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SOME LOCAL SPECTRAL PROPERTIES OF 3×3-BLOCK MATRIX OF LINEAR RELATIONS

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Abstract In this paper, we study some local spectral properties of 3×3-block matrix linear relations. More precisely, we give a necessary and sufficient conditions for linear relations matrix to have the single valued property in terms of the surjective spectrum and the analytic residuum of its diagonal entries. On the following, by means of it, we illustrate a new characterization of spectra and local spectra of linear relations matrix.

1 Introduction

In 1952, N. Dunford, has introduced the notion of single-valued extension property (SVEP) and some results of local spectral theory for bounded operators in Banach spaces, [10]. An operator T is said to have the SVEP at $\lambda_0 \in \mathbb{C}$, if the only analytic function which satisfies

$$(\lambda - T)f(\lambda) = 0$$

is $f = 0$ (see [11]). In addition, P. Aiena treated with this notion in [1, 2, 3, 4]. More precisely, he establish several results of Fredholm theory, based on the SVEP. Further, A. Ammar, A. Bouchekoua and A. Jeribi [7] and M. Mnif and A.-A. Ouled-Hmed [12], extended this concept to linear relation and studied some properties of the local spectral theory of linear relations. Further, let T is a bounded linear relation and x be a vector in X. The local resolvent of T at x denoted by $\rho_T(x)$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ and an analytic function $f : U \longrightarrow X$ such that the equation

$$(\lambda_0 - T)f(\lambda_0) = x + T(0), \text{ holds for all } \lambda_0 \in U,$$

when $T(0)$ is the multivalued part of T. The complement of $\rho_T(x)$ in $\mathbb{C}$ is called the local spectrum of T at x and denoted by $\sigma_T(x)$.
In [8], A. Ammar et all investigated certain properties of local spectral theory for upper triangular $2{\times}2$ -block matrix of linear relations. Also, in [**?**] A. Ammar et all checked up on a few properties of the local spectra of a 2×2-block matrix of linear relations.
In our paper, we treat some results about the local spectral theory of 3×3-block matrix of linear relations which is defined by

$$\mathcal{T}_0 = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix},$$

on $X_1 \oplus X_2 \oplus X_3$ where $T_{ii} \in \mathcal{LR}(X_i)$, and $T_{ij} \in \mathcal{LR}(X_j, X_i)$ for all $i \in \{1, 2, 3\}$, $j \in \{2, 3\}$ and X_1, X_2, X_3 are complex Banach spaces.

We organize our paper as the following: In section 2, we study some preliminary and auxiliary results of the local spectral theory of linear relations that we will be needed to prove our main results. In section 3, we investigate some results of the local spectral theory of 3×3-block matrix of linear relations involving SVEP. Furthermore, we treat some local spectral properties of $\mathcal{T}$ and $\mathcal{T}_0$ through the surjective spectrum and the analytic residuum of its entries. Besides, under necessary and sufficient conditions, we prove that $\mathcal{T}$ has the single valued extension property.

2 Preliminaries and auxiliary results

Throughout the paper, X and Y are two Banach spaces over the complex plane $\mathbb{C}$. We are going to recall some facts about the linear relations needed eventually. A linear relation $T : X \longrightarrow Y$ is a mapping from a subspace $\mathcal{D}(T) = \{x \in X : Tx \neq \emptyset\}$ of X, called the domain of T, into the collection of nonempty subsets of Y such that

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for all nonzero scalars $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(T)$. We denote the class of linear relations from X into Y by $\mathcal{LR}(X,Y)$ and as useful we write $\mathcal{LR}(X) = \mathcal{LR}(X,X)$. A linear relation $T \in \mathcal{LR}(X,Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \{(x,y) \in X \times Y : x \in \mathcal{D}(T) \text{ and } y \in Tx\}.$$

If M and N are subspaces of X, then

$$M^{\perp} = \{x' \in X^* \text{ such that } x'(x) = 0 \text{ for all } x \in M\}$$

and

$$N^{\top} = \{x \in X \text{ such that } x'(x) = 0 \text{ for all } x' \in N\},$$

where X^* is the dual of X. The adjoint T^* of T is defined by

$$G(T^*) = G(-T^{-1})^{\perp} \subset X^* \times Y^*, \text{ where}$$

$$\langle (y,x),(y',x')\rangle = \langle x,x'\rangle + \langle y,y'\rangle = x'(x) + y'(y).$$

The null, the range spaces and the multivalued part of T are defined respectively by:

$$\begin{aligned} \mathcal{N}(T) &= \{x \in \mathcal{D}(T) : (x,0) \in G(T)\}, \\ \mathcal{R}(T) &= \{y : (x,y) \in G(T)\} \text{ and} \\ T(0) &= \{y : (0,y) \in G(T)\}. \end{aligned}$$

We note that T is single valued (or operator) if, and only if, $T(0) = \{0\}$. T is said to be injective if $\mathcal{N}(T) = \{0\}$ and surjective if $\mathcal{R}(T) = Y$. If T is injective and surjective we say that T is bijective. T is called to be bounded, if $\mathcal{D}(T) = X$ and $\|T\|_{\mathcal{LR}(X,Y)} < \infty$. The set of all bounded linear relations from X into Y by $\mathcal{BR}(X,Y)$. If $X = Y$, then $\mathcal{BR}(X,X) = \mathcal{BR}(X)$. T is said to be closed if its graph is closed. The class of all closed linear relations is denote by $\mathcal{CR}(X,Y)$ and as useful we write $\mathcal{CR}(X,X) := \mathcal{CR}(X)$. Also, the set of all bounded and closed linear relations from X into Y is denoted by $\mathcal{BCR}(X,Y)$. When $X = Y$, we have $\mathcal{BCR}(X,Y) = \mathcal{BCR}(X)$.

Let us recall some properties of linear relations which shown in [9].

Proposition 2.1. *Let $T \in \mathcal{LR}(X,Y)$. Then,*
(i) *For $x \in \mathcal{D}(T)$, we have $y \in Tx$ if, and only if, $Tx = y + T(0)$.*
(ii) *In particular, $0 \in Tx$ if, and only if, $Tx = T(0)$.*

Definition 2.2. Let $T \in \mathcal{LR}(X,Y)$. A linear operator E is called a selection of T if

$$T = E + T - T \text{ and } \mathcal{D}(T) = \mathcal{D}(E).$$

If E is a selection of T, then we have, for all $x \in \mathcal{D}(T)$

$$Tx = Ex + T(0).$$

Now, we recollect some definitions, notations and properties of spectral and Fredholm theory of a linear relation.
Let $T \in \mathcal{LR}(X)$. The resolvent set of T is defined as:

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open and has dense range}\}.$$

Hence, the spectrum of T is defined by $\sigma(T) = \mathbb{C}\backslash\rho(T)$.
A scalar λ such that $\mathcal{N}(\lambda - T) \neq \{0\}$ is called an eigenvalue of T. So, the point spectrum of T is the set $\sigma_p(T)$ consisting of the eigenvalues of T.

Proposition 2.3. [9, 11, Proposition VI.1.11] *Let X be a normed space and let $T \in \mathcal{BR}(X)$. Then, $\sigma(T) = \sigma(T^*)$.*

The sets of upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm and Fredholm linear relations which are, respectively, defined as

$$\begin{aligned}
\Phi_+(X,Y) &= \{T \in \mathcal{BR}(X,Y) : \dim(\mathcal{N}(T)) < \infty \text{ and } \mathcal{R}(T) \text{ is closed }\} \\
\Phi_-(X,Y) &= \{T \in \mathcal{BR}(X,Y) : \operatorname{codim}(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed }\} \\
\Phi_\pm(X,Y) &= \Phi_+(X,Y) \cup \Phi_-(X,Y) \\
\Phi(X,Y) &= \Phi_+(X,Y) \cap \Phi_-(X,Y).
\end{aligned}$$

When, $X = Y$, we denote by $\Phi_+(X,X) = \Phi_+(X)$, $\Phi_-(X,X) = \Phi_-(X)$ and $\Phi(X,X) = \Phi(X)$.
The essential spectra of a linear relation T are defined as follows:

$$\begin{aligned}
\sigma_{e_1}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_+(X)\} \\
\sigma_{e_2}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_-(X)\} \\
\sigma_{e_3}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi_\pm(X)\} \\
\sigma_{e_4}(T) &= \{\lambda \in \mathbb{C} : \lambda - T \notin \Phi(X)\}.
\end{aligned}$$

We point out that,

$$\sigma_{e_3}(T) = \sigma_{e_1}(T) \cap \sigma_{e_2}(T) \subseteq \sigma_{e_4}(T) \subseteq \sigma(T).$$

Theorem 2.4. [5] *Let S, $T \in \mathcal{LR}(X)$. Then, we have*
(i) If S, $T \in \Phi_+(X)$, then $ST \in \Phi_+(X)$ and $TS \in \Phi_+(X)$.
(ii) If S, $T \in \Phi_-(X)$, with TS (resp. ST) is closed, then $TS \in \Phi_-(X)$ (resp. $ST \in \Phi_-(X)$).
(iii) If S, T are everywhere defined and $TS \in \Phi_+(X)$ then $S \in \Phi_+(X)$.
(iv) If S, T are everywhere defined such that $TS \in \Phi(X)$ and $ST \in \Phi(X)$, then $S \in \Phi(X)$ and $T \in \Phi(X)$.

In this case, we can define the upper triangular 3×3-block matrix of linear relations

$$\mathcal{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix}$$

on $X_1 \oplus X_2 \oplus X_3$ where $T_{ii} \in \mathcal{LR}(X_i)$, and $T_{ij} \in \mathcal{LR}(X_j, X_i)$ for all $i \in \{1,2,3\}$, $j \in \{2,3\}$ and X_1, X_2, X_3 are complex Banach spaces.

Let us recall a fundamental result of the adjoint of $\mathcal{T}$, which we will need to prove our main result.

Theorem 2.5. [5, Theorem 2.1] *Let $T_{ii} \in \mathcal{BR}(X_i)$, for $i \in \{1,2,3\}$. Then, the adjoint of $\mathcal{T}$ is:*

$$\mathcal{T} = \begin{pmatrix} T_{11}^* & 0 & 0 \\ T_{23}^* & T_{22}^* & 0 \\ T_{12}^* & T_{13}^* & T_{33}^* \end{pmatrix},$$

where T_{ij}, for all $i \neq j \in \{1,2,3\}$ are bounded linear relations from X_j to X_i.

In the rest of this section, to orient the reader, we summarize some of the definitions and the fundamental concepts and facts of the local spectral theory.

Proposition 2.6. [7] *Let X be a Banach space, $x, y \in X$ and let $T \in \mathcal{BR}(X)$. Hence, $\rho_T(x)$ is open, whereas the spectrum $\sigma_T(x)$ is closed.*

Definition 2.7. Let $T \in \mathcal{BR}(X)$, T is said to have the single valued extension property at $\lambda \in \mathbb{C}$, abbreviated SVEP at λ, if for every neighborhood U of λ the only analytic operator function $f : U \longrightarrow X$ which satisfies the equation

$$(T - \mu)f(\mu) = T(0)$$

is the constant function $f \equiv 0$.

Definition 2.8. Let $T \in \mathcal{BR}(X)$. The analytic residuum S_T is the set of $\lambda_0 \in \mathbb{C}$ for which there exist a neighborhood V_{λ_0} and $f : V_{\lambda_0} \longrightarrow \mathcal{D}(T)$, a nonzero analytic function such that

$$(T - \lambda)f(\lambda) = T(0), \quad \text{for all } \lambda \in V_{\lambda_0}.$$

Remark 2.9. Let $T \in \mathcal{BR}(X)$. T has the SVEP if, and only if, $S_T = \varnothing$ (the empty set).

Proposition 2.10. [7] *Let $T \in \mathcal{CR}(X)$. T have SVEP at λ if, and only if, $\lambda - T$ have SVEP at 0, for every $\lambda \in \mathbb{C}$.*

Theorem 2.11. [7] *Let X be a Banach space and let $T \in \mathcal{CR}(X)$ with closed range. If T is not one-one, then T does not have the SVEP at 0.*

Let $T \in \mathcal{LR}(X)$. The surjective spectrum of T is defined by

$$\sigma_{su}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not surjective }\}.$$

We can find the following result in [12].

Proposition 2.12. *Let X be a Banach space and let $T \in \mathcal{BR}(X)$. Then,*

$$\sigma_{su}(T) = \bigcup_{x \in X} \sigma_T(x).$$

Theorem 2.13. [8, Theorem 2.1] *Let an analytic function of the linear relation $T : U \longrightarrow \mathcal{LR}(X,Y)$ on an open set $U \subseteq \mathbb{C}$ for which the mapping $T(\lambda) : X \longrightarrow Y$ is surjective for all $\lambda \in U$. If T has a selection E satisfying $T(0) \subseteq \mathcal{R}(E)$, then for every analytic function of the linear relation $\mathcal{K} : U \longrightarrow Y$, there exists an analytic function $f : U \longrightarrow X$ such that*

$$T(\lambda)f(\lambda) = \mathcal{K}(\lambda), \quad \textit{for all } \lambda \in U.$$

For a survey findings related to local spectral theory of linear relations, the reader is referred to [5, 8, 12].

In the following result, we characterize the ponctuel spectrum of linear relation matrix $\mathcal{T}$.

Proposition 2.14. *Let $T_{ii} \in \mathcal{BR}(X_i)$, and T_{ij} are bouneded operators from X_j, to X_i for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$. Then,*

$$\sigma_p(T_{11}) \subset \sigma_p(\mathcal{T}) \subset \sigma_p(T_{11}) \cup \sigma_p(T_{22}) \cup \sigma_p(T_{33}).$$

Proof. To prove that $\sigma_p(\mathcal{T}) \subset \sigma_p(T_{11}) \cup \sigma_p(T_{22}) \cup \sigma_p(T_{33})$, we assume that $\lambda \notin \sigma_p(T_{11}) \cup \sigma_p(T_{22}) \cup \sigma_p(T_{33})$. Then, $(\lambda - T_{11})$, $(\lambda - T_{22})$ and $(\lambda - T_{33})$ are injective. Let $(x, y, z) \in \mathcal{N}(\lambda - \mathcal{T})$. Hence,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in (\lambda - \mathcal{T}) \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Also, since $(\lambda - T_{33})$ is injective, then $z = 0$. Further, based on the hypotheses $(\lambda - T_{22})$ is injective and T_{23} is an operator, we conclude that $y = 0$. Again, using the fact that $(\lambda - T_{11})$ is injective, T_{12} and T_{13} are operators, we infer that $x = 0$. Thus, $\mathcal{N}(\lambda - \mathcal{T}) = \{(0, 0, 0)\}$, which leads to conclude that $\lambda - \mathcal{T}$ is injective.
Conversely, let $\lambda \notin \sigma_p(\mathcal{T})$, hence $\lambda - \mathcal{T}$ is injective. Therefore, $0 \in (\lambda - T_{11})x$ and so

$$\begin{cases} 0 \in (\lambda - T_{11})x - T_{12}(0) - T_{13}(0) \\ 0 \in (\lambda - T_{22})0 - T_{23}(0) \\ 0 \in (\lambda - T_{33})(0). \end{cases}$$

As a result, $(x, 0, 0) \in \mathcal{N}(\lambda - \mathcal{T})$, which implies that $x = 0$. As a consequence, $\lambda \notin \sigma_p(T_{11})$. □

3 Some local spectral results of linear relations matrices

In the beginning of this section, we discuss some properties of the local spectral theory of $\mathcal{T}$ involving the surjective spectrum and the analytic residuum of its diagonal entries.

3.1 Some local spectral properties of linear relation matrix through surjective spectrum and analytic residuum

In the further Proposition, we illustrate a new result of the local spectrum of $\mathcal{T}$ in terms of the surjective spectrum.

Proposition 3.1. *Let $T_{ii} \in \mathcal{BR}(X_i)$, and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1, 2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$), then*

$$\sigma_{\mathcal{T}}(x) \subseteq \sigma_{su}(T_{11}) \cup \sigma_{su}(T_{22}) \cup \sigma_{T_{33}}(w), \quad \textit{for all } x = \begin{pmatrix} y \\ z \\ w \end{pmatrix} \in X_1 \times X_2 \times X_3.$$

Proof. Let $\lambda_0 \notin \sigma_{su}(T_{11}) \cup \sigma_{su}(T_{22}) \cup \sigma_{T_{33}}(w)$. Then, there exist three open neighborhoods V_1, V_2 and V_3 of $\lambda_0 \in \mathbb{C}$ and an analytic operator function $f_3 : V_3 \longrightarrow X_3$ such that

$$(T_{33} - \lambda) f_3(\lambda) = w + T_{33}(0), \quad \text{for all } \lambda \in V_3, \quad V_2 \cap \sigma_{su}(T_{22}) = \varnothing \text{ and } V_1 \cap \sigma_{su}(T_{11}) = \varnothing.$$

Now, suppose that $V = \bigcap_{i \in \{1,2,3\}} V_i$. Hence, we get

$$(T_{33} - \lambda) f_3(\lambda) = w + T_{33}(0), \quad \text{for all } \lambda \in V.$$

In addition, by referring to Theorem 2.13, we infer that there exist two analytic functions $f_2 : V \longrightarrow X_2$ and $f_1 : V \longrightarrow X_1$ such that

$$\begin{cases} (T_{22} - \lambda) f_2(\lambda) = z - T_{23} f_1(\lambda) + T_{22}(0), \\ (T_{11} - \lambda) f_3(\lambda) = y - T_{12} f_2(\lambda) - T_{13} f_1(\lambda) + T_{11}(0), \quad \text{for all } \lambda \in V. \end{cases}$$

Furthermore, the nonzero analytic function $\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix} : V \longrightarrow X_3 \times X_2 \times X_1$ defined by:

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \end{pmatrix}(\lambda) = \begin{pmatrix} f_3(\lambda) \\ f_2(\lambda) \\ f_1(\lambda) \end{pmatrix}$$

such that

$$(\mathcal{T}-\lambda)\begin{pmatrix} f_3(\lambda) \\ f_2(\lambda) \\ f_1(\lambda) \end{pmatrix} = \begin{pmatrix} y+T_{11} \\ z+T_{22} \\ w+T_{33} \end{pmatrix}$$
$$= x+\mathcal{T}(0), \quad \text{for all } \lambda \in V.$$

Therefore, $\lambda_0 \notin \sigma_{\mathcal{T}}(x)$. □

As a direct consequence of Proposition 3.1, we get the following result:

Corollary 3.2. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$). Then,*

$$\sigma_{su}(\mathcal{T}) \subseteq \bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii}).$$

Proposition 3.3. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$, $j = 2,3$ and $i \neq j \in \{1,2\}$ such that $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$ and $T_{23}(0) \subset T_{22}(0)$. Then,*

$$\sigma_{T_{33}}(w) \cup \sigma_{T_{22}}(z) \subseteq \sigma_{\mathcal{T}}\begin{pmatrix} y \\ z \\ w \end{pmatrix}, \quad \textit{for all } y \in \mathcal{D}(T_{11}),\ z \in \mathcal{D}(T_{22}),\ w \in \mathcal{D}(T_{33}). \tag{3.1}$$

Proof. Let $\lambda_0 \notin \sigma_{\mathcal{T}}\begin{pmatrix} y \\ z \\ w \end{pmatrix}$. Hence, there exist an open neighborhood V_{λ_0} and a nonzero analytic function $f : V_{\lambda_0} \longrightarrow \mathcal{D}(T_{11}) \times \mathcal{D}(T_{22}) \times \mathcal{D}(T_{33})$ such that

$$(\mathcal{T}-\lambda)f(\lambda) = \begin{pmatrix} y \\ z \\ w \end{pmatrix} + \mathcal{T}(0), \quad \text{for all } \lambda \in V_{\lambda_0}.$$

So, for $f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ with $f_1 : V_{\lambda_0} \longrightarrow \mathcal{D}(T_{11})$, $f_2 : V_{\lambda_0} \longrightarrow \mathcal{D}(T_{22})$ and $f_3 : V_{\lambda_0} \longrightarrow \mathcal{D}(T_{33})$ which are analytic functions, we have

$$\begin{cases} (T_{11}-\lambda)f_1(\lambda) + T_{12}f_2(\lambda) + T_{13}f_3(\lambda) = y + T_{11}(0) \\ (T_{22}-\lambda)f_2(\lambda) + T_{23}f_3(\lambda) = z + T_{22}(0), \\ (T_{33}-\lambda)f_3(\lambda) = w + T_{33}(0), \quad \text{for all } \lambda \in V_{\lambda_0}. \end{cases} \tag{3.2}$$

Hence, $\lambda \notin \sigma_{T_{33}}(w)$. This allows us to deduce that $\sigma_{T_{33}}(w) \subseteq \sigma_{\mathcal{T}}\begin{pmatrix} y \\ z \\ w \end{pmatrix}$. Moreover, by using the fact that $f_3(\lambda) \in \mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$, we get

$$T_{23}f_3(\lambda) = T_{23}(0). \tag{3.3}$$

By combining (3.2) and (3.3) we obtain

$$\begin{aligned} z + T_{22}(0) &= (T_{22}-\lambda)f_2(\lambda) + T_{23}(0) \\ &= (T_{22}-\lambda)(f_2(\lambda)+0) + T_{23}(0) \\ &= (T_{22}-\lambda)f_2(\lambda) + (T_{22}-\lambda)(0) + (T_{23}-\lambda)(0), \quad \text{for all } \lambda \in V_{\lambda_0}. \end{aligned}$$

Since, $T_{23}(0) \subset T_{22}(0)$, then we have

$$(T_{22}-\lambda)f_2(\lambda) = z + T_{22}(0), \quad \text{for all } \lambda \in V_{\lambda_0}.$$

As a consequence, $\sigma_{T_{22}}(z) \subseteq \sigma_{\mathcal{T}}\begin{pmatrix} y \\ z \\ w \end{pmatrix}$. □

Remark 3.4. It is clear that $S_{T_{11}} \subseteq S_{\mathcal{T}} \subseteq \bigcup_{i\in\{1,2,3\}} S_{T_{ii}}$. Hence, by using of Corollary 3.2, we obtain that $\sigma(\mathcal{T}) \subseteq \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii})$.

Proposition 3.5. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. Then, for all $x \in V_{\lambda_0}$,*

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{T_{11}}(x) = S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{\mathcal{T}}\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}.$$

Proof. Let $\lambda_0 \notin S_{T_{ii}} \cup \sigma_{\mathcal{T}}\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$, for all $i \in \{2,3\}$. Then, there exist an open neighborhood V_{λ_0} of λ_0 in $\mathbb{C}$ and a nonzero analytic function $f_i : V_{\lambda_0} \longrightarrow X_1 \times X_2 \times X_3$ such that

$$(\mathcal{T} - \lambda)f_i(\lambda) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \mathcal{T}(0), \text{ for all } \lambda \in V_{\lambda_0}.$$

Let $f_1 : V_{\lambda_0} \longrightarrow X_1$, $f_2 : V_{\lambda_0} \longrightarrow X_2$ and $f_3 : V_{\lambda_0} \longrightarrow X_3$ be analytic functions such that $f_i = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$. Thus,

$$\begin{cases} (T_{11} - \lambda)f_1(\lambda) + T_{12}f_2(\lambda) + T_{13}f_3(\lambda) = x + T_{11}(0) \\ (T_{22} - \lambda)f_2(\lambda) + T_{23}f_3(\lambda) = T_{22}(0), \\ (T_{33} - \lambda)f_3(\lambda) = T_{33}(0), \quad \text{for all } \lambda \in V_{\lambda_0}. \end{cases} \tag{3.4}$$

Since $\lambda_0 \notin S_{T_{33}}$, it follows from (3.4) that $f_3 = 0$ on V_{λ_0}. In addition, using both of hypothesis $\lambda_0 \notin S_{T_{22}}$ and $T_{22}(0) \subseteq T_{23}(0)$ allows us to conclude that $f_2 = 0$ on V_{λ_0}. Again, by referring to (3.4), we infer that

$$(T_{11} - \lambda)f_1(\lambda) = x + T_{11}(0) \text{ for all } \lambda \in V_{\lambda_0}.$$

Thus, $\lambda_0 \notin \sigma_{T_{11}}(x)$. It is easy to see the converse inclusion. □

Corollary 3.6. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{22} and T_{33} have the SVEP. Then, for all $x \in V_{\lambda_0}$,*

$$\sigma_{T_{11}}(x) = \sigma_{\mathcal{T}}\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}.$$

3.2 Some spectral properties of linear relation matrix through The SVEP

We begin this subsection by giving a new characterization of the surjective spectrum of $\mathcal{T}$ by means of SVEP. We first need the following result on the surjective spectrum of $\mathcal{T}$.

Lemma 3.7. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$ and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$. If T_{11} (resp. T_{22}) has a selection E_1 (resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$ (resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$) and T_{33} has the SVEP, then*

$$\sigma_{su}(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii}).$$

Proof. Applying Corollary 3.6, we can deduce that

$$\sigma_{T_{11}}(x) = \sigma_{\mathcal{T}}\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

Since $\sigma_{su}(T) = \bigcup\limits_{x\in X} \sigma_T(x)$, then we have

$$\begin{aligned} \sigma_{su}(T_{11}) &= \bigcup_{y\in\mathcal{D}(T_{11})} \sigma_{T_{11}}(y) \\ &= \bigcup_{y\in\mathcal{D}(T_{11})} \sigma_{\mathcal{T}}\begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}, \quad \big(\text{from } (3.5)\big) \\ &\subseteq \sigma_{su}(\mathcal{T}). \end{aligned}$$

In addition, by referring to Proposition 3.3, we infer that

$$\begin{aligned} \sigma_{su}(T_{33}) &\subseteq \bigcup_{w\in\mathcal{D}(T_{33})} \sigma_{T_{33}}\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} \\ &\subseteq \bigcup_{w\in\mathcal{D}(T_{33})} \sigma_{\mathcal{T}}\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} \\ &\subseteq \sigma_{su}(\mathcal{T}). \end{aligned}$$

Again, by using of Proposition 3.3, we obtain

$$\sigma_{su}(T_{22}) \subseteq \sigma_{su}(\mathcal{T}).$$

So, we get

$$\bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii}) \subseteq \sigma_{su}(\mathcal{T}).$$

Conversely, by applying Corollary 3.2, we can deduce that

$$\sigma_{su}(\mathcal{T}) \subseteq \sigma_{su}(T_{11}) \cup \sigma_{su}(T_{22}) \cup \sigma_{su}(T_{33}).$$

This end the proof. □

One of our main results is to find an equality between the spectrum of $\mathcal{T}$ and the spectrum of its diagonal entries invelvig SVEP which is the follwing.

Theorem 3.8. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$ and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$) and T_{33} has the SVEP, then*

$$\sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}).$$

Proof. In view of Lemma 3.7, we obtain

$$\sigma_{su}(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii}).$$

Using the fact that T_{33} has the SVEP, we infer that

$$\bigcup_{i\in\{1,2\}} \sigma_{su}(T_{ii}) \cup \sigma(T_{33}) \subseteq \sigma(\mathcal{T}).$$

Accordingly,

$$\bigcup_{i\in\{1,2\}} S_{T_{ii}} \bigcup_{i\in\{1,2\}} \sigma_{su}(T_{ii}) \cup \sigma(T_{33}) \subseteq \bigcup_{i\in\{1,2\}} S_{T_{ii}} \cup \sigma(\mathcal{T}).$$

Also, applying Corollary 3.6, we have

$$\begin{aligned}\bigcup_{i\in\{1,2\}} S_{T_{ii}} \bigcup_{i\in\{1,2\}} \sigma_{su}(T_{ii}) \cup \sigma(T_{33}) &= \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}) \\ &\subseteq \bigcup_{i\in\{1,2\}} S_{T_{ii}} \cup \sigma(\mathcal{T}).\end{aligned}$$

Furthermore, since $S_{T_{11}} \cup S_{T_{22}} \subseteq S_{\mathcal{T}}$, then

$$\bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}) \subset \sigma(\mathcal{T}).$$

Conversely, it follows from Remark 3.4, we get $\sigma_{su}(\mathcal{T}) \subset \bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii})$. □

In what follows, under some new conditions, we prove that the upper triangular matrix of linear relations also has the SVEP.

Proposition 3.9. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j = 2,3$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{ii} have the SVEP, for all $i \in \{1,2,3\}$, then $\mathcal{T}$ also has the SVEP.*

Proof. Let $f : V_{\lambda_0} \longrightarrow X_1 \times X_2 \times X_3$ be an analytic function on an open neighborhood V_{λ_0} of λ_0 in $\mathbb{C}$ satisfying

$$(\mathcal{T} - \lambda)f(\lambda) = \mathcal{T}(0), \quad \text{for all } \lambda \in V.$$

Let $f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ with $f_1 : V_{\lambda_0} \longrightarrow X_1$, $f_2 : V_{\lambda_0} \longrightarrow X_2$ and $f_3 : V_{\lambda_0} \longrightarrow X_3$ be an analytic functions. Hence, we get

$$\begin{cases} (T_{11} - \lambda)f_1(\lambda) + T_{12}f_2(\lambda) + T_{13}f_3(\lambda) = T_{11}(0) \\ (T_{22} - \lambda)f_2(\lambda) + T_{23}f_3(\lambda) = T_{22}(0), \\ (T_{33} - \lambda)f_3(\lambda) = T_{33}(0), \quad \text{for all } \lambda \in V_{\lambda_0}. \end{cases}$$

Based on the hypotheses T_{33} has the SVEP, we can conclude that $f_3(\lambda) = 0$, for all $\lambda \in V$. Therefore, we have

$$(T_{22} - \lambda)f_2(\lambda) = T_{22}(0), \quad \text{for all } \mu \in V.$$

The fact that T_{22} has the SVEP and $T_{22}(0) \subseteq T_{23}(0)$, allows us to deduce that $f_2(\lambda) = 0$ on V. So, we obtain

$$(T_{11} - \lambda)f_1(\lambda) = T_{11}(0).$$

Again, since T_{11} has the SVEP, $T_{11}(0) \subseteq T_{12}(0)$ and $T_{11}(0) \subseteq T_{13}(0)$, then $f_1(\lambda) = 0$ on V. Consequently, $\mathcal{T}$ has the SVEP. □

An immediate consequence of the Proposition 3.9 is the following:

Corollary 3.10. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j = 2,3$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{ii} have the SVEP, for all $i \in \{1,2,3\}$, then*

$$\sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}). \tag{3.6}$$

Proof. By referring to Theorem 3.7, we infer that

$$\sigma_{su}(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma_{su}(T_{ii}) \tag{3.7}$$

Also, since T_{ii} have the SVEP, for all $i \in \{1,2,3\}$, we can obtain that

$$S_{T_{11}} = S_{T_{22}} = S_{T_{33}} = \varnothing.$$

This is equivalent to

$$\sigma_{su}(T_{11}) = \sigma(T_{11}), \quad \sigma_{su}(T_{22}) = \sigma(T_{22}) \quad \text{and} \quad \sigma_{su}(T_{33}) = \sigma(T_{33}). \tag{3.8}$$

In view of Proposition 3.9, we deduce that $\mathcal{T}$ also has the SVEP. This leads to

$$\sigma_{su}(\mathcal{T}) = \sigma(\mathcal{T}). \tag{3.9}$$

As a consequence, by combining (3.7), (3.8) and (3.9), we conclude that (3.6) holds. □

Proposition 3.11. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$). Then,*

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}) = \sigma_{su}(T_{11}) \cup \sigma(T_{22}) \cup \sigma(T_{33}).$$

Proof. By using of Corollary 3.2, we obtain

$$\sigma_{su}(\mathcal{T}) \subseteq \bigcup_{\in\{1,2,3\}} \sigma_{su}(T_{ii}).$$

Hence,

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}) \subseteq S_{T_{22}} \cup S_{T_{33}} \cup \bigcup_{\in\{1,2,3\}} \sigma_{su}(T_{ii}).$$

Consequently,

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}) \subseteq \sigma_{su}(T_{11}) \cup \bigcup_{\in\{2,3\}} \sigma(T_{ii}).$$

Conversely, by referring to Proposition 3.5, we infer that

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{T_{11}}(x) = S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{\mathcal{T}} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(T_{11}) \subseteq S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}).$$

Using the fact that $\sigma_{su}(T_{22}) \subseteq \sigma_{su}(\mathcal{T})$ and $\sigma_{su}(T_{33}) \subseteq \sigma_{su}(\mathcal{T})$, we can deduce that

$$\begin{aligned} \sigma_{su}(T_{22}) \cup \sigma_{su}(T_{33}) \cup S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(T_{11}) &= \sigma_{su}(T_{11}) \cup \sigma(T_{22}) \cup \sigma(T_{33}) \\ &\subseteq S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}). \end{aligned}$$

This complete the proof. □

In the following theorem, we will study a new characterization of $\sigma(\mathcal{T})$ by means of its adjoint $\mathcal{T}^*$.

Theorem 3.12. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$). Then,*

$$(S_{T_{11}^*} \cup S_{T_{22}} \cup S_{T_{33}}) \bigcup \sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}).$$

Proof. In the one hand, we show that

$$(S_{T_{22}} \cup S_{T_{33}}) \bigcup \sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}).$$

Then, by referring to Remark 3.4 and Proposition 3.11, we infer that

$$[S_{T_{11}} \cup \sigma_{su}(T_{11})] \bigcup_{i\in\{2,3\}} \sigma(T_{ii}) \subseteq S_{T_{22}} \cup S_{T_{33}} \cup \sigma_{su}(\mathcal{T}) \cup S_{\mathcal{T}}.$$

It follows that,

$$\bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}) \subseteq S_{T_{22}} \cup S_{T_{33}} \cup \sigma(\mathcal{T}).$$

In the other hand, in view of Theorem 2.5, we get that the adjoint of $\mathcal{T}$ is:

$$\mathcal{T} = \begin{pmatrix} T_{11}^* & 0 & 0 \\ T_{23}^* & T_{22}^* & 0 \\ T_{12}^* & T_{13}^* & T_{33}^* \end{pmatrix}.$$

Using the same reasoning as before, we have

$$S_{T_{11}^*} \cup \sigma(\mathcal{T}^*) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}^*).$$

In view of Proposition 2.3, we obtain

$$S_{T_{11}^*} \cup \sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii})$$

which proves the statement. □

Corollary 3.13. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$) and $\left(S_{T_{11}^*} \cup S_{T_{22}} \cup S_{T_{33}}\right) = \emptyset$. Then,*

$$\sigma(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma(T_{ii}).$$

Moving forward, we show an equivalence between the SVEP of $\mathcal{T}$ and the SVEP of its diagonal entries.

Theorem 3.14. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$ for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$). Then,*

$$S_{\mathcal{T}} = \emptyset \text{ if, and only if, } S_{T_{11}} = S_{T_{22}} = S_{T_{33}} = \emptyset.$$

In particular, $\mathcal{T}$ has the SVEP if, and only if, T_{ii} for all $i \in \{1,2,3\}$ have the SVEP.

Proof. Suppose that $S_{\mathcal{T}} = \emptyset$, then $\mathcal{T}$ has the SVEP. Hence, $\sigma(\mathcal{T}) = \sigma_{su}(\mathcal{T})$. Based on the fact that

$$\sigma_{su}(T_{22}) \subseteq \sigma_{su}(\mathcal{T}) \text{ and } S_{T_{22}} \subseteq S_{\mathcal{T}},$$

we can conclude that $S_{T_{11}} = \emptyset$. The same reasoning, we can deduce that

$$S_{T_{11}} = S_{T_{33}} = \emptyset.$$

Conversely, if T_{ii} for all $i \in \{1,2,3\}$ have the SVEP. Then, using Proposition 3.9, we obtain $\mathcal{T}$ which has the SVEP. □

The next theorem shows that an inclusion between the essential spectrum of $\mathcal{T}$ and the essential spectrum of its entries by using the analytic residuum.

Theorem 3.15. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$ and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$), then*

$$\bigcup_{i\in\{1,2,3\}} \sigma_{e_4}(T_{ii}) \subset \sigma_{e_4}(\mathcal{T}) \cup \left(S_{T_{11}^*} \cup S_{T_{22}} \cup S_{T_{33}}\right).$$

Proof. Let $\lambda \in \mathbb{C}$, then we have

$$\begin{aligned}\mathcal{T}-\lambda &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_{33}-\lambda \end{pmatrix}\begin{pmatrix} I & 0 & T_{13} \\ 0 & I & T_{23} \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & T_{22}-\lambda & 0 \\ 0 & 0 & I \end{pmatrix}\\ &\quad \begin{pmatrix} I & T_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} T_{11}-\lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}. \end{aligned} \tag{3.10}$$

Now, let us suppose that $\lambda \notin \sigma_{e_4}(\mathcal{T}) \cup \left(S_{T_{11}^*} \cup S_{T_{22}} \cup S_{T_{33}}\right)$. Hence, $\mathcal{T}-\lambda$ is a Fredholm. Our purpose is to prove that $\lambda \notin \bigcup_{i\in\{1,2,3\}} \sigma_{e_4}(T_{ii})$. By (3.10), we have $\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T_{33}-\lambda \end{pmatrix}$ is lower semi Fredholm, then $T_{33}-\lambda$ is so. Now, to prove that $T_{33}-\lambda$ is a Fredholm, it remains to show that $\dim(\mathcal{N}(T_{33}-\lambda))$ is finite. Since $\lambda \notin S_{T_{33}}$, then by using of Proposition 2.10 and Theorem 2.11, we obtain that $\dim(\mathcal{N}(T_{33}-\lambda)) = 0$. Thus, $\lambda \notin \sigma_{e_4}(T_{33})$. The same reasoning as before, we show that $\lambda \notin \sigma_{e_4}(T_{22})$. Let us prove that $T_{11}-\lambda$ is Fredholm linear relation. Using the fact that $\mathcal{T}-\lambda$ is a Fredholm and by (3.10), we infer that $\begin{pmatrix} T_{11}-\lambda & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ is upper semi Fredholm. Therefore, $T_{11}-\lambda$ is upper semi Fredholm. It follows from the fact $\lambda \notin S_{T_{11}^*}$ and again by using of Proposition 2.10 and Theorem 2.11 that $\mathcal{N}(T_{11}^*-\lambda) = \{0\}$. Since T_{11} is bounded linear relation, then

$$\begin{aligned}\{0\} &= \mathcal{N}(T_{11}^*-\lambda)\\ &= \mathcal{N}((T_{11}-\lambda)^*)\\ &= \mathcal{R}(T_{11}-\lambda)^{\perp}.\end{aligned}$$

This implies that $\operatorname{codim}(T) < \infty$. Accordingly, $T_{11}-\lambda$ is a Fredholm linear relation. □

As a direct consequence of Theorem 3.15, we get:

Corollary 3.16. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{ii}(0) \subseteq T_{ij}(0)$ for $j \neq i \in \{1,2\}$ and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{23})$. If T_{11}(resp. T_{22}) has a selection E_1(resp. E_2) which satisfies $T_{11}(0) \subseteq \mathcal{R}(E_1)$(resp. $T_{22}(0) \subseteq \mathcal{R}(E_2)$), and T_{11}^*, T_{22} and T_{33} have a SVEP, then*

$$\sigma_{e_4}(\mathcal{T}) = \bigcup_{i\in\{1,2,3\}} \sigma_{e_4}(T_{ii}).$$

We finish this section by giving a new characterization of the local spectrum of 3×3-block matrix of linear relations.

Theorem 3.17. *Let $T_{ii} \in \mathcal{BR}(X_i)$ and $T_{ij} \in \mathcal{BR}(X_j, X_i)$, for all $i \in \{1,2,3\}$ and $j \in \{2,3\}$ such that $T_{12}(0)\cap T_{13}(0) \subset T_{11}(0)$, $T_{21}(0)\cap T_{23}(0) \subset T_{22}(0)$ and $T_{31}(0)\cap T_{32}(0) \subset T_{33}(0)$. If $\mathcal{D}(T_{11}) \subset \mathcal{N}(T_{23})\cap\mathcal{N}(T_{31})$, $\mathcal{D}(T_{22}) \subset \mathcal{N}(T_{12})\cap\mathcal{N}(T_{32})$, and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{13})\cap\mathcal{N}(T_{23})$. Then,*

$$\sigma_{\mathcal{T}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sigma_{T_{11}}(x)\cup\sigma_{T_{22}}(y)\cup\sigma_{T_{33}}(z), \ \textit{for all} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in X_1\times X_2\times X_3.$$

Proof. First, it should be noted that the result is equivalent to show that

$$\rho_T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \rho_{T_{11}}(x)\cap\rho_{T_{22}}(y)\cap\rho_{T_{33}}(z), \ \text{for all} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in X_1\times X_2\times X_3.$$

Let $\lambda \in \rho_{T_{11}}(x) \cap \rho_{T_{22}}(y) \cap \rho_{T_{33}}(z)$, then there exist there exist three open neighborhoods V_1, V_2 and V_3 of $\lambda \in \mathbb{C}$ and three analytic functions $f_1 : V_1 \longrightarrow X_1$, $f_2 : V_2 \longrightarrow X_2$ and $f_3 : V_3 \longrightarrow X_3$ such that for all $\lambda_0 \in V_1 \cap V_2 \cap V_3$, we have

$$\begin{cases} (T_{11}-\lambda_0)f_1(\lambda_0) = x + T_{11}(0), \\ (T_{22}-\lambda_0)f_2(\lambda_0) = x + T_{22}(0), \\ (T_{33}-\lambda_0)f_3(\lambda_0) = x + T_{33}(0). \end{cases} \tag{3.11}$$

Now, let $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} : V_1 \cap V_2 \cap V_3 \longrightarrow X_1 \times X_2 \times X_3$. Then, $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ is an analytic function and for all $\lambda_0 \in V_1 \cap V_2 \cap V_3$, we obtain

$$\begin{aligned} (\mathcal{T}_0 - \lambda_0)\begin{pmatrix} f_1(\lambda_0) \\ f_2(\lambda_0) \\ f_3(\lambda_0) \end{pmatrix} &= \begin{pmatrix} T_{11}-\lambda_0 & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda_0 & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda_0 \end{pmatrix}\begin{pmatrix} f_1(\lambda_0) \\ f_2(\lambda_0) \\ f_3(\lambda_0) \end{pmatrix} \\ &= \begin{pmatrix} (T_{11}-\lambda_0)f_1(\lambda_0) + T_{12}f_2(\lambda_0) + T_{13}f_3(\lambda_0) \\ T_{21}f_1(\lambda_0) + (T_{22}-\lambda_0)f_2(\lambda_0) + T_{23}f_3(\lambda_0) \\ T_{31}f_1(\lambda_0) + T_{32}f_2(\lambda_0) + (T_{33}-\lambda_0)f_3(\lambda_0) \end{pmatrix} \\ &= \begin{pmatrix} x + T_{11}(0) + T_{12}f_2(\lambda_0) + T_{13}f_3(\lambda_0) \\ T_{21}f_1(\lambda_0) + y + T_{22}(0) + T_{23}f_3(\lambda_0) \\ T_{31}f_1(\lambda_0) + T_{32}f_2(\lambda_0) + y + T_{33}(0) \end{pmatrix} \quad \text{(from (3.11))}. \end{aligned}$$

Based on the hypotheses $\mathcal{D}(T_{11}) \subset \mathcal{N}(T_{23}) \cap \mathcal{N}(T_{31})$, $\mathcal{D}(T_{22}) \subset \mathcal{N}(T_{12}) \cap \mathcal{N}(T_{32})$, and $\mathcal{D}(T_{33}) \subset \mathcal{N}(T_{13}) \cap \mathcal{N}(T_{23})$ and by referring to Proposition 2.1, we infer that

$$\begin{aligned} (\mathcal{T}_0 - \lambda_0)\begin{pmatrix} f_1(\lambda_0) \\ f_2(\lambda_0) \\ f_3(\lambda_0) \end{pmatrix} &= \begin{pmatrix} x + T_{11}(0) + T_{12}(0) + T_{13}(0) \\ T_{21}(0) + y + T_{22}(0) + T_{23}(0) \\ T_{31}(0) + T_{32}(0) + y + T_{33}(0) \end{pmatrix} \\ &= \begin{pmatrix} T_{11}(0) + T_{12}(0) + T_{13}(0) \\ T_{21}(0) + T_{22}(0) + T_{23}(0) \\ T_{31}(0) + T_{32}(0) + T_{33}(0) \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \mathcal{T}_0 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

Hence,

$$\rho_{T_{11}}(x) \cap \rho_{T_{22}}(y) \cap \rho_{T_{33}}(z) \subset \rho_T \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

For the reverse inclusion, let us suppose that $\lambda \in \rho_T \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then there exist an open neighborhoods V of $\lambda \in \mathbb{C}$ and an analytic function $\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} : V \longrightarrow X_1 \times X_2 \times X_3$ such that for all $\lambda_0 \in V$, we have

$$(\mathcal{T}_0 - \lambda_0)\begin{pmatrix} f_1(\lambda_0) \\ f_2(\lambda_0) \\ f_3(\lambda_0) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \mathcal{T}_0 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is equivalent to

$$\begin{cases}(T_{11}-\lambda_0)f_1(\lambda_0)+T_{12}f_2(\lambda_0)+T_{13}f_3(\lambda_0)=x+T_{11}(0)+T_{12}(0)+T_{13}(0),\\ T_{21}f_1(\lambda_0)+(T_{22}-\lambda_0)f_2(\lambda_0)+T_{23}f_3(\lambda_0)=T_{21}(0)+y+T_{22}(0)+T_{23}(0),\\ T_{31}f_1(\lambda_0)+T_{32}f_2(\lambda_0)+(T_{33}-\lambda_0)f_3(\lambda_0)=T_{31}(0)+T_{32}(0)+y+T_{33}(0).\end{cases}$$

Since, $T_{12}(0)\cap T_{13}(0)\subset T_{11}(0)$, then

$$(T_{11}-\lambda_0)f_1(\lambda_0)+T_{12}f_2(\lambda_0)+T_{13}f_3(\lambda_0)=x+T_{11}(0).$$

Also, it follows from $f_2(\lambda_0)\in\mathcal{D}(T_{22})\subset\mathcal{N}(T_{12})$ and $f_3(\lambda_0)\in\mathcal{D}(T_{33})\subset\mathcal{N}(T_{13})$ that

$$\begin{aligned}x+T_{11}(0) &= (T_{11}-\lambda_0)f_1(\lambda_0)+T_{12}f_2(\lambda_0)+T_{13}f_3(\lambda_0)\\ &= (T_{11}-\lambda_0)f_1(\lambda_0)+T_{12}(0)+T_{13}(0)\\ &= (T_{11}-\lambda_0)f_1(\lambda_0)+(T_{11}-\lambda_0)(0)+(T_{12}-\lambda_0)(0)+(T_{13}-\lambda_0)(0)\\ &= (T_{11}-\lambda_0)f_1(\lambda_0).\end{aligned}$$

Thus, $\lambda\in\rho_{T_{11}}(x)$. The same reasoning allows us to conclude that $\lambda\in\rho_{T_{22}}(y)$ and $\lambda\in\rho_{T_{33}}(z)$.
□

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