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GENERALIZATIONS OF PRIME RADICAL IN NONCOMMUTATIVE RINGS

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Abstract *Let R be a noncommutative ring with identity. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function where $\mathcal{S}(R)$ denotes the set of all subsets of R. The aim of this paper is to generalize the concept of prime radical $\sqrt{I}$ of an ideal I of R to ϕ-prime radical $\mathcal{P}_\phi(I)$. A proper ideal Q of R is called ϕ-prime if whenever $a, b \in R$, $aRb \subseteq Q$ and $aRb \nsubseteq \phi(Q)$ implies that either $a \in Q$ or $b \in Q$. In this paper, first we study the properties of several generalizations of prime ideals of R. Then, we verify that $\mathcal{P}_\phi(I)$ is equal to the intersection of all minimal ϕ-prime ideals of R containing I, and we show that this notion inherits many of the essential properties of the usual notion of prime radical of an ideal.*

1 Introduction

The first generalization of prime ideals in commutative rings is introduced in 2003 by Anderson's celebrated work [2]. A proper ideal I of a commutative ring R is weakly prime if $0 \neq ab \in I$ for some $a, b \in R$, then $a \in I$ or $b \in I$. Afterwards, in 2008, Anderson and Bataineh introduced ϕ-prime ideals in commutative rings. In [1], they define a function $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ which maps an ideal of R to an ideal of R or $\emptyset$. A proper ideal I of R is said to be a ϕ-prime ideal of R whenever if $ab \in I - \phi(I)$ for some $a, b \in R$, then $a \in I$ or $b \in I$. They gave a proof showing that I is ϕ-prime if and only if whenever J, K are ideals of R with $JK \subseteq I$ and $JK \nsubseteq \phi(I)$ imply that $J \subseteq I$ or $K \subseteq I$ (that is, I is strongly ϕ-prime), [1, Theorem 13]. For some of the different generalizations of prime ideals refer to [3]-[14].

Afterwards, in [9], Groenewald studied weakly prime ideals in noncommutative rings and the notion of a weakly prime radical of an ideal is introduced. A proper ideal I of R is said to be weakly prime if $a, b \in R$ such that $0 \neq aRb \subseteq I$, then $a \in I$ or $b \in I$.

Motivated and inspired from the above structures in the literature, we give the following definition. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function. We call a proper ideal of R a ϕ-prime ideal if $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$, then $a \in P$ or $b \in P$. Several characterizations and properties of this concept are studied in Section 3. At the end of this section, we show how to construct some interesting examples of ϕ-ideals using the method of idealization (Theorem 2.21). In Section 4, we introduce and study the notion of ϕ-m-system to generalize the concept of prime radical of an ideal to ϕ-prime radical. We call a subset S of a ring R a ϕ-m-system if for A and B ideals of R such that $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$ and $AB \nsubseteq \phi(R \backslash S)$ then $AB \cap S \neq \emptyset$. In Theorem 3.4, we obtain a relationship between ϕ-prime ideals and ϕ-m-system that if P is an ideal of R maximal with respect to the property that P is disjoint from S where $S \subseteq R$ is a ϕ-m-system, then P is a ϕ-prime ideal. Then, we introduce ϕ-prime radical of A, denoted by $\mathcal{P}_\phi(A)$, by the set of $\{a \in R :$ every ϕ-m-system containing a meets $A\}$. We show that the intersection of all the minimal ϕ-prime ideals of R containing the ideal A of R is equal to the ϕ-prime radical $\mathcal{P}_\phi(A)$. (Theorem 3.6)

Furthermore, we call the set of all ϕ-prime ideals of R the ϕ-prime spectrum of R and denoted *by $Spec(R)$ or simply X. Also, we have: $X_{\phi_\emptyset} \subseteq X_{\phi_0} \subseteq X_{\phi_\omega} \subseteq \cdots \subseteq X_{\phi_{n+1}} \subseteq X_{\phi_n} \subseteq \cdots \subseteq X_{\phi_2} \subseteq X_{\phi_1} = \mathcal{S}^*(R)$. In particular, if $\phi = \phi_\emptyset$, then $Spec_\phi(R) = Spec(R)$ and if $\phi = \phi_1$, then $Spec(R) = \mathcal{S}^*(R)$.*

2 ϕ-prime ideals of a noncommutative ring

Definition 2.1. Let $\phi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function. We call a proper ideal P of a ring R a ϕ-prime ideal if $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$, then $a \in P$ or $b \in P$.

We shall denote the following notations which are used for the rest of the paper. Let R be a ring (not necessarily commutative) and $\phi_\alpha : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function where $S(R)$ denotes the set of subsets of R and if $I \subseteq S(R)$ is an ideal of R, then $\phi(I)$ is an ideal. Some generalized forms of prime ideals correspond to ϕ_α are presented as follows.

$\phi_\emptyset$	$\phi(I) = \emptyset$	*prime ideal*
ϕ_0	$\phi(I) = 0$	*weakly prime ideal*
ϕ_2	$\phi(I) = I^2$	*almost prime ideal*
ϕ_n	$\phi(I) = I^n$	*n-almost prime ideal*
ϕ_ω	$\phi(I) = \bigcap_{n=0}^{\infty} I^n$	*ω-prime ideal*
ϕ_1	$\phi(I) = I$	*any ideal*

For two functions ϕ, $\psi : \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$, we write an order $\phi \leq \psi$ when $\phi(I) \subseteq \psi(I)$ for all ideals I of R. Note that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$ $(*)$.

The several equivalent characterizations of ϕ-prime ideals of rings are presented in the following.

Theorem 2.2. *Let P be a proper ideal of a ring R. Then the following statements are equivalent.*

(i) P is a ϕ-prime ideal of R.

(ii) For all $x \in R \backslash P$, $(P : Rx) = \{p \in R : pRx \subseteq P\} = P \cup (\phi(I) : Rx)$.

(iii) For all $x \in R \backslash P$, $(P : Rx) = P$ or $(P : Rx) = (\phi(I) : Rx)$.

(iv) For ideals A and B of R, $AB \subseteq P$ and $AB \nsubseteq \phi(P)$ implies $A \subseteq P$ or $B \subseteq P$.

(v) If J, K are right (left) ideals of R such that $JK \subseteq P$ and $JK \nsubseteq \phi(P)$, then $J \subseteq P$ or $K \subseteq P$. (In this case, we call P a ϕ-prime right ideal)

(vi) $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$.

Proof. (1) $\Rightarrow$ (2) Let $y \in (P : Rx)$ where $x \in R \backslash P$. Now $yRx \subseteq P$. If $yRx \nsubseteq \phi(P)$, then we have $y \in P$. If $yRx \subseteq \phi(P)$, then $y \in (\phi(P) : Rx)$ as P is ϕ-prime. Hence, $(P : Rx) \subseteq P \cup (\phi(P) : Rx)$. As the reverse containment always holds for any ideal P, we have the equality.

(2) $\Rightarrow$ (3) Since P and $(\phi(P) : Rx)$ are both ideals, $(P : Rx) = P \cup (\phi(P) : Rx)$ implies clearly $(P : Rx) = P$ or $(P : Rx) = (\phi(P) : Rx)$.

(3) $\Rightarrow$ (1) Let $x, y \in R$ such that $xRy \subseteq P$ and $yRx \nsubseteq \phi(P)$. Suppose $y \in R \backslash P$. Then, $(P : Ry) \neq (\phi(P) : Ry)$ and from (3), we have $(P : Ry) = P$. Hence $x \in P$, as needed.

(1) $\Rightarrow$ (4) Let A and B be ideals of R with $AB \subseteq P$. Suppose that $A \nsubseteq P$ and $B \nsubseteq P$. We show that $AB \subseteq \phi(P)$. Let $a \in A$. First, suppose that $a \notin P$. Then $aRB \subseteq P$ gives $B \subseteq (P : Ra)$. Now $B \nsubseteq P$; so $(P : Ra) = (\phi(P) : Ra)$. Hence $aB \subseteq \phi(P)$. Next, choose $a \in A \cap P$ and $a^{'} \in A \backslash P$.Then $a + a^{'} \notin A \backslash P$. So by the first case, $a^{'}B$, $(a + a^{'})B \subseteq \phi(P)$. Let $b \in B$. Then $ab = (a + a^{'})b - a^{'}b \in \phi(I)$ which means $aB \subseteq \phi(P)$. Thus $AB \subseteq \phi(P)$.

(4) $\Rightarrow$ (1) Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. Now, since R is a ring with identity $aRb \subseteq (RaR)(RbR) \subseteq P$ and $(RaR)(RbR) \nsubseteq \phi(P)$. From (4), we have either $a \in RaR \subseteq P$ or $b \in RbR \subseteq P$.

(4) $\Rightarrow$ (5) Assume (4) holds. Suppose that J, K are right (left) ideals of R such that $JK \subseteq P$ and $JK \nsubseteq \phi(P)$. Let $\langle J \rangle, \langle K \rangle$ be the ideals generated by J, K respectively. Then $\langle J \rangle \langle K \rangle \subseteq P$ and $\langle J \rangle \langle K \rangle \nsubseteq \phi(P)$, whence $J \subseteq \langle J \rangle \subseteq P$ or $K \subseteq \langle K \rangle \subseteq P$.

(5) $\Rightarrow$ (1) Assume (5) holds. Suppose $aRb \subseteq P$ and $aRb \nsubseteq \phi(P)$. Since R has an identity, $(aR)(bR) \subseteq P$ and $(aR)(bR) \nsubseteq \phi(P)$, we conclude $a \in aR \subseteq P$ or $b \in bR \subseteq P$.

(1) $\Rightarrow$ (5) Suppose that $AB \subseteq P$, and $AB \nsubseteq \phi(P)$, for right ideals A and B of R. Since R has an identity, $AR = A$, and $(RA)(RB) = RAB \subseteq RP = P$ for ideals RA and RB. On the other hand, if $(RA)(RB) \subseteq \phi(P)$, then $AB \subseteq RAB = (RA)(RB) \subseteq \phi(P)$, a contradiction. Thus $(RA)(RB) \nsubseteq \phi(P)$, and by (2) we have either $A \subseteq RA \subseteq P$ or $B \subseteq RB \subseteq P$ and we are done.

(1) $\Leftrightarrow$ (6) is straightforward. □

Corollary 2.3. *Let P be an ideal of a ring R. Then the following are equivalent.*

(i) P is a ϕ-prime ideal of R.

(ii) For any ideals I, J of R with $P \subset I$ and $P \subset J$, we have either $IJ \subseteq \phi(P)$ or $IJ \nsubseteq P$.

(iii) For any ideals I, J of R with $I \nsubseteq P$ and $J \nsubseteq P$, we have either $IJ \subseteq \phi(P)$ or $IJ \nsubseteq P$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear

(2) $\Rightarrow$ (3) Let I, J be ideals of R with $I \nsubseteqq P$ and $J \nsubseteqq P$. Suppose that $i \in I$ and $j \in J$ such that $ij \notin \phi(P)$. Since $I \nsubseteq P$ and $J \nsubseteq P$, there exist $i_1 \in I$ and $j_1 \in J$ such that $i_1, j_1 \notin P$. Now $P \subset \langle i_1 \rangle + \langle i \rangle + P$ and $P \subset \langle j_1 \rangle + \langle j \rangle + P$. Furthermore, $(\langle i_1 \rangle + \langle i \rangle + P)(\langle j_1 \rangle + \langle j \rangle + P) \nsubseteq \phi(P)$. Hence from our assumption, we have $(\langle i_1 \rangle + \langle i \rangle + P)(\langle j_1 \rangle + \langle j \rangle + P) \nsubseteqq P$ and it follows that $P + \langle i_1 \rangle (\langle j_1 \rangle + \langle j \rangle) + \langle i \rangle (\langle j_1 \rangle + \langle j \rangle) \nsubseteq P$. For this to be true, we must have $IJ \nsubseteq P$. □

We define a useful concept, namely "twin-zero", for a ϕ-prime ideal in a noncommutative ring.

Definition 2.4. Let I be a ϕ-prime ideal of R. We say (a, b) is a twin-zero of I if $aRb \subseteq \phi(I)$, $a \notin I$, and $b \notin I$.

Note that if I is a ϕ-prime ideal of R that is not a prime ideal, then I has a twin-zero (a, b) for some $a, b \in R$.

Lemma 2.5. *Let I be a ϕ-prime ideal of R and suppose that (a, b) is a twin-zero of I for some $a, b \in R$. Then aI, $Ib \subseteq \phi(I)$.*

Proof. Suppose that $aI \nsubseteq \phi(I)$. Then there exists $i \in I$ such that $ai \notin \phi(I)$. Hence $aR(b+i) \subseteq I$ and $aR(b + i) \nsubseteq \phi(I)$. Since $a \notin I$ and I is ϕ-prime, we have $b + i \in I$, and hence $b \in I$, a contradiction. Thus $aI \subseteq \phi(I)$. Now, suppose $Ib \nsubseteq \phi(I)$. Then there exists $t \in I$ such that $tb \notin \phi(I)$. Hence $(a + t)Rb \subseteq I$ and $(a + t)Rb \nsubseteq \phi(I)$. Since $b \notin I$ and I is ϕ-prime, we have $a + t \in I$, and hence $a \in I$, a contradiction. Thus $Ib \subseteq \phi(I)$. □

Theorem 2.6. *Let R be a ring and P an ideal of R. If P is a ϕ-prime ideal but not prime, then $P^2 \subseteq \phi(I)$.*

Proof. Let (a, b) be a twin-zero of P. Suppose that $p_1p_2 \notin \phi(P)$ for some $p_1, p_2 \in P$. Then by Lemma 2.5, we have $(a + p_1)(b + p_2) \in (a + p_1)R(b + p_2) \subseteq P$ and $(a + p_1)R(b + p_2) \nsubseteq \phi(P)$ Thus $(a + p_1) \in P$ or $(b + p_2) \in P$ and hence $a \in P$ or $b \in P$ which is a contradiction since (a, b) is a twin-zero of P. Therefore $P^2 \subseteq \phi(P)$. □

In view of Theorem 2.6, one can say in other words that if an ideal P of a ring R with $P^2 \nsubseteq \phi(P)$, then P is prime if and only if P is ϕ-prime.

Corollary 2.7. *Let P be a ϕ-prime ideal of a ring R where $\phi \leq \phi_3$. Then P is ω-prime.*

Proof. If P is prime, then P is ϕ-prime for each ϕ and there is nothing to prove. Suppose P is not prime. Then by Theorem 2.6, $P^2 \subseteq \phi(P) \subseteq P^3$. Hence $\phi(P) = P^n$ for each $n \geq 2$, and so P is almost prime for each $n \geq 2$. Thus P is ω-prime. □

It should be noted that a proper ideal P with a property that $\phi(P) = P^2$ need not be ϕ-prime. Take an ideal $P = \begin{bmatrix} 0 & \mathbb{R} \\ 0 & 0 \end{bmatrix}$ of $R = \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}$ and $\phi(P) = \{0\}$. Clearly $P^2 = \{0\} = \phi(P)$, *but P is not ϕ-prime since* $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \subseteq \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix} \in P$ *and* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \nsubseteq \phi(P)$ *with* $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin P$ *and* $\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \notin P$.

Lemma 2.8. *Let I be a ϕ-prime ideal of a ring R and suppose that (a, b) is a twin-zero of I. If $aRr \subseteq I$ for some $r \in R$, then $aRr \subseteq \phi(I)$.*

Proof. Suppose that $aRr \subseteq I$ and $aRr \nsubseteq \phi(I)$ for some $r \in R$. Then $r \in I$ as ϕ-prime and (a, b) is a twin-zero of I. Now, since $aRr \subseteq aI$, we have that $aRr \subseteq \phi(I)$ from Theorem 2.5, a contradiction. □

Theorem 2.9. *Let I be a ϕ-prime ideal of R and suppose that $AB \subseteq I$ for some ideals A, B of R. If I has a twin-zero (a, b) for some $a \in A$ and $b \in B$, then $AB \subseteq \phi(I)$.*

Proof. Suppose that I has a twin-zero (a, b) for some $a \in A$ and $b \in B$ and assume that $cd \notin \phi(I)$ for some $c \in A$ and $d \in B$. Since $cRd \subseteq AB \subseteq I$ and $cd \in cRd \nsubseteq \phi(I)$ and I ϕ-prime, we have $c \in I$ or $d \in I$. Without loss of generality, we may assume that $c \in I$. Since $I^2 \subseteq \phi(I)$ by Theorem 2.6 and $cd \in I$ and $cd \notin \phi(I)$, we conclude that $d \notin I$. Since $aRd \subseteq AB \subseteq I$ it follows from Lemma 2.8 that $aRd \subseteq \phi(I)$. Now, since $(a + c)Rd \subseteq AB \subseteq I$ and $cd \in cRd \nsubseteq \phi(I)$, we have $(a+c)Rd \subseteq I$ and $(a+c)Rd \nsubseteq \phi(I)$. Since I is ϕ-prime, we have $(a+c) \in I$ since $d \notin I$. Hence $a \in I$, a contradiction. Thus $AB \subseteq \phi(I)$. □

Proposition 2.10. *Any ϕ-prime ideal P in a ring R contains a minimal ϕ-prime ideal.*

Proof. Apply Zorn's Lemma to the family of ϕ-prime ideals of R contained in P. It suffices to check that for any chain of ϕ-prime ideals $\{P_i : i \in I\}$ in P, the intersection $P^{'} = \cap P_i$ is ϕ-prime. Let A and B be ideals of R such that $AB \subseteq P'$ and $AB \nsubseteq \phi(P')$. Suppose that $A \nsubseteq P'$ and $B \nsubseteq P'$. Then there exist $a \in A \backslash P'$ and $b \in A \backslash P'$. Hence $a \notin P_i$ and $b \notin P_j$ for some i, $j \in I$. If, say $P_i \subseteq P_j$, then both a, b are outside P_i. Since P_i is ϕ-prime we have $aRb \subseteq \phi(P_i)$ or $aRb \nsubseteq P_i$. On the other hand, since $aRb \subseteq AB \subseteq P' \subseteq P_i$ we must have $aRb \subseteq \phi(P_i)$. Hence, (a, b) is a twin zero for P_i. Now, Theorem 2.9 implies that $AB \subseteq \phi(P_i) \subseteq \phi(P)$ which contradicts to our assumption. Thus $A \subseteq P'$ or $B \subseteq P'$, and therefore P' is a ϕ-prime ideal. □

Theorem 2.11. *Let R be a Noetherian ring and I a proper ideal of R. Then, the set of minimal ϕ-prime ideals containing I is finite.*

Proof. Assume on the contrary that the claim is false and choose an ideal $I \neq R$ maximal concerning the property that $I \neq R$ and that there are infinitely many ϕ-prime ideals containing I. This is possible as R is Noetherian. Then clearly I is not a ϕ-prime ideal, so there exist elements $a, b \in R$ such that $\langle a \rangle \langle b \rangle \subseteq I$ and $\langle a \rangle \langle b \rangle \nsubseteq \phi(I)$ but $a \notin I$ and $b \notin I$. Let $J = I + \langle a \rangle$ and $K = I + \langle b \rangle$. Now, J and K properly contain I. Furthermore, $\langle a \rangle \langle b \rangle \subseteq JK = (I + \langle a \rangle)(I + \langle b \rangle) \subseteq I$ and $JK = (I + \langle a \rangle)(I + \langle b \rangle) \nsubseteq \phi(I)$. Since I is ϕ-prime we must have $J \subseteq I$ or $K \subseteq I$. Note that any ϕ-prime ideal containing I must contain either J or K. In particular, any ϕ-prime minimal over I is minimal over either J or K. But each of J and K has only finitely many minimal ϕ-primes (by choice of I), a contradiction. □

Proposition 2.12. *For a ring R, the following statements are equivalent.*

(i) Every proper right ideal of R is ϕ-prime.

(ii) For any right ideals J and K of R with $JK \neq \phi(JK)$, $JK = J$ or $JK = K$.

Proof. $(1) \Rightarrow (2)$. Let J, K be right ideals of R and $JK \neq \phi(JK)$. If JK is proper, then it is ϕ-prime by our assumption. Thus $JK \subseteq JK$ and $JK \nsubseteq \phi(JK)$ implies that $J \subseteq JK$ or $K \subseteq JK$. Thus $JK = J$ or $JK = K$.

$(2) \Rightarrow (1)$. Let I be a proper right ideal of R. Suppose that $JK \subseteq I$ and $JK \nsubseteq \phi(I)$. Since $\phi(JK) \subseteq \phi(I)$, we have $JK \neq \phi(JK)$ and (2) implies that $J = JK \subseteq I$ or $K = JK \subseteq I$. □

In view of the proposition above, we have the following.

Corollary 2.13. *Let R be a ring in which every ideal of R is a ϕ-prime right ideal. Then $I^2 = I$ or $I^2 = \phi(I)$ for any right ideal I of R.*

Recall that a ring R with unity is said to be a local ring if it contains a unique maximal right ideal M. We will denote it by (R, M). Recall that M is the unique (two sided) maximal ideal of R.

Proposition 2.14. *Let (R, M) be a local ring, and let I be a right ideal of R such that $M^2 \subseteq \phi(I)$. Then I is a ϕ-prime right ideal. In particular, if (R, M) is a local ring such that $M^2 = 0$, then every proper ideal of R is a ϕ-prime right ideal.*

Proof. Suppose that J, K are two right ideals of R. Since $JK \subseteq M^2 \subseteq \phi(I)$, I is a ϕ-prime right ideal. The "in particular" case is straightforward. □

Example 2.15. Let (R, M) be a local ring and P be a right ideal of R such that $P \cap M^2 \subseteq \phi(P)$ $(P \cap M^2 = 0)$. Then, P is a ϕ-prime right ideal of R. Observe that if A and B are right ideals of R such that $AB \subseteq P$, then $AB \subseteq P \cap M^2 \subseteq \phi(P)$ $(AB = 0 \subseteq \phi(P))$.

Next, we discuss the behavior of ϕ-prime right ideals of a ring under an epimorphism.

Proposition 2.16. *Let $f : R \to S$ be a ring epimorphism, $\phi : S(R) \to S(R)$ a function such that $\phi(f(I)) = f(\phi(I))$.*

(i) If I is a ϕ-prime right ideal of S where $\ker f \subseteq I$, then $f^{-1}(I)$ is a ϕ-prime right ideal of R.

(ii) If I is a be a ϕ-prime right ideal of R and $\ker f \subseteq \phi(I)$, then $f(I)$ is a ϕ-prime right ideal of S.

Proof. (1) Let J, K be two right ideals of S and $JK \subseteq f^{-1}(I)$ and $JK \nsubseteq \phi(f^{-1}(I))$. Then $f(J)f(K) = f(JK) \subseteq I$. Since $\phi(f(I)) = f(\phi(I))$, we have $f(J)f(K) \nsubseteq \phi(I)$. It follows either $f(J) \subseteq f(I)$ or $f(K) \subseteq f(I)$ and since as $\ker f \subseteq I$, we conclude that either $J \subseteq f^{-1}(I)$ or $K \subseteq f^{-1}(I)$, as needed.

(2) Let $J := f(J_1), K := f(K_1)$ be two right ideals of S and $JK = f(J_1K_1) \subseteq f(I)$ and $JK \nsubseteq \phi(f(I))$. Then $J_1K_1 = f^{-1}(JK) \subseteq I$. Since $\phi(f(I)) = f(\phi(I))$ and $\ker f \subseteq \phi(I)$, we have $J_1K_1 = f^{-1}(J)f^{-1}(K) \nsubseteq \phi(I)$. Hence, $J_1 \subseteq I$ or $K_1 \subseteq I$, and thus $J \subseteq f(I)$ or $K \subseteq f(I)$, as needed. □

Corollary 2.17. *Let I and J be two right ideals of R with $I \subseteq J$. If I is a ϕ-prime right ideal ideal of R, then I/J is a ϕ-prime right ideal of R/J.*

Let R and S be noncommutative rings. It is well known that the prime ideals of $R \times S$ have the form $P \times S$ or $R \times Q$ where P is a prime ideal of R and Q is a prime ideal of S. We next generalize this result to ϕ-prime ideals.

Theorem 2.18. *Let R_1 and R_2 be noncommutative rings and let ϕ_i $\mathcal{S}(R_i) \to \mathcal{S}(R_i) \cup \{\emptyset\}$ be functions. Let $\phi = \phi_1 \times \phi_2$. Then a ϕ-prime ideal of $R_1 \times R_2$ has exactly one of the following three forms:*

(i) $I_1 \times I_2$ where I_i is a proper ideal of R_i with $\phi_i(I_i) = I_i$ $(i = 1, 2)$.

(ii) $I_1 \times R_2$ where I_1 is a ϕ_1-prime of R_1 which must be prime if $\phi_2(R_2) \neq R_2$.

(iii) $R_1 \times I_2$ where I_2 is a ϕ_2-prime of R_2 which must be prime if $\phi_1(R_1) \neq R_1$.

Proof. We first note that an ideal of $R_1 \times R_2$ having one of these three types is ϕ-prime. Case (1) is clear since $I_1 \times I_2 = \phi_1(I_1) \times \phi_2(I_2)$. If I_1 is prime, certainly $I_1 \times R_2$ is prime and hence ϕ-prime. So suppose that I_1 is ϕ_1-prime and $\phi_2(R_2) = R_2$. Suppose $(a_1, b_1)R(a_2, b_2) \subseteq I_1 \times R_2$ and $(a_1, b_1)R(a_2, b_2) \nsubseteq \phi(I_1 \times R_2) = \phi_1(I_1) \times \phi_2(R_2) = \phi_1(I_1) \times R_2$ for $a_1, a_2 \in R_1$ and $b_1, b_2 \in R_2$. Hence $a_1R_1a_2 \subseteq I_1$ and $a_1R_1a_2 \nsubseteq \phi(I_1)$. Since I_1 is ϕ_1-prime $a_1 \in I_1$ or $a_2 \in I_1$. Hence $(a_1, b_1) \in I_1 \times R_2$ or $(a_2, b_2) \in I_1 \times R_2$. Hence $I_1 \times R_2$ is ϕ-prime. The proof for Case (3) is similar. Next, suppose that $I_1 \times I_2$ is ϕ-prime. Let $aR_1b \subseteq I_1$ and $aR_1b \nsubseteq \phi_1(I_1)$ for $a, b \in R_1$. Then $(a, 0)R(b, 0) = (aR_1b, 0R_20) \subseteq I_1 \times I_2$ and $(a, 0)R(b, 0) = (aR_1b, 0R_20) \nsubseteq$

$\phi_1(I_1)\times\phi_2(I_2)=\phi(I_1\times I_2)$. Hence $(a,0)\in I_1\times I_2$ or $(b,0)\in I_1\times I_2$ since $I_1\times I_2$ is ϕ-prime. Therefore $a\in I_1$ or $b\in I_1$ and we have I_1 is ϕ_1-prime. Likewise, I_2 is ϕ_2-prime. Suppose that $I_1\times I_2\neq\phi_1(I_1)\times\phi_2(R_2)$. Say $I_1\neq\phi_1(I_1)$. Let $p\in I_1-\phi_1(I_1)$ and $q\in I_2$. Then $(p,1)R(1,q)=(pR_11,1R_2q)\subseteq I_1\times I_2$ and $(p,1)R(1,q)=(pR_11,1R_2q)\nsubseteq\phi_1(I_1)\times\phi_2(I_2)=\phi(I_1\times I_2)$. Hence $(p,1)\in I_1\times I_2$ or $(1,q)\in I_1\times I_2$ since $I_1\times I_2$ is ϕ-prime. So $I_2=R_2$ or $I_1=R_1$. Suppose that $I_2=R_2$. So $I_1\times R_2$ is ϕ-prime where I_1 is ϕ_1-prime. It remains to show that if $\phi_2(R_2)\neq R_2$, then I_1 is prime. Let $aR_1b\subseteq I_1$ for $a,b\in R_1$. Now $1\notin\phi_2(R_2)$. Then $(a,1)R(b,1)=(aR_1b,1R_21)\subseteq I_1\times R_2$ and $(a,1)R(b,1)=(aR_1b,1R_21)\nsubseteq\phi_1(I_1)\times\phi_2(R_2)=\phi(I_1\times R_2)$. Hence $(a,1)\in I_1\times R_2$ or $(b,1)\in I_1\times R_2$. Thus, $a\in I_1$ or $b\in I_1$. Hence I_1 is a prime ideal and we are done. □

We next give a way to construct ϕ-prime ideals J where $\phi_\omega\leq\phi$.

Theorem 2.19. *Let T and S be noncommutative rings and I be a weakly prime ideal of T. Then $J=I\times S$ is a ϕ-prime ideal of $R=T\times S$ for each ϕ with $\phi_\omega\leq\phi\leq\phi_1$.*

Proof. If I is a weakly prime ideal of T, then $J=I\times S$ need not be a weakly prime ideal of $R=T\times S$; indeed J is weakly prime if and only if J (or equivalently, I) is actually prime [9, Theorem 1.18]. However, J is ϕ-prime for each ϕ with $\phi_\omega\leq\phi$. If I is actually prime, then J is prime and hence is ϕ-prime for all ϕ. Suppose that I is not prime. Then $I^2=0$. So $J^2=0\times S$ and hence $\phi_\omega(J)=0\times S$. Then if $(x_1,x_2)R(y_1,y_2)\subseteq J$ and $(x_1,x_2)R(y_1,y_2)\nsubseteq\phi_\omega(J)$. Hence $(x_1,x_2)R(y_1,y_2)\subseteq I\times\ S$ and $(x_1,x_2)R(y_1,y_2)\nsubseteq 0\times S\Rightarrow x_1Ty_1\subseteq I$ and $x_1Ty_1\nsubseteq 0$. Hence $x_1\in I$ or $y_1\in I\Rightarrow(x_1,x_2)\in J$ or $(y_1,y_2)\in J$. So J is ϕ_ω-prime and hence ϕ-prime. □

Proposition 2.20. *Let $R=R_1\times R_2$, where R_1, R_2 are nonzero rings with identity elements. Then every proper ideal of R is ϕ-prime if and only if $\phi_i(J_i)=J_i$ for any proper ideal J_i of R_i $(i=1,2)$.*

Proof. Suppose that every proper ideal of R is ϕ-prime. Let $I=J_1\times J_2$ be a proper ideal of R where J_i is an ideal of R_i $(i=1,2)$. If both J_1 and J_2 are proper, then $\phi_1(J_1)=J_1$ and $\phi_2(J_2)=J_2$ by Theorem 2.18(1). Assume that $J_1=R_1$. Then J_2 must be a ϕ-prime ideal by Theorem 2.18(2). Assume on the contrary that there exists $b\in J_2\backslash\phi_2(J_2)$ which implies that $(R_1\times<b>)(0\times R_2)\subseteq 0\times J_2$ and $(R_1\times<b>)(0\times R_2)\nsubseteq\phi(0)\times\phi(J_2)=\phi(0\times J_2)$. Since $0\times J_2$ is also ϕ-prime from our assumption, we conclude that either $R_1\times<b>\subseteq 0\times J_2$ or $0\times R_2\subseteq 0\times J_2$ which yields $R_1=\{0\}$ or $J_2=R_2$, a contradiction. Thus $\phi_2(J_2)=J_2$. In case of $J_2=R_1$, we conclude that $\phi_1(J_1)=J_1$ by a similar argument above. The converse part is clear by Theorem 2.18. □

We end this section by showing how to construct some interesting examples of ϕ-ideals using the Method of Idealization. In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R\boxplus M$ with $(R\boxplus M,+)=(R,+)\oplus(M,+)$ and the multiplication is given by $(r,m)(s,n)=(rs,rn+ms)$. $R\boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M\simeq 0\boxplus M$ is a nilpotent ideal of $R\boxplus M$ of index 2. We also have $R\simeq R\boxplus 0$ and the latter is a subring of $R\boxplus M$. Note also that $R\boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M\\ 0 & R\end{bmatrix}$ via the mapping $(r,m)\mapsto\begin{bmatrix} r & m\\ 0 & r\end{bmatrix}$. We will require some knowledge about the ideal structure of $R\boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I\boxplus N$ is an ideal of $R\boxplus M$ if and only if $IM+MI\subseteq N$. Let $\psi_1:\mathcal{L}(R)\longrightarrow\mathcal{L}(R)\cup\{\emptyset\}$ and $\psi_2:\mathcal{L}(R\boxplus M)\longrightarrow\mathcal{L}(R\boxplus M)\cup\{\emptyset\}$ be two functions such that $\psi_2(I\boxplus M)=\psi_1(I)\boxplus M$ for a proper ideal I of R.

It follows from [13] that the prime ideals of $R\boxplus M$ are exactly the ideals of the form $I\boxplus M$ where I is a prime ideal of R.

Theorem 2.21. *Let R be a ring, M an $R-R-$bimodule and I a proper ideal of R. Then $I\boxplus M$ is a ψ_2 prime ideal of $R\boxplus M$ if and only if I is a ψ_1 prime ideal of R*

Proof. Suppose $I \boxplus M$ is a ψ_2 prime ideal of $R \boxplus M$. Let $aRb \subseteq I$ and $aRb \nsubseteq \psi_1(I)$ where $a, b \in R$. Now $(a,0)R \boxplus M(b,0) \subseteq I \boxplus M$ and $(a,0)R \boxplus M(b,0) \nsubseteq \psi_2(I \boxplus M) = \psi_1(I) \boxplus M$. $I \boxplus M$ a ψ_2-prime ideal gives $(a,0) \in I \boxplus M$ or $(a,0) \in I \boxplus M$. Hence $a \in I$ or $b \in I$. So I is ψ_1 prime.

Suppose I is a ψ_1-prime ideal of R. Let $(a,n),(b,m) \in R \boxplus M$ such that $(a,n)R \boxplus M(b,m) \subseteq I \boxplus M$ and $(a,n)R \boxplus M(b,m) \nsubseteq \psi_2(I \boxplus M) = \psi_1(I) \boxplus M)$ Hence $aRb \subseteq I$ and $aRb \nsubseteq \psi_1(I)$. Since I is a ψ_1-prime, we have $a \in I$ or $b \in I$. Hence $(a,n) \in I \boxplus M$ or $(b,m) \in I \boxplus M$, we are done. □

3 ϕ-prime radical

Let $\phi : \mathcal{S}(R) \longrightarrow \mathcal{S}(R)$ be a function from the set of subsets of the ring R such that if A is an ideal of R,then $\phi(A)$ is an ideal.

Definition 3.1. A subset S of a ring R is a ϕ-m-system if for A and B ideals of R such that $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$ and $AB \nsubseteq \phi(R \backslash S)$ then $AB \cap S \neq \emptyset$.

Lemma 3.2. *A proper ideal P of R is a ϕ-prime ideal if and only if $S = R \backslash P$ is an ϕ-m-system.*

Proof. Suppose $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$ and $AB \nsubseteq \phi(R \backslash S)$. If $AB \cap S = \emptyset$ then $AB \subseteq P$ and since $AB \nsubseteq \phi(R \backslash S) = \phi(R \backslash (R \backslash P)) = \phi(P)$ and P a ϕ-prime ideal gives $A \subseteq P$ or $B \subseteq P$ a contradiction. Hence $AB \cap S \neq \emptyset$ and we have S an ϕ-m-system.

Conversely, let A, B be ideals such that $AB \subseteq P$ and $AB \nsubseteq \phi(P) = \phi(R \backslash S)$. If $A \nsubseteq P$ and $B \nsubseteq P$, then $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$. Now, since $AB \nsubseteq \phi(P) = \phi(R \backslash S)$ and S an $\phi - m$-system we get $AB \cap S = AB \cap (R \backslash P) \neq \emptyset$, a contradiction. □

Proposition 3.3. *Let R be a ring and P be a proper ideal of R and let $S := R \backslash P$. Then the following statements are equivalent.*

(i) P is ϕ-prime ideal of R.

(ii) S is a ϕ-m-system.

(iii) For left ideals A, B of R, if $A \cap S \neq \emptyset$, $B \cap S \neq \emptyset$ and $AB \nsubseteq \phi(R \backslash S)$ then $AB \cap S \neq \emptyset$.

(iv) For right ideals A, B of R if $A \cap S \neq \emptyset$, $B \cap S \neq \emptyset$ and $AB \nsubseteq \phi(R \backslash S)$, then $AB \cap S \neq \emptyset$.

(v) For each $a, b \in R$, if $a, b \in S$ and $aRb \nsubseteq \phi(R \backslash S)$, then $aRb \cap S \neq \emptyset$.

Proof. $(1) \Leftrightarrow (2)$ follows from Lemma 3.2.

$(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ follows from Theorem 2.2. □

Theorem 3.4. *Let $S \subseteq R$ be a ϕ-m-system, and let P an ideal of R maximal with respect to the property that P is disjoint from S. Then P is a ϕ-prime ideal.*

Proof. Since $P \cap S = \emptyset$,we have $P = R - S$. Suppose $AB \subseteq P$ and $AB \nsubseteq \phi(P) = \phi(R - S)$ where A and B are ideals of R. If $A \nsubseteq P$ and $B \nsubseteq P$, then by the maximal property of P, we have, $(P + A) \cap S \neq \emptyset$ and $(P + B) \cap S \neq \emptyset$. Furthermore, $AB \subseteq (P + A)(P + B) \subseteq P$ and $(P+A)(P+B) \nsubseteq \phi(P) = \phi(R-S)$. Thus, since S is a ϕ-m-system $(P+A)(P+B) \cap S \neq \emptyset$ and it follows that $(P+A)(P+B) \nsubseteq P$. For this to happen, we must have $AB \nsubseteq P$, a contradiction. Thus, P must be a ϕ-prime ideal. □

It is well-known that for an ideal I of a ring R, prime radical of I is $\mathcal{P}(I) = \bigcap\{P : I \subseteq P$ and P a prime ideal of $R\}$ and $\mathcal{P}(R) = \bigcap\{P : P$ a prime ideal of $R\}$ where $\mathcal{P}(R)$ is the prime radical of R.Now, we are ready to generalize the notion of prime radical $\mathcal{P}(I)$ for any ideal I of R.

Definition 3.5. Let R be a ring. For an ideal A of R, if there is a ϕ-prime ideal containing A, then we define ϕ-prime radical by the set of $\{a \in R :$ every ϕ-m-system containing a meets $A\}$,denoted by $\mathcal{P}_\phi(A)$. If there is no ϕ-prime ideal containing A, then we put $\mathcal{P}_\phi(A) = R$.

Note that, for an ideal A of R, A and $\mathcal{P}_\phi(A)$ are contained in precisely the same ϕ-prime ideals of R.

Theorem 3.6. *Let A be an ideal of the ring R Then either $\mathcal{P}_\phi(A) = R$ or $\mathcal{P}_\phi(A)$ equals the intersection of all the ϕ-prime ideals of R containing A.*

Proof. Suppose that $\mathcal{P}_\phi(A) \neq R$. This means that $\{P \mid P$ is a ϕ-prime ideal of R and $A \subseteq P\} \neq \emptyset$. We first prove that $\mathcal{P}_\phi(A) \subseteq \{P|P$ is a ϕ-prime ideal of R and $A \subseteq P\}$. Let $m \in \mathcal{P}_\phi(A)$ and P be any ϕ-prime ideal of R containing A. Consider the ϕ-m-system $R\backslash P$. This ϕ-m-system cannot contain m, for otherwise it meets A and hence also P. Therefore, we have $m \in P$. Conversely, assume $m \notin \mathcal{P}_\phi(A)$. Then, by Definition 3.5, there exists a ϕ-m-system S containing m which is disjoint from A. By Zorn's Lemma, there exists an ideal $P \supseteq A$ which is maximal with respect to being disjoint from S. By Proposition 3.4, P is a ϕ-prime ideal of R and we have $m \notin P$, as desired. □

Theorem 3.7. *Let A be an ideal of the ring R. Then $\mathcal{P}_\phi(A)$ equals the intersection of all the minimal ϕ-prime ideals of R containing A.*

Proof. This follows from Theorem 3.6 and Proposition 2.10. □

For the following examples, let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be a function. Fot the ϕ-prime radical of the ideal Q of the ring R we take $\phi(Q) = 0$ i.e. a ϕ-prime radical of an ideal is a weakly prime radical of the ideal.

Example 3.8. Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_4,\ b \in \{0, 2\} \right\}$.
Then, R has proper ideals

$$P_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \right\},$$

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\},$$

$$M = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \right\}$$

where $P_1^2 = P_2^2 = M^2 = \{0\}$. Now, P_1 is a ϕ_0-prime ideal which is not a prime ideal since $P_2^2 = \{0\} \subseteq P_1$ but $P_2 \nsubseteq P_1$. Also, observe that $\mathcal{P}_{\phi_0}(P_1) = P_1$ and $\mathcal{P}_{\phi_0}(P_2) = R$ and $\mathcal{P}_{\phi_2}(P_1) = P_1 \cap M = P_1, \mathcal{P}_{\phi_2}(P_2) = P_2 \cap M = P_2$ and $\mathcal{P}_{\phi_2}(M) = M$.

Example 3.9. Let R be the noncommutative ring of endomorphisms of a countably infinite dimensional vector space. R is a prime ring with exactly one nonzero proper ideal P. Every ideal of $S_1 = R \boxplus P$ is ϕ_0-prime: the maximum ideal $P_1 = P \boxplus P$ is idempotent and the nonzero minimal ideal $P_2 = 0 \boxplus P$ is nilpotent, both of which are prime. Let $S_2 = S_1 \boxplus P_2$. Every ideal of S_2 is ϕ_0-prime: The maximum ideal $Q_1 = P_1 \boxplus P_2$ is idempotent and the three nonzero nilpotent ideals are $Q_2 = P_2 \boxplus P_2$, $Q_3 = 0 \boxplus P_2$, and $Q_4 = P_2 \boxplus 0$. Q_3 and Q_4 are not prime ideals since $0 = Q_2^2 \subseteq Q_3$ and $0 = Q_2^2 \subseteq Q_4$. For the ϕ_0-prime and prime radicals of the ideal Q_3 we have $\mathcal{P}_{\phi_0}(Q_3) = Q_3 \cap Q_2 \cap Q_1 = Q_3$ and $\mathcal{P}(Q_3) = Q_2 \cap Q_1 = Q_2$.

The ϕ-prime radical satisfies the following properties analogous to prime radical of an ideal.

Proposition 3.10. *Let R be a ring, $: \mathcal{S}(R) \to \mathcal{S}(R) \cup \{\emptyset\}$ be a function.*

(i) If I, J are ideals of R with $I \subseteq J$, then $\mathcal{P}_\phi(I) \subseteq \mathcal{P}_\phi(J)$.

(ii) $\mathcal{P}_\phi(I_1 I_2 \cdots I_n) \subseteq (I_1 \cap I_2 \cap \cdots \cap I_n) \subseteq \mathcal{P}_\phi(I_1) \cap \mathcal{P}_\phi(I_2) \cap \cdots \cap \mathcal{P}_\phi(I_n)$ for all ideals $I_1, ..., I_n$ of R.

(iii) $\mathcal{P}_\phi(\mathcal{P}_\phi(I)) = \mathcal{P}_\phi(I)$.

(iv) If $\mathcal{P}_\phi(I) = R$, then $I = R$

Proof. (1) Let Q be a ϕ-prime ideal containing J. Since $I \subseteq J$, Q also contains I. Thus $\mathcal{P}_\phi(I) \subseteq \bigcap\limits_{\substack{Q_\alpha\ \phi\text{-prime} \\ J \subseteq Q_\alpha}} Q_\alpha = \mathcal{P}_\phi(J)$.

(2) Since $I_1I_2\cdots I_n \subseteq I_1\cap I_2\cap\cdots\cap I_n$, we have $\mathcal{P}_\phi(I_1I_2\cdots I_n)\subseteq\mathcal{P}_\phi(I_1\cap I_2\cap\cdots\cap I_n)$ by (1). Also, since $I_i\subseteq\mathcal{P}_\phi(I_i)$ for each $i=1,2,...,n$, we have clearly $(I_1\cap I_2\cap\cdots\cap I_n)\subseteq\mathcal{P}_\phi(I_1)\cap\mathcal{P}_\phi(I_2)\cap\cdots\cap\mathcal{P}_\phi(I_n)$.

(3) If Q is a ϕ-prime ideal containing I, then it contains also $\bigcap\limits_{\substack{Q_\alpha\ \phi\text{-prime}\\ I\subseteq Q_\alpha}} Q_\alpha=\mathcal{P}_\phi(I)$. Thus $\mathcal{P}_\phi(\mathcal{P}_\phi(I))\subseteq\mathcal{P}_\phi(I)$. The inverse inclusion follows from (2) as $I\subseteq\mathcal{P}_\phi(I)$.

(4) Since $\mathcal{P}_\phi(I)\subseteq\mathcal{P}(I)$, we have $\mathcal{P}(I)=R$ which implies $I=R$. □

Proposition 3.11. *Let I be an ideal of a ring R and $\phi,\psi:\mathcal{S}(R)\to\mathcal{S}(R)\cup\{\emptyset\}$ be two functions with $\psi\leq\phi$.*

(i) $\mathcal{P}_\phi(I)\subseteq\mathcal{P}_\psi(I)$. In particular, $\mathcal{P}_{\phi_1}(I)\subseteq\mathcal{P}_{\phi_2}(I)\subseteq\cdots\subseteq\mathcal{P}_{\phi_n}(I)\subseteq\mathcal{P}_{\phi_{n+1}}(I)\subseteq\mathcal{P}_\phi(I)\subseteq\mathcal{P}_{\phi_0}(I)$.

(ii) $\mathcal{P}_\phi(\mathcal{P}_\psi(I))=\mathcal{P}_\psi(\mathcal{P}_\phi(I))=\mathcal{P}_\psi(I)$. In particular, $\mathcal{P}_\phi(\mathcal{P}(I))=\mathcal{P}(\mathcal{P}_\phi(I))=\mathcal{P}(I)$.

Proof. (1) As $\psi\leq\phi$, any ψ-prime ideal is a ϕ-prime ideal. Hence, $\mathcal{P}_\phi(I)\subseteq\mathcal{P}_\psi(J)$. The "in particular" statement follows from the order $\phi_\emptyset\leq\phi_0\leq\phi_\omega\leq\cdots\leq\phi_{n+1}\leq\phi_n\leq\cdots\leq\phi_2\leq\phi_1$.

(2) From (1), we have $\mathcal{P}_\psi(I)\subseteq\mathcal{P}_\phi(\mathcal{P}_\psi(I))\subseteq\mathcal{P}_\psi\mathcal{P}_\psi(I)$ and we have $\mathcal{P}_\psi\mathcal{P}_\psi(I)=\mathcal{P}_\psi(I)$ by 3.10(3). Thus $\mathcal{P}_\phi(\mathcal{P}_\psi(I))=\mathcal{P}_\psi(I)$. Similarly, as $\mathcal{P}_\phi(\mathcal{P}_\psi(I))\subseteq\mathcal{P}_\psi(\mathcal{P}_\psi(I))\subseteq\mathcal{P}_\psi(I)\subseteq$ and we have $\mathcal{P}_\psi\mathcal{P}_\psi(I)=\mathcal{P}_\psi(I)$ □

Theorem 3.12. *Let $\psi_1:\mathcal{S}(R)\longrightarrow\mathcal{S}(R)\cup\{\emptyset\}$ and $\psi_2:\mathcal{S}(R\boxplus M)\longrightarrow\mathcal{S}(R\boxplus M)\cup\{\emptyset\}$ be two functions such that $\psi_2(I\boxplus M)=\psi_1(I)\boxplus M$ for a proper ideal I of R. For the ring R we have $\mathcal{P}_{\psi_2}(I\boxplus M)=\mathcal{P}_{\psi_1}(I)\boxplus M$.*

Proof. Let Q be a ψ_2-prime ideal of $R\boxplus M$ containing $I\boxplus M$. Since Q contains $0\boxplus M$, $Q=P\boxplus M$ where P is a ψ_1-prime ideal of R containing I by Theorem 2.21. Hence $\mathcal{P}_{\psi_1}(I)\boxplus M\subseteq\mathcal{P}_{\psi_2}(I\boxplus M)$. Also, if P is a ψ_1-prime ideal of R containing I, then $P\boxplus M$ is a ψ_1-prime ideal containing $I\boxplus M$. Thus $\mathcal{P}_{\psi_2}(I\boxplus M)\subseteq\mathcal{P}_{\psi_1}(I)\boxplus M$ and we are done. □

Proposition 3.13. *Let R be a ring and $I\in\mathcal{S}^*(R)$. Then either $\mathcal{P}_\phi(I)=\mathcal{P}(I)$ or $(\mathcal{P}_\phi(I))^2\subseteq\phi(P)$ for some ϕ-prime ideal P of R containing I. In particular, if I is an n-almost prime ideal, then $\mathcal{P}_{\phi_n}(I)=\mathcal{P}(I)$ or $(\mathcal{P}_{\phi_n}(I))^2\subseteq P^n$, and $\mathcal{P}_{\phi_0}(I)=\mathcal{P}(I)$ or $(\mathcal{P}_{\phi_0}(I))^2=\{0\}$.*

Proof. If every ϕ-prime ideal of R containing I is prime, then clearly $\mathcal{P}_\phi(I)=\mathcal{P}(I)$. Now let P be a ϕ-prime ideal of R containing I which is not prime and let $x,y\in\mathcal{P}_\phi(I)$. Then $x,y\in P$ and hence $xy\in P^2\subseteq\phi(P)$, by Proposition 2.6. Thus $(\mathcal{P}_\phi(I))^2\subseteq\phi(P)$. The "in particular" part follows by considering $\phi=\phi_0$. □

Proposition 3.14. *Let R be a ring, $I\in\mathcal{S}^*(R)$ and $\phi:\mathcal{S}(R)\to\mathcal{S}(R)\cup\{\emptyset\}$ be a function such that $\phi_\omega\leq\phi\leq\phi_3$. Then $\mathcal{P}_\phi(I)=\mathcal{P}_{\phi_\omega}(I)$.*

Proof. Since $\phi_\omega\leq\phi$, $\mathcal{P}_{\phi_\omega}(I)\subseteq\mathcal{P}_\phi(I)$. Let P be a ϕ-prime ideal of R containing I. Since $\phi\leq\phi_3$ by Corollary 2.7, P is a ϕ_ω-prime ideal and so $\mathcal{P}_{\phi_\omega}(I)\subseteq\mathcal{P}_\phi(I)$. □

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