{j} \frac{q_{(2n+1)(2m-2j)}}{q_{2m-2j}} + 2^{2m-1} + \binom{2m}{m}(n + \frac{1}{2})$$

and

$$\sum_{k=0}^{n} l_{2k+1}^{2m} = \left(\frac{a}{b}\right)^{m} \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \frac{q_{2(n+1)(2m-2j)}}{q_{2m-2j}} + (-1)^m \left(\frac{a}{b}\right)^{m} \binom{2m}{m}(n+1).$$

□

Note that when $a = b = 1$, we obtain the sums of the powers of the classical Lucas numbers (see [15]).

The sums of the powers of bi-periodic Lucas numbers can be expressed in terms of the powers of the numbers themselves.

**Theorem 3.2.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=0}^{n} l_{2k}^{2m+1} = \sum_{k=0}^{n} l_{2n+1}^{2k+1} \sum_{j=0}^{m-k} \left(\frac{b}{a}\right)^k \frac{2m+1-2j}{2k+1} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{1}{l_{2m+1-2j}} + 4^m, \tag{3.5}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m+1} = \left(\frac{a}{b}\right)^{m+1} \sum_{k=0}^{m} l_{2n+2}^{2k+1} \sum_{j=0}^{m-k} \frac{(-1)^{m-k}}{2k+1} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{2m+1-2j}{l_{2m+1-2j}} - 2\left(\frac{a}{b}\right)^{m+1} \sum_{j=0}^{m} \frac{(-1)^j}{l_{2m+1-2j}} \binom{2m+1}{j}, \tag{3.6}$$

$$\sum_{k=0}^{n} l_{2k}^{2m} = \sum_{k=0}^{m-1} l_{2n+1}^{2k+1} \sum_{j=0}^{m-k-1} \left(\frac{b}{a}\right)^{k} \binom{2m}{j} \binom{m-j+k}{2k+1} \frac{q_{2n+1}}{q_{2m-2j}} + 2^{2m-1} + \binom{2m}{m}(n+\frac{1}{2}), \tag{3.7}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m} = \left(\frac{a}{b}\right)^{m} q_{2n+2} \sum_{k=0}^{m-1} l_{2n+2}^{2k+1} \sum_{j=0}^{m-k-1} \frac{(-1)^{m-k-1}}{q_{2m-2j}} \binom{2m}{j} \binom{m-j+k}{2k+1} + (-1)^m \left(\frac{a}{b}\right)^{m} \binom{2m}{m}(n+1). \tag{3.8}$$

*Proof.* Taking $x = \frac{\alpha^{2n+1}}{\sqrt{ab}(ab)^n}$, $y = \frac{\beta^{2n+1}}{\sqrt{ab}(ab)^n}$, and $x = \frac{\alpha^{2(n+1)}}{(ab)^{n+1}}$, $y = \frac{\beta^{2(n+1)}}{(ab)^{n+1}}$ in (1.5), respectively, we obtain the following formulas

$$l_{(2n+1)(2m+1-2j)} = \sum_{k=0}^{m-j} \left(\frac{b}{a}\right)^{k} \frac{2m+1-2j}{2k+1} \binom{m-j+k}{2k} l_{2n+1}^{2k+1}, \tag{3.9}$$

$$l_{2(n+1)(2m+1-2j)} = \sum_{k=0}^{m-j} (-1)^{m-j-k} \frac{2m+1-2j}{2k+1} \binom{m-j+k}{2k} l_{2n+2}^{2k+1}. \tag{3.10}$$

Substituting equations (3.9), (3.10), (2.19), and (2.20) into (3.1), (3.2), (3.3), and (3.4), respectively, yields the desired results. □

The sums of the powers of bi-periodic Lucas numbers can be expressed in terms of the powers of the bi-periodic Fibonacci numbers.

**Theorem 3.3.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=0}^{n} l_{2k}^{2m+1} = l_{2n+1} \sum_{k=0}^{m} q_{2n+1}^{2k} \sum_{j=0}^{m-k} (-1)^{m-j-k} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{(ab+4)^k}{l_{2m+1-2j}} + 4^m, \tag{3.11}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m+1} = l_{2n+2} \sum_{k=0}^{m} q_{2n+2}^{2k} \sum_{j=0}^{m-k} (-1)^j \left(\frac{a}{b}\right)^{m+1-k} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{(ab+4)^k}{l_{2m+1-2j}} - 2\left(\frac{a}{b}\right)^{m+1} \sum_{j=0}^{m} \frac{(-1)^j}{l_{2m+1-2j}} \binom{2m+1}{j}, \tag{3.12}$$

$$\sum_{k=0}^{n} l_{2k}^{2m} = l_{2n+1} \sum_{k=0}^{m-1} q_{2n+1}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^{m-j-k-1} \binom{2m}{j} \binom{m-j+k}{2k+1} \frac{(ab+4)^k}{q_{2m-2j}} + 2^{2m-1} + \binom{2m}{m}(n+\frac{1}{2}), \tag{3.13}$$

$$\sum_{k=0}^{n} l_{2k+1}^{2m} = l_{2n+2} \sum_{k=0}^{m-1} q_{2n+2}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^j \left(\frac{a}{b}\right)^{m-k} \binom{2m}{j} \binom{m-j+k}{2k+1} \frac{(ab+4)^k}{q_{2m-2j}} + (-1)^m \left(\frac{a}{b}\right)^{m} \binom{2m}{m}(n+1). \tag{3.14}$$

*Proof.* Taking $x = \frac{\alpha^{2n+1}}{\sqrt{ab}(ab)^n}$, $y = -\frac{\beta^{2n+1}}{\sqrt{ab}(ab)^n}$, and $x = \frac{\alpha^{2(n+1)}}{(ab)^{n+1}}$, $y = -\frac{\beta^{2(n+1)}}{(ab)^{n+1}}$ in (1.4), respectively, we get

$$l_{(2n+1)(2m+1-2j)} = l_{2n+1} \sum_{k=0}^{m-j} (-1)^{m-j-k} \binom{m-j+k}{2k} (ab+4)^k q_{2n+1}^{2k}, \tag{3.15}$$

$$l_{2(n+1)(2m+1-2j)} = l_{2(n+1)} \sum_{k=0}^{m-j} \binom{m-j+k}{2k} \left(\frac{b}{a}\right)^k (ab+4)^k q_{2n+2}^{2k}. \tag{3.16}$$

By replacing (3.15), (3.16), (2.11), and (2.12) in (3.1), (3.2), (3.3), and (3.4), respectively, we obtain results. □

When substituting $a = b = 1$ in (3.8), (3.11), (3.12), and (3.13), we get the following results.

**Corollary 3.4.** *For $n \geq 0$ and $m \geq 1$, we have*

$$\sum_{k=0}^{n} L_{2k+1}^{2m} = \sum_{k=0}^{m-1} L_{2n+2}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^{m-k-1} \frac{F_{2n+2}}{F_{2m-2j}} \binom{2m}{j} \binom{m-j+k}{2k+1} + (-1)^m \binom{2m}{m} (n+1),$$

$$\sum_{k=0}^{n} L_{2k}^{2m+1} = \sum_{k=0}^{m} F_{2n+1}^{2k} \sum_{j=0}^{m-k} (-1)^{m-j-k} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{5^k L_{2n+1}}{L_{2m+1-2j}} + 4^m,$$

$$\sum_{k=0}^{n} L_{2k+1}^{2m+1} = \sum_{k=0}^{m} F_{2n+2}^{2k} \sum_{j=0}^{m-k} (-1)^{j} \binom{2m+1}{j} \binom{m-j+k}{2k} \frac{5^k L_{2n+2}}{L_{2m+1-2j}} - \sum_{j=0}^{m} \frac{2(-1)^j}{L_{2m+1-2j}} \binom{2m+1}{j},$$

$$\sum_{k=0}^{n} L_{2k}^{2m} = \sum_{k=0}^{m-1} F_{2n+1}^{2k+1} \sum_{j=0}^{m-k-1} (-1)^{m-j-k-1} \binom{2m}{j} \binom{m-j+k}{2k+1} \frac{5^k L_{2n+1}}{F_{2m-2j}} + \binom{2m}{m} (n + \frac{1}{2}) + 2^{2m-1}.$$

**Remark 3.5.** Note that if we take $a = b = 1$ in (3.5), (3.6), (3.7), and (3.14), we obtain closed formulas for the sums of the powers of the classical Lucas numbers (see [15]).

## 4 Generating Function

In this section, we establish the generating functions corresponding to the sequences $(q_n^m)$ and $(l_n^m)$.

**Theorem 4.1.** *For a fixed $m \geq 1$, we have for $m$ odd*

$$\sum_{n\geq 0} q_n^m z^n = \frac{z}{(a^2b^2+4ab)^{(m-1)/2}} \sum_{k=0}^{\frac{m-1}{2}} \binom{m}{k} \frac{(-1)^k a^{m-1} q_{2(m-2k)} z + (ab)^{(m-1)/2} q_{m-2k}(1-z^2)}{1 - l_{2(m-2k)} z^2 + z^4}$$

*and for $m$ even*

$$\sum_{n\geq 0} q_n^m z^n = \frac{1}{(a^2b^2+4ab)^{m/2}} \sum_{k=0}^{\frac{m}{2}-1} \binom{m}{k} \frac{(-1)^k a^m (2 - l_{2(m-2k)} z^2) + (ab)^{m/2} l_{m-2k} z (1-z^2)}{1 - l_{2(m-2k)} z^2 + z^4} + \binom{m}{m/2} \frac{(-1)^{m/2} a^m + (ab)^{m/2} z}{(a^2b^2-4ab)^{m/2}(1-z^2)}.$$

*Proof.* From Binet's formula for the bi-periodic Fibonacci sequence, we obtain

$$\begin{aligned}
\sum_{n\geq 0} q_n^m z^n &= \sum_{n\geq 0} q_{2n}^m z^{2n} + \sum_{n\geq 0} q_{2n+1}^m z^{2n+1} \\
&= \frac{a^m}{(\alpha-\beta)^m} \sum_{n\geq 0} \left(\frac{1}{ab}\right)^{mn} \left(\alpha^{2n} - \beta^{2n}\right)^m z^{2n} \\
&\quad + \frac{1}{(\alpha-\beta)^m} \sum_{n\geq 0} \left(\frac{1}{ab}\right)^{mn} \left(\alpha^{2n+1} - \beta^{2n+1}\right)^m z^{2n+1}.
\end{aligned}$$

According to the binomial theorem, it follows that

$$\begin{aligned}\sum_{n\geq 0} q_n^m z^n &= \frac{a^m}{(\alpha-\beta)^m}\sum_{n\geq 0}\left(\frac{1}{ab}\right)^{mn}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\alpha^{2nk}\beta^{2n(m-k)}z^{2n} \\ &\quad + \frac{1}{(\alpha-\beta)^m}\sum_{n\geq 0}\left(\frac{1}{ab}\right)^{mn}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\alpha^{k(2n+1)}\beta^{(m-k)(2n+1)}z^{2n+1} \\ &= \frac{a^m}{(\alpha-\beta)^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{n\geq 0}\left(\left(\frac{1}{ab}\right)^m \alpha^{2k}\beta^{2(m-k)}z^2\right)^n \\ &\quad + \frac{z}{(\alpha-\beta)^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\alpha^k\beta^{m-k}\sum_{n\geq 0}\left(\left(\frac{1}{ab}\right)^m \alpha^{2k}\beta^{2(m-k)}z^2\right)^n \\ &= \frac{1}{(\alpha-\beta)^m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{a^m+\alpha^k\beta^{m-k}z}{1-\left(\frac{1}{ab}\right)^m\alpha^{2k}\beta^{2(m-k)}z^2}.\end{aligned}$$

Consider the case when $m$ is odd

$$\begin{aligned}\sum_{n\geq 0} q_n^m z^n &= \sum_{k=0}^{(m-1)/2}\frac{(-1)^k}{(\alpha-\beta)^m}\binom{m}{k}\left(\frac{a^m+\alpha^{m-k}\beta^k z}{1-\left(\frac{1}{ab}\right)^m\alpha^{2(m-k)}\beta^{2k}z^2} - \frac{a^m+\alpha^k\beta^{m-k}z}{1-\left(\frac{1}{ab}\right)^m\alpha^{2k}\beta^{2(m-k)}z^2}\right) \\ &= \frac{1}{(\alpha-\beta)^m}\sum_{k=0}^{(m-1)/2}(-1)^k\binom{m}{k}\left(\frac{a^m\frac{(\alpha\beta)^{2k}}{(ab)^m}\left(\alpha^{2(m-2k)}-\beta^{2(m-2k)}\right)z^2}{1-\left(\frac{1}{ab}\right)^{m-2k}\left(\alpha^{2(m-2k)}+\beta^{m-2k}\right)z^2+z^4}\right. \\ &\quad \left. + \frac{z\left(\alpha^{m-2k}-\beta^{m-2k}\right)\left((\alpha\beta)^k+\frac{(\alpha\beta)^{m+k}}{(ab)^m}z^2\right)}{1-\left(\frac{1}{ab}\right)^{m-2k}\left(\alpha^{2(m-2k)}+\beta^{m-2k}\right)z^2+z^4}\right) \\ &= \frac{z}{(\alpha-\beta)^{m-1}}\sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}\frac{(-1)^k a^{m-1}q_{2(m-2k)}z+(ab)^{(m-1)/2}q_{m-2k}(1-z^2)}{1-l_{2(m-2k)}z^2+z^4}.\end{aligned}$$

Consider the case when $m$ is even

$$\begin{aligned}\sum_{n\geq 0} q_n^m z^n &= \sum_{k=0}^{\frac{m}{2}-1}\frac{(-1)^k}{(\alpha-\beta)^m}\binom{m}{k}\left(\frac{a^m+\alpha^{m-k}\beta^k z}{1-\left(\frac{1}{ab}\right)^m\alpha^{2(m-k)}\beta^{2k}z^2} + \frac{a^m+\alpha^k\beta^{m-k}z}{1-\left(\frac{1}{ab}\right)^m\alpha^{2k}\beta^{2(m-k)}z^2}\right) \\ &\quad + (-1)^{m/2}\binom{m}{m/2}\frac{a^m+(\alpha\beta)^{m/2}z}{(\alpha-\beta)^m(1-(\alpha\beta)^m z^2)} \\ &= \frac{1}{(\alpha-\beta)^m}\sum_{k=0}^{\frac{m}{2}-1}\binom{m}{k}\left(\frac{a^m(-1)^k\left(2-\frac{1}{(ab)^{m-2k}}\left(\alpha^{2(m-2k)}+\beta^{2(m-2k)}\right)z^2\right)}{1-\left(\frac{1}{ab}\right)^{m-2k}\left(\alpha^{2(m-2k)}+\beta^{m-2k}\right)z^2+z^4}\right. \\ &\quad \left. + \frac{(ab)^k z\left(\alpha^{m-2k}+\beta^{m-2k}\right)(1-z^2)}{1-\left(\frac{1}{ab}\right)^{m-2k}\left(\alpha^{2(m-2k)}+\beta^{m-2k}\right)z^2+z^4}\right) + \binom{m}{m/2}\frac{(-1)^{m/2}a^m+(ab)^{m/2}z}{(\alpha-\beta)^m(1-z^2)} \\ &= \frac{1}{(\alpha-\beta)^m}\sum_{k=0}^{\frac{m}{2}-1}\binom{m}{k}\frac{(-1)^k a^m(2-l_{2(m-2k)}z^2)+(ab)^{m/2}l_{m-2k}z(1-z^2)}{1-l_{2(m-2k)}z^2+z^4} \\ &\quad + \binom{m}{m/2}\frac{(-1)^{m/2}a^m+(ab)^{m/2}z}{(\alpha-\beta)^m(1-z^2)}.\end{aligned}$$

□

**Theorem 4.2.** *For a fixed* $m\geq 1$, *we have for* $m$ *odd*

$$\sum_{n\geq 0} l_n^m z^n = \sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}\frac{2-l_{2(m-2k)}z^2+(-1)^k(a/b)^{(m-1)/2}l_{m-2k}z(1+z^2)}{1-l_{2(m-2k)}z^2+z^4}$$

*and for $m$ even*

$$\sum_{n\geq 0} l_n^m z^n = \sum_{k=0}^{\frac{m}{2}-1} \binom{m}{k} \frac{2 - l_{2(m-2k)}z^2 + (-1)^k (a/b)^{m/2} l_{m-2k} z(1-z^2)}{1 - l_{2(m-2k)}z^2 + z^4} + \binom{m}{m/2} \frac{1 + (-1)^{m/2}(a/b)^{m/2} z}{1 - z^2}.$$

*Proof.* Consider

$$\sum_{n\geq 0} l_n^m z^n = \sum_{n\geq 0} l_{2n}^m z^{2n} + \sum_{n\geq 0} l_{2n+1}^m z^{2n+1} = \sum_{k=0}^{m} \frac{1 + (1/b)^m \alpha^k \beta^{m-k} z}{1 - 1/(ab)^m \alpha^{2k} \beta^{2(m-k)}}.$$

By considering even and odd values of $m$ separately, we obtain results. □

## References


[1] N. Belaggoun and H. Belbachir, *Alternating sums of the powers of bi-periodic Fibonacci and Lucas numbers*, Montes Taurus J. Pure Appl. Math., **5(2)**, 39-50, (2023).

[2] H. Belbachir and F. Bencherif, *Sums of products of generalized Fibonacci and Lucas numbers*. arXivpreprint arXiv:0708.2347, (2007).

[3] H. Belbachir, F. Bencherif, *On Some Properties on Bivariate Fibonacci and Lucas Polynomials*, J. Integer Seq., **11(2)**, (2008).

[4] G. Bilgici, *Two generalizations of Lucas sequence*, Appl. Math. Comput., **245(2)**, 526–538, (2014).

[5] L. Carlitz, *A Fibonacci array*, Fibonacci Quart., **1(2)**, 17–28, (1963).

[6] L. Comtet, *Advanced Combinatorics: The art of finite and infinite expansions*, Springer Science & Business Media, New York, (1974).

[7] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's formula*, Integers., **9(A48)**, 639–654, (2009).

[8] H. W. Gould, *The Girard-Waring power sum formulas for symmetric functions, and Fibonacci sequences*, Fibonacci Quart., **37**, 135–140, (1999).

[9] A. F. Horadam, *The Generalized Fibonacci Sequences*, The American Math. Monthly, **68(5)**, 455-459, (1961).

[10] E. Kılıç, N. Ömür, I. Akkus, and Y. T. Ulutas, *Various Sums Including the Generalized Fibonacci and Lucas numbers*, Palest. J. Math., **4(2)**, (2015).

[11] H. H. Leung, *Some binomial-sum identities for the generalized bi-periodic Fibonacci sequences*, Notes Number Theory Discrete Math., **26(1)**, 199–208, (2020).

[12] R. S. Melham, *Some conjectures concerning sums of odd powers of Fibonacci and Lucas numbers*, Fibonacci Quart., **46/47(4)**, 312–315, (2008/2009).

[13] K. Ozeki, *On melham's sum*, Fibonacci Quart., **46/47(2)**, 107–110, (2008/2009).

[14] Y. K. Panwar, B. Singh, and V. K. Gupta, *Generalized Fibonacci sequences and its properties*, Palest. J. Math., **3(1)**, 141–147, (2014).

[15] H. Prodinger, *On a sum of melham and its variants*, Fibonacci Quart., **46(47)** (2009/2008).

[16] E. Tan, *General Sum Formula for Bi-periodic Fibonacci and Lucas Numbers*, Integers, **17**, A42, (2017).

[17] E. Waring, *Miscellanea analytica, de aequationibus algebraicis, et curvarum proprietatibus*, Thurlbourn & Woodyer, New York, (1762).

[18] M. Wiemann and C. Cooper, *Divisibility of an F-L type convolution*, Applications of Fibonacci Numbers, **9(2)**, 267–287, (2004).

[19] O. Yayenie, *A note on generalized Fibonacci sequence*, Appl. Math. Comput., **217**, 5603–5611, (2011).

## Author information

N. Belaggoun, USTHB, Faculty of Mathematics, RECITS Laboratory, P.O. Box 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria
CERIST, Research Center on Scientific and Technical Information 05, Rue des 3 frères Aissou, Ben Aknoun, Algiers, Algeria.
E-mail: belaggounmanassil@gmail.com; nbelaggoun@usthb.dz

H. Belbachir, USTHB, Faculty of Mathematics, RECITS Laboratory, P.O. Box 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria.
E-mail: hacenebelbachir@gmail.com; hbelbachir@usthb.dz

Received: 2023-04-25
Accepted: 2023-12-03