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Existence results for non-coercive anisotropic Neumann boundary value problems with lower order terms

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Abstract *The aim of this paper is to prove the existence of renormalized solutions for the Neumann boundary value problem associated to a class of anisotropic elliptic equation with some lower order term and degenerate coercivity.*

1 Introduction

Recently, an increasing interest has turned towards the case of anisotropic elliptic problems in the study of nonlinear elliptic equations with lower order terms. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. It is well known that the lower order terms may affect the existence, uniqueness, regularity and asymptotic behavior of solutions of partial differential equations (see, for instance [11, 13]).

In this paper, we are going to prove the existence of renormalized solutions for a class of degenerate elliptic problem with Neumann boundary conditions and with two lower order terms, whose prototype is

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha|u|^{s-2}u = f(x,u) - div\ \phi(x,u) & in \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i = g(x) & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $R^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $D^i u = \dfrac{\partial u}{\partial x_i}$, the datums $f(x,s)$ and $\Phi(x,s)$ are two Carathéodory functions satisfying some growth conditions and $\vec{n} = (n_1,\ldots,n_N)$ is the unit outward normal on $\partial\Omega$. The technical difficulty of the problem is the combination of the anisotropic non-coercivity and the presence of the lower order terms. Let us begin with the paper of Boccardo et al. [12] in which they have proved existence and uniqueness results for nonlinear elliptic equation of the type

$$\begin{cases} -div\,(a\,(x,u,\nabla u)) + \lambda|u|^{p-2}u = f & in \quad \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.2}$$

where $f \in W_0^{-1,p'}(\Omega)$, $\lambda \geqq 0$. Moreover, the uniqueness result holds true under some additional conditions on p and λ, and fails for other conditions. Concerning the existence of renormalized solutions for elliptic equations with L^1-data, we refer the reader to [26], and for

Radon measure-data to the paper [16].
In [1], the authors prove the existence of entropy solutions for the non-coercive elliptic problem

$$\begin{cases} -div\left(b(|u|)|\nabla u|^{p-2}\nabla u\right) + d(|u|)|\nabla u|^p = f(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

We point out that the main difficulty in studying this kind of problems is due to fact that the differential operator is non-coercive. We refer also to the degenerated elliptic equation : Guibé et al. have considerate in [7] the following growth condition $|\phi(x,u)| \leq b(x)(1+|u|^{p-1})$. Also, Di-Nardo et al. have assumed in [18] that $|\phi(x,u)| \leq c(x)|u|^\gamma$ for $\gamma \leq p-1$.

We recall that the study of problems (1.2) has been the object of several papers, we refer for example to [3, 15, 19].

For the anisotropic case, Benboubker et al. in [5] considered the Neumann elliptic problem:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{p_0-2}u = f(x,u,\nabla u) & in \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i = g(x) & on \quad \partial\Omega, \end{cases}$$

where the Carathéodory function $f(x,u,\nabla u)$ verifies only some growth conditions and they proved the existence of weak solution for the data $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$, then they have proved the existence of renormalized solutions for $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

Furthermore, Al-Hamwi et al. in [2] have treated the existence of entropy solutions for an anisotropic quasilinear elliptic unilateral problem by using a penalization method in the approximate problems, we refer the reader also to [20], [21] and [23].

Concerning the existence of solutions for the linear Neumann problem, various existence results have been proved when the datum f belongs to $L^1(\Omega)$, see [3], in the same paper the authors investigate the solvability of the Neumann problem for semilinear equations. Later, the nonlinear case was treated in several papers and in different contests, for example in [10] the authors proved existence results of weak solutions by using a fixed point arguments for nonlinear elliptic Neumann problems with lower order term where the datum f belongs to $L^1(\Omega)$ and satisfies the compatibility condition $\int_\Omega f = 0$. Furthermore, they gave various definitions of solutions to the nonlinear elliptic problem and proved also the existence of renormalized solution which satisfies $med(u) = 0$.

As far as the existence and regularity results for (1.1) are concerned, there are two difficulties associated with this kind of problems. Firstly, from hypothesis (3.2), the operator

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,u,D^i u),$$

is well defined between $W^{1,\vec{p}}(\Omega)$ and its dual. However, by the assumption (3.3) this operator in general is not coercive. This lack of coerciveness does not allow one to use classical results to prove existence of solutions. The second difficulty appears due to the presence of the lower order term in the right-hand side, which have regularizing effects on solutions. To overcome these difficulties, we shall first introduce a term of penalization in the approximate problems and then establish some estimates for solutions by taking suitable test functions, and finally prove some convergence results to get the existence results.

This article is presented as follows. In Section 2, we recall some definitions and properties about anisotropic Sobolev spaces. In Section 3, we give our basic assumptions and some fundamental lemmas. Finally, in Section 4, we presente the definition of a renormalized solutions for problem (1.1) and we prove the Theorem 4.2, this result ensure the existence of at least one weak solution $u_n \in W^{1,\vec{p}}(\Omega)$ for the strongly nonlinear elliptic Neumann problem.

2 Preliminaries

In this section, we recall some definitions and basic properties of anisotropic Sobolev spaces.
Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$, and let $p_1, \ldots, p_N$

be N real constants, such that $1 < p_i < \infty$ for $i = 1, \dots, N$. We denote

$$\vec{p} = (1, p_1, \dots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \dots, N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \qquad \text{and} \qquad \underline{p}^+ = \max\{p_1, p_2, \dots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N \right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf. [25]).
Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

Proposition 2.1. *(cf. [24])*
Let $u \in W^{1,\vec{p}}(\Omega)$, we have
(i) *Poincaré Wirtinger inequality: there exists a constant $C_p > 0$, such that*

$$\|u - med(u)\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for} \quad i = 1, \dots, N,$$

with

$$med(u) = \frac{1}{|\Omega|} \int_{\Omega} |u| \, dx.$$

(ii) *Sobolev inequality : there exists an other constant $C_s > 0$, such that*

$$\|u - med(u)\|_q \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 2.2. *Let Ω be a bounded open set in $I\!R^N$ $(N \geq 2)$, we set*

$$s = \max(q, \underline{p}^+),$$

then, we have the following embedding :

- *if $\overline{p} < N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of this lemma follows from the Proposition 2.1, and the fact that the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Definition 2.3. Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.4. *Let $u \in \mathcal{T}^{1,\vec{p}}(\Omega)$. For any $i \in \{1,\dots,N\}$, there exists a unique measurable function $v_i : \Omega \mapsto I\!R$ such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \quad \text{a.e. in} \quad \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 2.4 follows the usual techniques developed in [9] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [4, 8, 17, 18].

Moreover, we introduce the set $T^{1,\vec{p}}_{tr}(\Omega)$ as a subset of $T^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace may be defined (see also [3] in the case of $p_i = p$ for $i = 1,\dots,N$. More precisely, $T^{1,\vec{p}}_{tr}(\Omega)$ is the set of functions u in $T^{1,\vec{p}}(\Omega)$, such that : there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega)$ and a measurable function v on $\partial\Omega$ verifying

(a) $u_n \longrightarrow u$ *a.e. in* Ω,

(b) $DT_k(u_n) \longrightarrow DT_k(u)$ *in* $L^1(\Omega)$ *for every* $k > 0$.

(c) $u_n \longrightarrow v$ *a.e. on* $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [3].
Let $u \in W^{1,\vec{p}}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $\tau(u)$.
For any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $tr(u)$, the operator $tr(\cdot)$ satisfied the following properties

(i) *If $u \in T^{1,\vec{p}}_{tr}(\Omega)$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$.*

(ii) *If $\varphi \in W^{1,\vec{p}}(\Omega)$, then, for any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, we have $u - \varphi \in T^{1,\vec{p}}_{tr}(\Omega)$ and $tr(u-\varphi) = tr(u) - \tau(\varphi)$.*

In the case where $u \in W^{1,\vec{p}}(\Omega)$, $tr(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}}(\Omega) \subset T^{1,\vec{p}}_{tr}(\Omega) \subset T^{1,\vec{p}}(\Omega).$$

3 Essential assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, and let $p_1,\dots,p_N$ be N real constants, with $1 < p_i < \infty$ for $i = 1,\dots,N$.
We consider the quasilinear anisotropic elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha|u|^{s-2}u = f(x,u) - \sum_{i=1}^{N} D^i \phi_i(x,u) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} (a_i(x,T_n(u),\nabla u) - \phi_i(x,u)) \cdot n_i = g(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

with A is a Leray-Lions operator acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

where $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions, for $i = 1,\dots,N$, (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω) which satisfy the following conditions :

$$|a_i(x,s,\xi)| \leq \beta(K_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{3.2}$$

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \qquad \text{with} \quad b(|s|) \geq \frac{b_0}{(1+|s|)^{\lambda}}, \tag{3.3}$$

for a.e. $x \in \Omega$ *and all* $(s,\xi) \in I\!R \times I\!R^N$, *where* b_0 *is a positive constant and* $0 \le \lambda < \underline{p}^+ - 1$, *such that* $b(|\cdot|) : I\!R^+ \mapsto I\!R^+$ *is a decreasing function that belongs to* $L^1(I\!R) \cap L^\infty(I\!R)$.

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \qquad \textit{for} \quad \xi_i \neq \xi_i' \quad \textit{and for any} \quad i = 1,\ldots,N, \tag{3.4}$$

for almost every $x \in \Omega$ *and all* (s,ξ) *in* $I\!R \times I\!R^N$, *where the nonnegative functions* $K_i(\cdot)$ *are assumed to be in* $L^{p_i'}(\Omega)$, *with* α, β *are two strictly positive real constants.*
The nonlinear term $f(x,s)$ *and* $\phi(x,s) = (\phi_1(x,s),\ldots,\phi_N(x,s))$ *are some Carathéodory functions these verifying some growth condition.*
As a consequence of (3.3) *and the continuity of the function* $a_i(x,s,\cdot)$ *with respect to* ξ, *we have*

$$a_i(x,s,0) = 0.$$

We are going now to recall the following technical Lemma, useful to prove our main results.

Lemma 3.1. *(see [6]) Assuming that* (3.2) – (3.4) *hold true, and let* $(u_n)_{n\in I\!N}$ *be a sequence in* $W^{1,\vec{p}}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W^{1,\vec{p}}(\Omega)$ *and*

$$\begin{aligned}&\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\,dx \longrightarrow 0 \qquad \textit{as} \quad n \to \infty,\end{aligned} \tag{3.5}$$

then $u_n \to u$ *strongly in* $W^{1,\vec{p}}(\Omega)$ *for a subsequence.*

3.1 Existence of weak solutions for L^∞−data

We consider the quasilinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^N D^i a_i(x,T_n(u),\nabla u) + \alpha|u|^{s-2}u = F(x,u) - \sum_{i=1}^N D^i\Phi_i(x,u) & \textit{on} \quad \Omega, \\ \sum_{i=1}^N (a_i(x,T_n(u),\nabla u) - \Phi_i(x,u)) \cdot n_i = G(x) & \textit{in} \quad \partial\Omega, \end{cases} \tag{3.6}$$

with

$$G(x) \in L^\infty(\partial\Omega) \qquad \textit{and} \qquad |F(x,s)| \le C_0 \qquad \textit{and} \qquad \sum_{i=1}^N |\Phi_i(x,u)| \le C_1, \tag{3.7}$$

for any $x \in \Omega$ *and* $s \in I\!R$, *where* C_0 *and* C_1 *are two positive constants.*

Definition 3.2. A measurable function u is called weak solution for the quasilinear anisotropic elliptic equation (3.6), if $u \in W^{1,\vec{p}}(\Omega)$, $|u|^s \in L^1(\Omega)$, and u verifies the following equality

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,dx + \alpha\int_\Omega |u|^{s-2}uv\,dx \\ &= \int_{\partial\Omega} Gv\,d\sigma + \int_\Omega F(x,u)v\,dx + \sum_{i=1}^N \int_\Omega \Phi_i(x,u)D^i v\,dx,\end{aligned} \tag{3.8}$$

for every $v \in W^{1,\vec{p}}(\Omega)$.

Theorem 3.3. *Assuming that* (3.2) – (3.4) *and* (3.7) *hold true. Then, there exists at least one weak solution for the quasilinear elliptic problem* (3.6), *such that* $u \in L^s(\Omega)$.

Proof of Theorem 3.3

Step 1: Approximate problem

We consider the approximate problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + \alpha |T_m(u_m)|^{s-2} T_m(u_m) + \frac{1}{m}|u_m|^{\underline{p}-2} u_m \\ \qquad = F(x, u_m) - \sum_{i=1}^{N} D^i \Phi_i(x, u_m) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} (a_i(x, T_n(u_m), \nabla u_m) - \Phi_i(x, u_m)) \cdot n_i = G(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{3.9}$$

We consider the two operators A_m and H acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, defined by the formula

$$\langle A_m u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v\, dx + \alpha \int_\Omega |T_m(u)|^{s-2} T_m(u) v\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} uv\, dx, \tag{3.10}$$

and

$$\langle Hu, v\rangle = -\int_\Omega F(x, u) v\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u) D^i v\, dx - \int_{\partial\Omega} G(x) v\, d\sigma, \tag{3.11}$$

for any $u, v \in W^{1,\vec{p}}(\Omega)$.

Lemma 3.4. *The operator $B_m = A_m + H$ acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, is bounded and pseudo-monotone. Moreover, B_m is coercive in the following sense*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \text{for} \quad v \in W^{1,\vec{p}}(\Omega).$$

Using the Hölder's inequality and the growth condition (3.2), we can show that the operator A_m is bounded, and since

$$\begin{aligned} | < Hu, v > | &= \Big| -\int_\Omega F(x, u) v\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u) D^i v\, dx - \int_{\partial\Omega} G(x) v d\sigma \Big| \\ &\le \int_\Omega |F(x, u)|\, |v|\, dx + \sum_{i=1}^{N} \int_\Omega |\Phi_i(x, u)|\, |D^i v|\, dx + \int_{\partial\Omega} |G(x)|\, |v|\, d\sigma \\ &\le C_0 \int_\Omega |v|\, dx + C_1 \sum_{i=1}^{N} \int_\Omega |D^i v|\, dx + \|G(\cdot)\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |v|\, d\sigma \\ &\le C_0 \|v\|_{L^1(\Omega)} + C_1 \sum_{i=1}^{N} \|D^i v\|_{L^1(\Omega)} + \|G(\cdot)\|_{L^\infty(\partial\Omega)} \|v\|_{L^1(\partial\Omega)} \\ &\le C_2 \|v\|_{1,\vec{p}} \qquad\qquad \textit{for any } u, v \in W^{1,\vec{p}}(\Omega). \end{aligned} \tag{3.12}$$

We conclude that the operator B_m is bounded. For the coercivity we have

$$\begin{aligned}\langle B_m u, u\rangle &= \sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u) D^i u\,dx + \alpha \int_\Omega |T_m(u)|^{s-1}|u|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx \\ &\quad - \int_\Omega F(x,u)u\,dx - \sum_{i=1}^N \int_\Omega \Phi_i(x,u) D^i u\,dx - \int_{\partial\Omega} G(x) u\,d\sigma \\ &\geq \sum_{i=1}^N \int_\Omega b(|T_n(u)|)|D^i u|^{p_i}\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx - C_0\|u\|_{L^1(\Omega)} \\ &\quad -C_1 \sum_{i=1}^N \|D^i u\|_{L^1(\Omega)} - \|G(\cdot)\|_{L^\infty(\partial\Omega)}\|u\|_{L^1(\partial\Omega)} \\ &\geq \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^N \int_\Omega |D^i u|^{p_i}\,dx + \frac{C_3}{m}\|u\|^{\underline{p}}_{L^1(\Omega)} - C_2\|u\|_{1,\vec{p}} \\ &\geq C_4 \|u\|^{\underline{p}}_{1,\vec{p}} - C_2\|u\|_{1,\vec{p}}\,,\end{aligned}$$

we conclude that

$$\frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p}}} \geq \frac{C_4\|u\|^{\underline{p}}_{1,\vec{p}} - C_2\|u\|_{1,\vec{p}}}{\|u\|_{1,\vec{p}}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}} \to \infty.$$

It remains to show that B_m is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W^{1,\vec{p}}(\Omega), \\ B_m u_k \rightharpoonup \chi_m & \text{in } (W^{1,\vec{p}}(\Omega))', \\ \limsup_{k\to\infty}\langle B_m u_k, u_k\rangle \leq \langle \chi_m, u\rangle. \end{cases} \tag{3.13}$$

We will show that

$$\chi_m = B_m u \quad \text{and} \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle \chi_m, u\rangle \ \text{as} \ k\to+\infty.$$

In view of the compact embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ and $W^{1,\vec{p}}(\Omega) \hookrightarrow L^1(\partial\Omega)$, there exists a subsequence still denoted $(u_k)_{k\in\mathbb{N}^*}$ such that $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$ and $u_k \to u$ weakly in $L^1(\partial\Omega)$.
As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}}(\Omega)$, using the growth condition (3.2), it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}^*}$ is bounded in $L^{p_i'}(\Omega)$, and there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{weakly in} \quad L^{p_i'}(\Omega) \ \text{as} \ k\to\infty. \tag{3.14}$$

We have $(F(x,u_k))_{k\in\mathbb{N}^*}$ is uniformly bounded in $L^{\underline{p}'}(\Omega)$, and $F(x,u_k) \to F(x,u)$ almost everywhere in Ω, in view of Lebesgue dominated convergence theorem we conclude that

$$F(x,u_k) \to F(x,u) \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{3.15}$$

Similarly, since $(\Phi_i(x,u_k))_{k\in\mathbb{N}^*}$ is uniformly bounded in $L^{p_i'}(\Omega)$ for $i = 1,\dots,N$, and $\Phi_i(x,u_k) \to \Phi_i(x,u)$ almost everywhere in Ω, it follows that

$$\Phi_i(x,u_k) \to \Phi_i(x,u) \quad \text{strongly in} \quad L^{p_i'}(\Omega) \quad \text{for } i = 1,\dots,N. \tag{3.16}$$

Moreover, since $u_k \to u$ a.e. in Ω, and in view of Lebesgue dominated convergence theorem we conclude that

$$\alpha|T_m(u_k)|^{s-2}T_m(u_k) \longrightarrow \alpha|T_m(u)|^{s-2}T_m(u) \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{3.17}$$

Also, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$, it follows that

$$\frac{1}{m}|u_k|^{\underline{p}-2}u_k \longrightarrow \frac{1}{m}|u|^{\underline{p}-2}u \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{3.18}$$

Thus, for any $v \in W^{1,\vec{p}}(\Omega)$ *we have*

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\
&= \lim_{k\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v\, dx + \alpha \int_\Omega |T_m(u_k)|^{p_0-2} T_m(u_k) v\, dx \\
&\quad + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}-2} u_k v\, dx - \int_\Omega F(x, u_k) v\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u_k) D^i v\, dx - \int_{\partial\Omega} G\, v\, d\sigma \Big) \\
&= \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i v\, dx + \alpha \int_\Omega |T_m(u)|^{s-2} T_m(u) v\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} u v\, dx \\
&\quad - \int_\Omega F(x, u) v\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u) D^i v\, dx - \int_{\partial\Omega} G v\, d\sigma.
\end{aligned} \tag{3.19}$$

In view of (3.13) *and* (3.19), *we conclude that*

$$\begin{aligned}
\limsup_{k\to\infty} \langle B_m(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big(\int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx + \alpha \int_\Omega |T_m(u_k)|^{s-1} |u_k|\, dx \\
&\quad + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}}\, dx - \int_\Omega F(x, u_k) u_k\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u_k) D^i u_k\, dx - \int_{\partial\Omega} G\, u_k\, d\sigma \Big) \\
&\le \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i u\, dx + \alpha \int_\Omega |T_m(u)|^{s-1} |u|\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}}\, dx \\
&\quad - \int_\Omega F(x, u) u\, dx - \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u) D^i u\, dx - \int_{\partial\Omega} G\, u\, d\sigma.
\end{aligned} \tag{3.20}$$

Thanks to (3.15) – (3.18) *we have*

$$\alpha \int_\Omega |T_m(u_k)|^{s-1} |u_k| dx + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}} dx \longrightarrow \alpha \int_\Omega |T_m(u)|^{s-1} |u| dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}} dx \;\; as \;\; k \to \infty, \tag{3.21}$$

and

$$\int_\Omega F(x, u_k) u_k dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u_k) D^i u_k dx \longrightarrow \int_\Omega F(x, u) u dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u) D^i u dx \;\; as \;\; k \to \infty, \tag{3.22}$$

and since $G \in L^\infty(\partial\Omega)$ *then*

$$\int_{\partial\Omega} G\, u_k\, d\sigma \longrightarrow \int_{\partial\Omega} G u\, d\sigma \qquad as \quad k \to \infty. \tag{3.23}$$

It follows that

$$\limsup_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \le \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i u\, dx. \tag{3.24}$$

On the other hand, in view of (3.4) *we have*

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u)\, dx \ge 0, \tag{3.25}$$

then

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx &\ge \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u)\, dx.
\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ *strongly in* $L^{p_i}(\Omega)$*, thus* $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ *strongly in* $L^{p_i'}(\Omega)$*, and using* (3.14) *we get*

$$\liminf_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{3.26}$$

Having in mind (3.24)*, we conclude that*

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx = \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{3.27}$$

Therefore, having in mind (3.21), (3.22) *and* (3.23) *we obtain*

$$\langle B_m u_k, u_k \rangle \longrightarrow \langle \chi_m, u \rangle \quad \text{as} \quad k \to \infty. \tag{3.28}$$

On the other hand, thanks to (3.27) *we can show that*

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx = 0.$$

We have $u_k \to u$ *strongly in* $L^{\underline{p}}(\Omega)$*, it follows that*

$$\begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2} u_n - |u|^{\underline{p}-2} u)(u_k - u) \, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u)(D^i u_n - D^i u) \, dx \to 0, \end{aligned} \tag{3.29}$$

in view of Lemma 3.1, we conclude that

$$u_k \to u \quad \text{in} \quad W^{1,\vec{p}}(\Omega) \qquad \text{and} \qquad D^i u_k \to D^i u \quad \text{a.e. in} \quad \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega) \quad \text{for} \quad i = 1, \ldots, N.$$

Having in mind (3.15) $-$ (3.18) *we obtain* $\chi_m = B_m u$*. Thus, the proof of the Lemma 3.4 is concluded.*

In view of Lemma 3.4*, there exists at least one weak solution* $u_m \in W^{1,\vec{p}}(\Omega)$ *of the problem* (3.9)*, i.e.*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i v \, dx + \alpha \int_\Omega |T_m(u_m)|^{s-2} T_m(u_m) v \, dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m v \, dx \\ &= \int_\Omega F(x, u_m) v \, dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u_m) D^i v \, dx + \int_{\partial\Omega} G \, v \, d\sigma, \end{aligned} \tag{3.30}$$

for any $v \in W^{1,\vec{p}}(\Omega)$*. For more detail, we refer the reader to (cf. [24], Theorem 8.2).*

Step 2: Weak convergence of the sequence $(u_m)_m$

By using $u_m \in W^{1,\vec{p}}(\Omega)$ *as a test function for the approximate problem* (3.9)*, we have*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i u_m \, dx + \alpha \int_\Omega |T_m(u_m)|^{s-1} \, |u_m| \, dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}} \, dx \\ &= \int_\Omega F(x, u_m) u_m \, dx + \sum_{i=1}^{N} \int_\Omega \Phi_i(x, u_m) D^i u_m \, dx + \int_{\partial\Omega} G \, u_m \, d\sigma. \end{aligned} \tag{3.31}$$

In view of (3.3) *and* (3.7), *we obtain*

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} b(|T_n(u_m)|)|D^i u_m|^{p_i}\,dx + \alpha\int_{\Omega}|T_m(u_m)|^{s-1}\,|u_m|\,dx + \frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}}\,dx\\
&\le \int_{\Omega} F(x,u_m)u_m\,dx + \sum_{i=1}^{N}\int_{\Omega}\Phi_i(x,u_m)D^i u_m\,dx + \int_{\partial\Omega}|G(x)|\,|u_m|\,d\sigma\\
&\le C_0\int_{\Omega}|u_m|\,dx + C_1\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|\,dx + \|G(\cdot)\|_{L^\infty(\Omega)}\int_{\partial\Omega}|u_m|\,d\sigma\\
&\le C_2\|u_m\|_{1,1}\\
&= C_2\Big(\int_{\Omega}|u_m|\,dx + \sum_{i=1}^{N}\int_{\Omega}|D^i u_m|\,dx\Big),
\end{aligned} \tag{3.32}$$

with C_2 *is a constant that doesn't depend on* m. *In view of Young's inequality it follows that*

$$\begin{aligned}
&\frac{b_0}{(1+n)^{\lambda}}\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i}\,dx + \alpha\int_{\{|u_m|\le m\}}|u_m|^{s}\,dx + m^{s-1}\int_{\{|u_m|>m\}}|u_m|\,dx\\
&\le C_3 + \frac{\alpha}{2}\int_{\{|u_m|\le m\}}|u_m|^{s}\,dx + C_2\int_{\{|u_m|>m\}}|u_m|\,dx + \frac{b_0}{2(1+n)^{\lambda}}\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i}\,dx.
\end{aligned} \tag{3.33}$$

By taking $m\ge 1$ *large enough* $\left(\text{for example } \dfrac{m^{s-1}}{2} > C_2\right)$, *we conclude that*

$$\begin{aligned}
&\frac{b_0}{2(1+n)^{\lambda}}\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i}\,dx + \frac{\alpha}{2}\int_{\Omega}|u_m|\,dx\\
&\le \frac{b_0}{2(1+n)^{\lambda}}\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i}\,dx + \frac{\alpha}{2}\int_{\{|u_n|\le m\}}|u_m|^{s}\,dx + \alpha\frac{m^{s-1}}{2}\int_{\{|u_n|>m\}}|u_m|\,dx + C_3\\
&\le C_4.
\end{aligned} \tag{3.34}$$

It follows that

$$\begin{aligned}
\|u_m\|_{1,\vec{p}} &= \|u_m\|_{1,1} + \sum_{i=1}^{N}\|D^i u_m\|_{L^{p_i}(\Omega)}\\
&\le \|u_m\|_{L^1(\Omega)} + C_5\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i}\,dx + C_6\\
&\le C_7.
\end{aligned} \tag{3.35}$$

with C_7 *is a constant that doesn't depend on* m. *Thus, the sequence* $(u_m)_m$ *is uniformly bounded in* $W^{1,\vec{p}}(\Omega)$, *and there exists a subsequence still denoted* $(u_m)_m$ *such that*

$$\begin{cases}
u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega),\\
u_m \longrightarrow u & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega,\\
u_m \longrightarrow u & \text{strongly in } L^{1}(\partial\Omega) \text{ and a.e. in } \partial\Omega.
\end{cases} \tag{3.36}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in } L^{\underline{p}'}(\Omega). \tag{3.37}$$

Moreover, in view of (3.34) *we conclude that* $(T_m(u_m))_m$ *is bounded uniformly in* $L^s(\Omega)$, *and since* $T_m(u_m)\to u$ *almost everywhere in* Ω, *we get*

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in } L^s(\Omega). \tag{3.38}$$

Having in mind (3.7) *and the fact that* $u_m \to u$ *a.e. in* Ω, *thanks to Lebesgue dominated convergence theorem we conclude that*

$$F(x,u_m) \longrightarrow F(x,u) \quad \text{strongly in } L^{\underline{p}'}(\Omega). \tag{3.39}$$

and

$$\Phi_i(x,u_m) \longrightarrow \Phi_i(x,u) \quad \textit{strongly in} \quad L^{p'_i}(\Omega) \qquad \textit{for} \quad i=1,\dots,N. \tag{3.40}$$

Step 3 : The convergence almost everywhere of the gradient

By taking $u_m - u$ as a test function for the approximated problem (3.9) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)(D^i u_m - D^i u)\,dx + \alpha\int_\Omega |T_m(u_m)|^{s-2}T_m(u_m)\,(u_m-u)\,dx\\
&+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-u)\,dx\\
&=\int_\Omega F(x,u_m)(u_m-u)\,dx + \sum_{i=1}^{N}\int_\Omega \Phi_i(x,T_n(u_m))(D^i u_m - D^i u)\,dx + \int_{\partial\Omega} G(x)\,(u_m-u)\,d\sigma,
\end{aligned} \tag{3.41}$$

it follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m - D^i u)\,dx\\
&+\alpha\int_\Omega \left(|T_m(u_m)|^{s-2}T_m(u_m) - |T_m(u)|^{s-2}T_m(u)\right)(u_m-u)\,dx\\
&\le \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m - D^i u|\,dx + \alpha\int_\Omega |T_m(u)|^{s-1}\,|u_m-u|\,dx\\
&+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}\,|u_m-u|\,dx + \int_\Omega |F(x,u_m)|\,|u_m-u|\,dx\\
&+\sum_{i=1}^{N}\int_\Omega |\Phi_i(x,T_n(u_m))|\,|D^i u_m - D^i u|\,dx + \int_{\partial\Omega}|G(x)|\,|u_m-u|\,d\sigma.
\end{aligned} \tag{3.42}$$

For the first term on the right-hand side of (3.42), we have $T_n(u_m) \to T_n(u)$ strongly in $L^{p_i}(\Omega)$ then

$$|a_i(x,T_n(u_m),\nabla u)| \longrightarrow |a_i(x,T_n(u),\nabla u)| \quad \textit{strongly in} \quad L^{p'_i}(\Omega),$$

and since $D^i u_m \rightharpoonup D^i u$ weakly in $L^{p_i}(\Omega)$, it follows that

$$\int_\Omega |a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m - D^i u|\,dx \longrightarrow 0 \qquad \textit{for any} \quad i=1,\dots,N. \tag{3.43}$$

Concerning the second and third terms on the right-hand side of (3.42), in view of (3.36) and (3.37) we conclude that

$$\int_\Omega |T_m(u)|^{s-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \textit{as} \quad m\to\infty, \tag{3.44}$$

and

$$\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \textit{as} \quad m\to\infty. \tag{3.45}$$

Moreover, we have $|F(x,u_m)| \to |F(x,u)|$ strongly in $L^{\underline{p}'}(\Omega)$, then

$$\int_\Omega |F(x,u_m)|\,|u_m-u|\,dx \longrightarrow 0 \qquad \textit{as} \quad m\to\infty. \tag{3.46}$$

and since $|\Phi_i(x,u_m)| \to |\Phi_i(x,u)|$ strongly in $L^{p'_i}(\Omega)$ for $i=1,\dots,N$, it follows that

$$\sum_{i=1}^{N}\int_\Omega |\Phi_i(x,u_m)|\,|D^i u_m - D^i u|\,dx \longrightarrow 0 \qquad \textit{as} \quad m\to\infty. \tag{3.47}$$

For the last term on the right-hand side of (3.42)*, we have* $G(x) \in L^\infty(\partial\Omega)$ *and* $u_m \rightharpoonup u$ *weakly in* $L^1(\partial\Omega)$*, then*

$$\int_{\partial\Omega} |G|\,|u_m - u|\,d\sigma \longrightarrow 0 \qquad as \quad m \to \infty. \tag{3.48}$$

By combining (3.42) *and* (3.43) – (3.48) *we conclude that*

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u))\,(D^i u_m - D^i u)\,dx\\ &+\alpha\int_\Omega \left(|T_m(u_m)|^{s-2}T_m(u_m) - |T_m(u)|^{s-2}T_m(u)\right)(u_m - u)\,dx \longrightarrow 0 \quad as \quad m\to\infty.\end{aligned} \tag{3.49}$$

and since $u_m \to u$ *strongly in* $L^{\underline{p}}(\Omega)$*. Thus, in view of Lemma 3.1, we conclude that*

$$\begin{cases} u_m \to u & \text{strongly in} \quad W^{1,\vec{p}}(\Omega),\\ D^i u_m \to D^i u & \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1,\dots,N.\end{cases} \tag{3.50}$$

Step 4 : Passage to the limit

In view of (3.50) *we have* $a_i(x,T_n(u_m),\nabla u_m) \longrightarrow a_i(x,T_n(u),\nabla u)$ *almost everywhere in* Ω*, then*

$$a_i(x,T_n(u_m),\nabla u_m) \rightharpoonup a_i(x,T_n(u),\nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega) \quad \text{for} \quad i=1,\dots,N. \tag{3.51}$$

Thus, by taking $v \in W^{1,\vec{p}}(\Omega)$ *as a test function for the approximate problem* (3.9)*, we obtain*

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^i v\,dx + \alpha\int_\Omega |T_m(u_m)|^{s-2}T_m(u_m)v\,dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m v\,dx\\ &= \int_\Omega F(x,u_m)v\,dx + \sum_{i=1}^{N}\int_\Omega \Phi_i(x,u_m)D^i v\,dx + \int_{\partial\Omega} G\,v\,d\sigma,\end{aligned} \tag{3.52}$$

In view of (3.37), (3.38), (3.39), (3.40) *and* (3.51)*, by letting* m *tends to infinity we conclude that*

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,dx + \alpha\int_\Omega |u|^{s-2}uv\,dx\\ &= \int_\Omega F(x,u)v\,dx + \sum_{i=1}^{N}\int_\Omega \Phi_i(x,u)D^i v\,dx + \int_{\partial\Omega} G\,v\,d\sigma.\end{aligned} \tag{3.53}$$

Thus, the proof of Theorem 3.3 is concluded.

4 Existence of renormalized solutions

We consider the quasilinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha|u|^{s-2}u = f(x,u) - \sum_{i=1}^{N} D^i\phi_i(x,u) & on \quad \Omega,\\ \sum_{i=1}^{N}(a_i(x,T_n(u),\nabla u) - \phi_i(x,u))\cdot n_i = g(x) & in \quad \partial\Omega,\end{cases} \tag{4.1}$$

where the nonlinear term $f(x,s)$ *is a Carathéodory function that verifies the growth condition*

$$|f(x,s)| \le f_0(x) + c(x)|s|^r, \tag{4.2}$$

where $f_0(\cdot) \in L^1(\Omega)$, *and* $1 < r < s-1$ *such that* $c(x) \in L^{\frac{s-1}{s-1-r}}(\Omega)$..
The Carathéodory function $\phi(\cdot,\cdot)\ :\ \Omega \times I\!R \longmapsto I\!R^N$ *satisfies*

$$\phi(x,s) = (\phi_1(x,s),\ldots,\phi_N(x,s)) \qquad \text{such that} \qquad |\phi_i(x,s)| \le c_i(x)(1+|s|)^{\sigma_i}, \tag{4.3}$$

where $0 < \sigma_i < \dfrac{s-1}{p_i'} - \dfrac{\lambda}{p_i}$ *and* $c_i(x)$ *are some positive functions in* $L^{\gamma_i}(\Omega)$ *with*

$$\gamma_i > \frac{p_i'(s-1)}{s-1-p_i'(\sigma_i+\frac{\lambda}{p_i})}.$$

Definition 4.1. A measurable function u is called a renormalized solution of the quasilinear elliptic problem (4.1) if $u \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$, with $|u|^{s-1} \in L^1(\Omega)$, ans $f(x,s) \in L^1(\Omega)$. such that

$$\lim_{h\to\infty} \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}} a_i(x,u,\nabla u)D^i u\,dx = 0,$$

where u verifies the following equality

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)(D^iuS'(u)\varphi + S(u)D^i\varphi\,)\,dx + \alpha\int_\Omega |u|^{s-2}uS(u)\varphi\,dx\\ &\quad = \int_\Omega f(x,u)S(u)\varphi\,dx + \sum_{i=1}^{N}\int_\Omega \phi_i(x,u)(D^iuS(u)\varphi + S(u)D^i\varphi)\,dx + \int_{\partial\Omega} gS(u)\varphi d\sigma,\end{aligned} \tag{4.4}$$

for every $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ and any smooth function $S(\cdot) \in W^{1,\infty}(\Omega)$ with a compact support.

The existence result is the following theorem.

Theorem 4.2. *Let* $g \in L^1(\partial\Omega)$. *Assuming that* (3.2) − (3.4) *and* (4.2) − (4.3) *hold true, then there exists at least one renormalized solution of the problem* (4.1).

Proof of Theorem 4.2

Step 1: Approximate problems

Let $n \in I\!N^*$, *we set* $f_n(x,s) = T_n(f(x,s))$ *and* $g_n = T_n(g(\cdot))$, *then the sequence* $(g_n(\cdot))_n$ *is bounded in* $L^\infty(\partial\Omega) \cap L^1(\partial\Omega)$ *such that*

$$g_n \longrightarrow g \qquad \text{strongly in} \quad L^1(\partial\Omega).$$

We consider the approximate problem :

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + \alpha|u_n|^{s-2}u_n = f_n(x,T_n(u_n)) - \sum_{i=1}^{N} D^i\phi_{i,n}(x,u_n) & \text{in } \Omega,\\ \displaystyle\sum_{i=1}^{N}(a_i(x,T_n(u_n),\nabla u_n) - \phi_{i,n}(x,u_n))n_i = g_n(x) & \text{on } \partial\Omega,\end{cases} \tag{4.5}$$

where $\phi_{i,n}(x,s) = T_n(\phi_i(x,T_n(s)))$ *for* $i = 1,\ldots,N$.
In view of the theorem 3.3, there exists at least one weak solution $u_n \in W^{1,\vec{p}}(\Omega)$ *for the anisotropic quasilinear elliptic Neumann problem* (4.5), *such that* $|u_n|^s \in L^1(\Omega)$.

Step 2: A priori Estimate

By taking $v = \varphi(u_n)$ *as a test function in the approximate problem* (4.5), *with*

$$\varphi(u_n) = \frac{1}{\theta-1}\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)sign(u_n) \quad \text{where} \quad 1 < \theta < \underline{p} \quad \text{small enough},$$

we have

$$\sum_{i=1}^{N}\int_{\Omega}\frac{a_i(x,T_n(u_n),\nabla u_n)D^iu_n}{(1+|u_n|)^{\theta}}\,dx+\alpha\int_{\Omega}|u_n|^{s-1}|\varphi(u_n)|\,dx$$
$$\leq\int_{\Omega}|f_n(x,u_n)||\varphi(u_n)|\,dx+\sum_{i=1}^{N}\int_{\Omega}\frac{\phi_i(x,T_n(u_n))D^iu_n}{(1+|u_n|)^{\theta}}\,dx+\int_{\partial\Omega}|g_n(x)||\varphi(u_n)|\,d\sigma.$$

In view of (3.3), (4.2) *and* (4.3), *using Young's inequality, we obtain*

$$\begin{aligned}
&b_0\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx+\alpha\int_{\Omega}|u_n|^{s-1}|\varphi(u_n)|\,dx\\
&\leq\int_{\Omega}\Big(|f_0(x)|+c(x)|u_n|^{r}\Big)|\varphi(u_n)|\,dx+\sum_{i=1}^{N}\int_{\Omega}\frac{c_i(x)(1+|T_n(u_n)|)^{\sigma_i}}{(1+|u_n|)^{\theta}}|D^iu_n|\,dx\\
&+\int_{\partial\Omega}|g_n(x)||\varphi(u_n)|\,d\sigma\\
&\leq\int_{\Omega}|f_0(x)|\,|\varphi(u_n)|\,dx+\int_{\partial\Omega}|g_n(x)|\,|\varphi(u_n)|\,d\sigma+\int_{\Omega}c(x)|u_n|^{r}|\varphi(u_n)|\,dx\\
&+\frac{b_0}{2}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx+C_0\int_{\Omega}|c_i(x)|^{p_i'}(1+|u_n|)^{p_i'(\sigma_i+\frac{\lambda}{p_i})-\theta}\,dx\\
&\leq C_1+C_2\int_{\Omega}|c(x)|^{\frac{s-1}{s-1-r}}\,dx+\frac{\alpha}{4}\int_{\Omega}|u_n|^{s-1}|\varphi(u_n)|\,dx+\frac{b_0}{2}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx\\
&+C_3\int_{\Omega}|c_i(x)|^{\frac{p_i'(s-1)}{s-1-p_i'(\sigma_i+\frac{\lambda}{p_i})+\theta}}\,dx+\frac{\alpha}{8(\theta-1)}\int_{\Omega}|u_n|^{s-1}\,dx.
\end{aligned}\tag{4.6}$$

Let $R=2^{\frac{1}{\theta-1}}-1$, *we have* $|\varphi(u_n)|\geq\dfrac{1}{2(\theta-1)}$ *on the set* $\{|u_n|\geq R\}$, *then*

$$\begin{aligned}
\frac{\alpha}{8(\theta-1)}\int_{\Omega}|u_n|^{s-1}\,dx&=\frac{\alpha}{8(\theta-1)}\int_{\{|iu_n|\leq R\}}|u_n|^{s-1}\,dx+\frac{\alpha}{8(\theta-1)}\int_{\{|iu_n|>R\}}|u_n|^{s-1}\,dx\\
&\leq\frac{\alpha|R|^{s-1}meas(\Omega)}{8(\theta-1)}+\frac{\alpha}{4}\int_{\{|u_n|>R\}}|u_n|^{s-1}|\varphi(u_n)|\,dx\\
&\leq C_4+\frac{\alpha}{4}\int_{\Omega}|u_n|^{s-1}|\varphi(u_n)|\,dx.
\end{aligned}$$

Thanks to (4.6), *we conclude that*

$$\frac{b_0}{2}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx+\frac{\alpha}{2}\int_{\Omega}|u_n|^{s-1}|\varphi(u_n)|\,dx\leq C_5.\tag{4.7}$$

It follows that

$$\frac{1}{(1+k)^{\theta+\lambda}}\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}|D^iu_n|^{p_i}\,dx\leq\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx\leq C_6.$$

then

$$\sum_{i=1}^{N}\int_{\Omega}|D^iT_k(u_n)|^{p_i}\,dx\leq C_6(1+k)^{\theta+\lambda}\qquad\textit{for any}\quad k>0.\tag{4.8}$$

where C_6 is a constant that doesn't depend on n and k. It follows that

$$\begin{aligned}\|T_k(u_n)\|_{1,\vec{p}} &= \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{L^{p_i}(\Omega)} + \|T_k(u_n)\|_{W^{1,1}(\Omega)} \\ &\leq \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\,dx + N + \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|\,dx + \int_\Omega |T_k(u_n)|\,dx \\ &\leq 2\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\,dx + N(1+meas(\Omega)) + k\cdot\ meas(\Omega) \\ &\leq C_8.k^{\theta+\lambda} \qquad\qquad \textit{for any} \quad 1<\theta<\underline{p}-\lambda.\end{aligned}$$

Thus, the sequence $(T_k(u_n))_n$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a measurable function $\nu_k \in W^{1,\vec{p}}(\Omega)$, such that :

$$\begin{cases} T_k(u_n) \rightharpoonup \nu_k & \textit{weakly in } \ W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to \nu_k & \textit{strongly in } \ L^{\underline{p}}(\Omega). \end{cases} \tag{4.9}$$

Moreover, thanks to (4.7) we have

$$\int_\Omega |u_n|^{s-1}\,dx \leq \int_{\{|u_n|\leq R\}} |u_n|^{s-1}\,dx + 2(\theta-1)\int_{\{|u_n|>R\}} |u_n|^{s-1}|\varphi(u_n)|\,dx \leq C_7, \tag{4.10}$$

thus, we obtain

$$k^{s-1}meas\{|u_n|>k\} = \int_{\{|u_n|>k\}} |u_n|^{s-1}\,dx \leq \int_\Omega |u_n|^{s-1}\,dx \leq C_8,$$

it follows that

$$meas\{|u_n|>k\} \leq \frac{C_8}{k^{s-1}} \longrightarrow 0 \qquad\qquad as \quad k\to\infty. \tag{4.11}$$

On the other hand, we have for every $\delta>0$,

$$meas\{|u_n-u_m|>\delta\} \leq meas\{|u_n|>k\} + meas\{|u_m|>k\} + meas\{|T_k(u_n)-T_k(u_m)|>\delta\}. \tag{4.12}$$

Let $\varepsilon>0$, thanks to (4.11) we can choose $k=k(\varepsilon)$ large enough such that

$$meas\{|u_n|>k\} \leq \frac{\varepsilon}{3} \qquad \textit{and} \qquad meas\{|u_m|>k\} \leq \frac{\varepsilon}{3} \tag{4.13}$$

Moreover, in view of (4.9), we have $T_k(u_n) \longrightarrow \nu_k$ strongly in $L^{\underline{p}}(\Omega)$ and a.e in Ω. Thus $(T_k(u_n))_{n\in N}$ is a Cauchy sequence in measure, and for any $k>0$ and $\delta,\varepsilon>0$, there exists $n_0 = n_0(k,\delta,\varepsilon)$ such that

$$meas\{|T_k(u_n)-T_k(u_m)|>\delta\} \leq \frac{\varepsilon}{3} \quad \textit{for all} \quad m,n\geq n_0(k,\delta,\varepsilon). \tag{4.14}$$

By combining (4.13) and (4.14), we conclude that, for all $\delta,\varepsilon>0$, there exists $n_0=n_0(\delta,\varepsilon)$ such that

$$meas\{|u_n-u_m|>\delta\} \leq \varepsilon \qquad \textit{for any} \quad m,n\geq n_0(\delta,\varepsilon).$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \textit{weakly in} \quad W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to T_k(u) & \textit{strongly in} \quad L^1(\Omega) \quad \textit{and} \quad \textit{a.e in} \quad \Omega, \\ T_k(u_n) \to T_k(u) & \textit{weakly in} \quad L^1(\partial\Omega). \end{cases} \tag{4.15}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \qquad \textit{strongly in} \quad L^{p_i}(\Omega) \qquad \textit{and} \quad \textit{a.e. in} \quad \Omega \quad \textit{for} \quad i=1,\dots,N. \tag{4.16}$$

Moreover, thanks to (4.10) *the sequence* $(|u_n|^{s-2}u_n)_n$ *is uniformly bounded in* $L^1(\Omega)$, *we conclude that*

$$|u_n|^{s-2}u_n \rightharpoonup |u|^{s-2}u \quad \text{weakly in} \quad L^1(\Omega). \tag{4.17}$$

Furthermore, thanks to (4.11) *we have* $\Big\|\dfrac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} \longrightarrow 0$ *as k tends to infinity, it follows necessary*

$$\begin{aligned}
\lim_{k\to\infty}\Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\partial\Omega)} &\leq \lim_{k\to\infty} C\Big\|\frac{T_k(u_n)}{k}\Big\|_{W^{1,1}(\Omega)} \\
&\leq C\lim_{k\to\infty}\Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C\lim_{k\to\infty}\sum_{i=1}^N\Big\|\frac{D^iT_k(u_n)}{k}\Big\|_{L^1(\Omega)} \\
&\leq C\lim_{k\to\infty}\Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C\lim_{k\to\infty}\sum_{i=1}^N\|1\|_{L^{p_i'}(\Omega)}\Big\|\frac{D^iT_k(u_n)}{k}\Big\|_{L^{p_i}(\Omega)} \\
&\longrightarrow 0 \qquad \text{as} \quad k\to\infty.
\end{aligned}$$

We conclude that

$$\frac{T_k(u_n)}{k} \longrightarrow 0 \quad \text{weak} - * \ \text{in } L^\infty(\partial\Omega). \tag{4.18}$$

Step 3: The equi-integrability of the sequence $(f(x, T_n(u_n)))_n$

Now, we shall show that

$$f_n(x, T_n(u_n)) \longrightarrow f(x,u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{4.19}$$

Let $h \geq 1$, *by taking* $\dfrac{T_h(u_n)}{h}$ *as a test function in the approximate problem* (4.5), *we have*

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^N\int_\Omega a_i(x, T_n(u_n), \nabla u_n)D^iT_h(u_n)\,dx + \frac{\alpha}{h}\int_\Omega |u_n|^{s-1}\,|T_h(u_n)|\,dx \\
&= \int_\Omega f_n(x, T_n(u_n))\frac{T_h(u_n)}{h}\,dx + \frac{1}{h}\sum_{i=1}^N\int_\Omega \phi_i(x, T_n(u_n))D^iT_h(u_n)\,dx + \int_{\partial\Omega} g_n(x)\frac{T_h(u_n)}{h}\,d\sigma.
\end{aligned} \tag{4.20}$$

In view of (3.3), (4.2) *and* (4.3), *and thanks to Young's inequality we obtain*

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^N\int_\Omega a_i(x, T_n(u_n), \nabla u_n)D^iT_h(u_n)\,dx + \frac{b_0}{2h}\sum_{i=1}^N\int_{\{|u_n|\leq h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx \\
&+\frac{\alpha}{h}\int_\Omega |u_n|^{s-1}|T_h(u_n)|\,dx \\
&\leq \int_\Omega |f_n(x, T_n(u_n))|\frac{|T_h(u_n)|}{h}\,dx + \frac{1}{h}\int_{\partial\Omega}|g_n(x)||T_h(u_n)|\,d\sigma \\
&+\frac{1}{h}\sum_{i=1}^N\int_{\{|u_n|\leq h\}}|\phi_i(x, T_n(u_n))||D^iu_n|\,dx \\
&\leq \int_\Omega (|f_0(x)| + |c(x)||u_n|^r)\frac{|T_h(u_n)|}{h}\,dx + \int_{\partial\Omega}|g_n(x)|\frac{|T_h(u_n)|}{h}\,d\sigma \\
&+\sum_{i=1}^N\frac{1}{h}\int_{\{|u_n|\leq h\}}c_i(x)(1+|u_n|)^{\sigma_i}|D^iu_n|\,dx \\
&\leq \int_\Omega |f_0(x)|\frac{|T_h(u_n)|}{h}\,dx + \int_{\partial\Omega}|g_n(x)|\frac{|T_h(u_n)|}{h}\,d\sigma + C_0\int_\Omega |c(x)|^{\frac{s-1}{s-1-r}}\frac{|T_h(u_n)|}{h}\,dx \\
&+\frac{\alpha}{2}\int_\Omega |u_n|^{s-1}\frac{|T_h(u_n)|}{h}\,dx + \frac{C_1}{h}\Big(1 + \sum_{i=1}^N\int_\Omega |c_i(x)|^{\frac{p_i'(s-1)}{s-1-p_i'(\sigma_i+\frac{\lambda}{p_i})}}\,dx\Big) \\
&+\frac{b_0}{4h}\sum_{i=1}^N\int_{\{|u_n|\leq h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^\lambda}\,dx.
\end{aligned} \tag{4.21}$$

We deduce that

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_h(u_n)\,dx+\frac{b_0}{4h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\lambda}}\,dx\\
&+\frac{\alpha}{2h}\int_{\Omega}|u_n|^{s-1}|T_h(u_n)|\,dx\\
&\le\int_{\Omega}|f_0(x)|\frac{|T_h(u_n)|}{h}\,dx+\int_{\partial\Omega}|g_n(x)|\frac{|T_h(u_n)|}{h}\,d\sigma\\
&+C_0\int_{\Omega}|c(x)|^{\frac{s-1}{s-1-r}}\frac{|T_h(u_n)|}{h}\,dx+\frac{C_1}{h}\Big(1+\sum_{i=1}^{N}\int_{\Omega}|c_i(x)|^{\frac{p_i'(s-1)}{s-1-p_i'(\sigma_i+\frac{\lambda}{p_i})}}\,dx\Big).
\end{aligned}\tag{4.22}$$

For the first term on the right-hand side of (4.22), we have meas$\{|u_n|\ge h\}\to 0$ *as* h *tends to infinity, it follows that* $\frac{T_h(u_n)}{h}\rightharpoonup 0$ *weak*$-*$ *in* $L^{\infty}(\Omega)$*, and since* $|f_0(x)|\in L^1(\Omega)$ *then*

$$\int_{\Omega}|f_0(x)|\frac{|T_h(u_n)|}{h}\,dx\longrightarrow 0\qquad as\quad h\to\infty.\tag{4.23}$$

Moreover, since $|c(x)|^{\frac{s-1}{s-1-r}}$ *belongs to* $L^1(\Omega)$*, then*

$$\int_{\Omega}|c(x)|^{\frac{s-1}{s-1-r}}\frac{|T_h(u_n)|}{h}\,dx\longrightarrow 0\qquad as\quad h\to\infty.\tag{4.24}$$

For the second term on the right-hand side of (4.22), thanks to (4.18) we have $\frac{|T_h(u_n)|}{h}\rightharpoonup 0$ *weak*$-*$ *in* $L^{\infty}(\partial\Omega)$*, and since* $g\in L^1(\partial\Omega)$ *it follows that*

$$\int_{\partial\Omega}|g_n(x)|\frac{|T_h(u_n)|}{h}\,d\sigma\longrightarrow 0\qquad as\quad h\to\infty.\tag{4.25}$$

Furthermore, we have $|c_i(x)|^{\frac{p_i'(s-1)}{s-1-p_i'(\sigma_i+\frac{\lambda}{p_i})}}$ *belongs to* $L^1(\Omega)$*, thus, in view of (4.22) and (4.23) – (4.25), we deduce that*

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0.\tag{4.26}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\lambda}}\,dx=0,\tag{4.27}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{s-1}\,dx=0.\tag{4.28}$$

In view of Young's inequality we deduce that

$$\begin{aligned}
\int_{\{|u_n|>h\}}|f_n(x,T_n(u_n))|\,dx&\le\int_{\{|u_n|>h\}}|f_0|+|c_0(x)||u_n|^{r}\,dx\\
&\le\int_{\{|u_n|>h\}}|f_0|\,dx+\int_{\{|u_n|>h\}}|c_0(x)|^{\frac{s-1}{s-r-1}}\,dx\\
&+\int_{\{|u_n|>h\}}|u_n|^{s-1}\,dx\longrightarrow 0\qquad as\quad h\to\infty.
\end{aligned}\tag{4.29}$$

It follows that : for all $\eta>0$*, there exists* $h(\eta)>0$ *such that*

$$\int_{\{|u_n|>h(\eta)\}}|f_n(x,T_n(u_n))|\,dx\le\frac{\eta}{2}.\tag{4.30}$$

On the other hand, let E be a measurable subset of Ω, we have

$$\int_E |f_n(x,T_n(u_n))|\,dx \leq \int_E |f_n(x,T_{h(\eta)}(u_n))|\,dx + \int_{\{|u_n|>h(\eta)\}} |f_n(x,T_n(u_n))|\,dx. \tag{4.31}$$

For any $\eta > 0$, there exists a positive constant $\beta(\eta) > 0$ such that

$$\int_E |f_n(x,T_{h(\eta)}(u_n))|\,dx \leq \frac{\eta}{2} \quad \text{for all } E\subset\Omega \quad \text{such that} \quad meas(E) \leq \beta(\eta). \tag{4.32}$$

By combining (4.30), (4.31) and (4.32), we conclude that

$$\int_E |f_n(x,T_n(u_n))|\,dx \leq \eta \qquad \text{for all} \quad E\subset\Omega \text{ such that } meas(E)\leq\beta(\eta). \tag{4.33}$$

It follows that the sequence $(f_n(x,T_n(u_n)))_n$ is uniformly equi-integrable, and since $f_n(x,T_n(u_n)) \to f(x,u)$ a.e. in Ω, and in view of Vitali's theorem, we conclude that

$$f_n(x,T_n(u_n)) \longrightarrow f(x,u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{4.34}$$

Step 4: Strong convergence of truncations.

In this step, we will show the convergence of the sequence $(D^i u_n)_n$ to $D^i u$ almost everywhere in Ω, for any $i=1,\ldots,N$.
We set

$$\varphi(u_n) = 1 - \frac{|T_{2h}(u_n) - T_h(u_n)|}{h},$$

By taking $(T_k(u_n)-T_k(u))\varphi(u_n)$ as a test function for the approximate problem (4.5), we have

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n|\leq 2h\}} a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\varphi(u_n)\,dx\\
&-\frac{1}{h}\int_{\{h\leq|u_n|\leq 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n.sign(u_n)(T_k(u_n)-T_k(u))\,dx\\
&+\alpha\int_\Omega |u_n|^{s-2}u_n(T_k(u_n)-T_k(u))\varphi(u_n)\,dx\\
&=\int_\Omega f_n(x,T_n(u_n))(T_k(u_n)-T_k(u))\varphi(u_n)\,dx + \int_{\partial\Omega} g_n(x)(T_k(u_n)-T_k(u))\varphi(u_n)\,d\sigma\\
&+\sum_{i=1}^N\int_\Omega \phi_{i,n}(x,u_n)(D^iT_k(u_n)-D^iT_k(u))\varphi(u_n)\,dx\\
&-\frac{1}{h}\sum_{i=1}^N\int_{\{h\leq|u_n|\leq 2h\}} \phi_{i,n}(x,u_n)D^iu_n.sign(u_n)(T_k(u_n)-T_k(u))\,dx
\end{aligned} \tag{4.35}$$

According to (4.3) we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&-\int_{\{k\leq|u_n|\leq 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u)\varphi(u_n)\,dx\\
&\leq\int_\Omega |f_n(x,T_n(u_n))|\,|T_k(u_n)-T_k(u)|\,dx + \alpha\int_\Omega |u_n|^{s-1}\,|T_k(u_n)-T_k(u)|\,dx\\
&+\int_{\partial\Omega}|g_n(x)|\,|T_k(u_n)-T_k(u)|\,d\sigma + \sum_{i=1}^N\int_{\{|u_n|\leq 2h\}} |c_i(x)|\,(1+|u_n|)^{\sigma_i}\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\\
&+\frac{1}{h}\sum_{i=1}^N\int_{\{h\leq|u_n|\leq 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)\,dx\\
&+\frac{1}{h}\sum_{i=1}^N\int_{\{h\leq|u_n|\leq 2h\}} c_i(x)(1+|u_n|)^{\sigma_i}|D^iu_n||T_k(u_n)-T_k(u)|\,dx.
\end{aligned} \tag{4.36}$$

For the first and second terms on the right-hand side of (4.36)*, We have* $T_k(u_n) \rightharpoonup T_k(u)$ *weak*$-\star$ *in* $L^\infty(\Omega)$*, and thanks to* (4.34) *we have* $f_n(x, T_n(u_n)) \longrightarrow f(x,u)$ *strongly in* $L^1(\Omega)$*, we conclude that*

$$\varepsilon_1(n) = \int_\Omega |f_n(x, T_n(u_n))|\, |T_k(u_n) - T_k(u)|\, dx \longrightarrow 0 \qquad as \qquad n \to \infty. \tag{4.37}$$

Moreover, in view of (4.17) *we have* $|u_n|^{s-1} \to |u|^{s-1}$ *strongly in* $L^1(\Omega)$*, it follows that*

$$\varepsilon_2(n) = \int_\Omega |u_n|^{s-1}\, |T_k(u_n) - T_k(u)|\, dx \longrightarrow 0 \qquad as \qquad n \to \infty. \tag{4.38}$$

Concerning the third term on the right-hand side of (4.36)*. We have* $|g_n(x)| \to |g(x)|$ *strongly in* $L^1(\partial\Omega)$ *and since* $T_k(u_n) - T_k(u) \rightharpoonup 0$ *weak*$-\star$ *in* $L^\infty(\partial\Omega)$*, then*

$$\varepsilon_3(n) = \int_{\partial\Omega} |g_n(x)|\, |T_k(u_n) - T_k(u)|\, d\sigma \longrightarrow 0 \qquad as \quad n \to \infty. \tag{4.39}$$

For the fourth term on the right-hand side of (4.36)*, we have* $c_i(x)(1 + |T_{2h}(u_n)|)^{\sigma_i} \longrightarrow c_i(x)(1 + |T_{2h}(u)|)^{\sigma_i}$ *strongly in* $L^{p_i'}(\Omega)$*, and since* $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ *weakly in* $L^{p_i}(\Omega)$ *for any* $i = 1, \ldots, N$*, it follows that*

$$\varepsilon_4(n) = \sum_{i=1}^N \int_{\{|u_n|\le 2h\}} |c_i(x)|\, (1 + |u_n|)^{\sigma_i}\, |D^iT_k(u_n) - D^iT_k(u)|\, dx \longrightarrow 0 \qquad as \quad n \to \infty. \tag{4.40}$$

Moreover, in view of (4.26) *we have*

$$\varepsilon_5(h) = \frac{1}{h}\sum_{i=1}^N \int_{\{h\le|u_n|\le 2h\}} a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) D^iT_{2h}(u_n)\, dx \longrightarrow 0 \qquad as \quad h \to \infty. \tag{4.41}$$

Concerning the last term on the right-hand side of (4.36)*, using Young's inequality we have*

$$\begin{aligned}
\varepsilon_5(h) \;&= \frac{1}{h}\sum_{i=1}^N \int_{\{h\le|u_n|\le 2h\}} c_i(x)(1 + |u_n|)^{\sigma_i}|D^iu_n||T_k(u_n) - T_k(u)|\, dx \\
&\le \frac{1}{h}\sum_{i=1}^N \int_{\{h\le|u_n|\le 2h\}} |c_i(x)|^{p_i'}(1 + |T_{2h}(u_n)|)^{p_i'(\sigma_i + \frac{\lambda}{p_i})}|T_k(u_n) - T_k(u)|\, dx \\
&\quad + \frac{2k}{h}\sum_{i=1}^N \int_{\{|u_n|\le 2h\}} \frac{|D^iu_n|^{p_i}}{(1 + |u_n|)^\lambda}\, dx.
\end{aligned}$$

We have $|c_i(x)|^{p_i'}(1+|T_{2h}(u_n)|)^{p_i'(\sigma_i+\frac{\lambda}{p_i})} \longrightarrow |c_i(x)|^{p_i'}(1+|T_{2h}(u)|)^{p_i'(\sigma_i+\frac{\lambda}{p_i})}$ *strongly in* $L^1(\Omega)$*, and since* $T_k(u_n) \rightharpoonup T_k(u)$ *weak-*$*$ *in* $L^\infty(\Omega)$*, thus*

$$\frac{1}{h}\sum_{i=1}^N \int_{\{h\le|u_n|\le 2h\}} |c_i(x)|^{p_i'}(1+|T_{2h}(u_n)|)^{p_i'(\sigma_i+\frac{\lambda}{p_i})}|T_k(u_n)-T_k(u)|\, dx \longrightarrow 0 \qquad as \quad n \to \infty.$$

In view of (4.27) *we obtain*

$$\varepsilon_5(n,h) = \frac{1}{h}\sum_{i=1}^N \int_{\{h\le|u_n|\le 2h\}} c_i(x)(1+|u_n|)^{\sigma_i}|D^iu_n||T_k(u_n)-T_k(u)|\, dx \longrightarrow 0 \;\; as \;\; n,h \to \infty. \tag{4.42}$$

By combining (4.36) *and* (4.37) – (4.42), *we conclude that*

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\right)(D^iT_k(u_n)-D^iT_k(u))\,dx\\&+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\\&-\sum_{i=1}^{N}\int_{\{k\le|u_n|\le 2h\}}a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\varphi(u_n)\,dx\\&\le\varepsilon_6(n,h).\end{aligned}\tag{4.43}$$

For the second term on the right-hand side of (4.43), *we have* $T_k(u_n)\to T_k(u)$ *strongly in* $L^{p_i}(\Omega)$, *then* $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ *strongly in* $L^{p_i'}(\Omega)$, *and since* $D^iT_k(u_n)$ *tends to* $D^iT_k(u)$ *weakly in* $L^{p_i}(\Omega)$, *we conclude that*

$$\varepsilon_7(n)=\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\longrightarrow 0\quad as\quad n\to\infty.\tag{4.44}$$

Concerning the third term on the left-hand side of (4.43), *we have* $(a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n)))_n$ *is bounded in* $L^{p_i'}(\Omega)$, *then there exists* $\nu_i\in L^{p_i'}(\Omega)$ *such that* $|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\rightharpoonup\nu_i$ *weakly in* $L^{p_i'}(\Omega)$, *and since* $|D^iT_k(u)|\in L^{p_i}(\Omega)$ *for any* $i=1,\dots,N$, *it follows that*

$$\begin{aligned}\varepsilon_7(n)&=\sum_{i=1}^{N}\left|\int_{\{k\le|u_n|\le 2h\}}a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\varphi(u_n)\,dx\right|\\&\le\sum_{i=1}^{N}\int_{\{k\le|u_n|\le 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\,|D^iT_k(u)|\,dx\longrightarrow 0\quad as\quad n\to\infty,\end{aligned}\tag{4.45}$$

By combining (4.43) *and* (4.44) – (4.45), *we conclude that*

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),D^iT_k(u_n))-a_i(x,T_k(u_n),D^iT_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\to 0,\tag{4.46}$$

as $n,h\to\infty$. *Moreover, since* $T_k(u_n)\to T_k(u)$ *strongly in* $L^{\underline{p}}(\Omega)$, *it follows that*

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\&+\int_{\Omega}(|T_k(u_n)|^{\underline{p}-2}T_k(u_n)-|T_k(u)|^{\underline{p}-2}T_k(u))(T_k(u_n)-T_k(u))\,dx\to 0\quad as\quad n\to\infty.\end{aligned}\tag{4.47}$$

In view of Lemma 3.1, *we conclude that*

$$\begin{cases}T_k(u_n)\to T_k(u) & \text{strongly in}\quad W^{1,\vec{p}}(\Omega),\\ D^iu_n\to D^iu & \text{a.e. in}\quad\Omega\quad\text{for}\quad i=1,\dots,N.\end{cases}\tag{4.48}$$

Moreover, since $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ *tends to* $a_i(x,u,\nabla u)D^iu$ *almost everywhere in* Ω, *and in view of Fatou's lemma and* (4.26), *we conclude that*

$$\begin{aligned}&\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}}a_i(x,u,\nabla u)D^iu\,dx\\&\le\lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\&\le\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0.\end{aligned}\tag{4.49}$$

Step 5 : Passage to the limit

Let $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$*, and let* $S(\cdot)$ *be a smooth function in* $W^{1,\infty}(I\!R)$ *such that supp* $(S(\cdot)) \subseteq [-M, M]$ *for some* $M \geq 0$*.*
By choosing $S(u_n)\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ *as a test function in the approximate problem* (4.5)*, we obtain*

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi)\, dx + \alpha \int_\Omega |u_n|^{s-2} u_n S(u_n)\varphi\, dx \\
&= \int_\Omega f_n(x, T_n(u_n)) S(u_n)\varphi\, dx + \int_\Omega g_n(x) S(u_n)\varphi\, d\sigma \\
&\quad + \sum_{i=1}^{N} \int_\Omega \phi_i(x, T_n(u_n))(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi)\, dx.
\end{aligned} \tag{4.50}$$

We begin by the first term on the left-hand side of (4.50)*, we have*

$$\begin{aligned}
&\int_\Omega a_i(x, T_n(u_n), \nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n) D^i \varphi)\, dx \\
&= \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi\right)\, dx.
\end{aligned}$$

In view of (3.2) *and* (4.48)*, we have* $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ *is uniformly bounded in* $L^{p_i'}(\Omega)$*, and since* $a_i(x, T_M(u_n), \nabla T_M(u_n))$ *tends to* $a_i(x, T_M(u), \nabla T_M(u))$ *almost everywhere in* Ω*, it follows that*

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \qquad \text{weakly in} \quad L^{p_i'}(\Omega),$$

and since $S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi$ *tends to* $S'(u)\varphi D^i T_M(u) + S(T_M(u)) D^i \varphi$ *strongly in* $L^{p_i}(\Omega)$*, we deduce that*

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n) D^i \varphi)\, dx \\
&= \lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi\right)\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \left(S'(u)\varphi D^i T_M(u) + S(T_M(u)) D^i \varphi\right)\, dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u) D^i \varphi\right)\, dx.
\end{aligned} \tag{4.51}$$

Concerning the second term on the right-hand side of (4.50)*, we have* $S(T_M(u_n))\varphi \to S(T_M(u))\varphi$ *weak*$-*$ *in* $L^\infty(\Omega)$*, and thanks to* (4.17) *we deduce that*

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{s-2} u_n S(T_M(u_n))\varphi\, dx = \int_\Omega |u|^{s-2} u S(u)\varphi\, dx. \tag{4.52}$$

Moreover thanks to (4.34)*, we have*

$$\lim_{n\to+\infty} \int_\Omega f_n(x, T_n(u_n)) S(T_M(u_n))\varphi\, dx = \int_\Omega f(x, T_M(u)) S(T_M(u))\varphi\, dx = \int_\Omega f(x, u) S(u)\varphi\, dx. \tag{4.53}$$

Similarly, we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ *weak*$-*$ *in* $L^\infty(\partial\Omega)$ *then*

$$\lim_{n\to\infty} \int_{\partial\Omega} g_n(x) S(T_M(u_n))\varphi\, d\sigma = \int_{\partial\Omega} g(x) S(T_M(u))\varphi\, d\sigma = \int_{\partial\Omega} g(x) S(u)\varphi\, d\sigma. \tag{4.54}$$

For the last term on the right-hand side of (4.50), *we have*

$$\sum_{i=1}^{N}\int_{\Omega}\phi_i(x,T_n(u_n))(D^iu_nS'(u_n)\varphi+S(u_n)D^i\varphi)\,dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\phi_i(x,T_M(u_n))(D^iT_M(u_n)S'(u_n)\varphi+S(T_M(u_n))D^i\varphi)\,dx,$$

since $\phi_i(x,T_M(u_n))\longrightarrow\phi_i(x,T_M(u)$ *strongly in* $L^{p_i'}(\Omega)$, *and similarly to* (4.51), *we show that*

$$\begin{aligned}&\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\phi_i(x,T_M(u_n))(D^iT_M(u_n)S'(u_n)\varphi+S(T_M(u_n))D^i\varphi)\,dx\\&=\sum_{i=1}^{N}\int_{\Omega}\phi_{i,n}(x,T_M(u))\left(S'(u)\varphi D^iT_M(u)+S(T_M(u))D^i\varphi\right)\,dx\\&=\sum_{i=1}^{N}\int_{\Omega}\phi_i(x,u)(D^iuS'(u)\varphi+S(u)D^i\varphi)\,dx.\end{aligned}\tag{4.55}$$

Hence, putting all the terms (4.50), *and* (4.51)-(4.55) *together, we obtain*

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,u,\nabla u)(D^iuS'(u)\varphi+S(u)D^i\varphi)\,dx+\alpha\int_{\Omega}|u|^{s-2}uS(u)\varphi\,dx\\&=\int_{\Omega}f(x,u)S(u)\varphi\,dx+\int_{\Omega}g(x)S(u)\varphi\,d\sigma+\sum_{i=1}^{N}\int_{\Omega}\phi_i(x,u)(D^iuS'(u)\varphi+S(u)D^i\varphi)\,dx,\end{aligned}\tag{4.56}$$

which conclude the proof of Theorem 4.2.

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