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A Visit to Zero Dimensional Rings and their spectrums

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Abstract For any commutative unitary ring R, we let $B(R)$ denote the set of idempotent elements of R. In 1976, Popescu and Vracui have proved that if R and S are von Neumann regular rings, then $\text{Spec}(R)$ and $\text{Spec}(S)$ are homeomorphic (with respect to their Zariski topologies) if and only if the Boolean algebras $B(R)$ and $B(S)$ are isomorphic. Our aim here is to improve their result by showing that it holds true even for zero dimensional rings.

1 Introduction

All rings and algebras considered in this paper are commutative and unitary. All ring homomorphisms are assumed to be unital. The *spectrum* of a ring R, denoted $\text{Spec}(R)$, is the set of all prime ideals of R, equipped with the *Zariski topology*. The open sets for this topology are the sets of the form $D_R(E) := \{P \in \text{Spec}(R) | E \nsubseteq P\}$, where E is a subset of R (cf. [2]). It is not hard to see that $D_R(E) = D_R(\langle E \rangle)$, where $\langle E \rangle$ is the ideal of R generated by E. Any closed set has the form $V_R(E) := \text{Spec}(R) \setminus D_R(E)$. The family $\{D_R(a) | a \in R\}$, where $D_R(a) := \{P \in \text{Spec}(R) | a \notin P\}$, constitutes a basis for this topology. The complement of $D_R(a)$ in $\text{Spec}(R)$ will be denoted by $V_R(a)$. Recall that a ring R is called *zero dimensional* if each prime ideal is maximal. A ring R is called *von Neumann regular* (in short, *regular*) if for every $x \in R$, there exists $y \in R$ such that $x = x^2 y$. Von Neumann regular rings are also called *absolutely flat* rings, because these rings are characterized by the fact that every R-module is flat. According to [5], R is regular if and only if R is reduced and zero dimensional.

Let $(R, +, \cdot)$ be a (nonzero) commutative ring with identity and set $B(R) := \{x \in R | x^2 = x\}$. It is well-known and easy to see that $(B(R), \vee, \wedge, ', 0_R, 1_R)$ is a Boolean algebra, where $x \vee y = x + y - xy, x \wedge y = xy$ and $x' = 1_R - x$ (the complement of x) for any $x, y \in B(R)$. Now, if we let $x \oplus y = (x - y)^2 = x + y - 2xy$ for any $x, y \in B(R)$, then $(B(R), \oplus, \cdot)$ is a Boolean ring (see for instance [10, p. 366] and [9, Theorem 2.9]). One can easily check that $B(R)$ is a subring of R if and only if R has characteristic 2 and that $B(R) = R$ if and only if R is a Boolean ring.

In [10, Corollaire 2.8], Popescu and Vracui have proved that if R and S are von Neumann regular rings, then $\text{Spec}(R)$ and $\text{Spec}(S)$ are homeomorphic (with respect to their Zariski topologies) if and only if the Boolean algebras $B(R)$ and $B(S)$ are isomorphic. So our aim here is to improve this result. We show in Lemma 2.1 that for any rings R and S, $B(R)$ and $B(S)$ are isomorphic rings if and only if they are isomorphic Boolean algebras. In Theorem 2.3, we prove that for any rings R and S, if $\text{Spec}(R)$ and $\text{Spec}(S)$ are homeomorphic, then $B(R) \cong B(S)$. If moreover, R and S are zero dimensional, then the converse holds true.

Any unexplained terminology is standard as in [6], [7] and [8].

2 Main results

Recall that if $(A, \vee, \wedge, ', 0_A, 1_A)$ and $(B, \vee, \wedge, ', 0_B, 1_B)$ are Boolean algebras, then a mapping $f : A \to B$ is called a *homomorphism of Boolean algebras*, if for all a, b in A, we have $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$, $f(0_A) = 0_B$ and $f(1_A) = 1_B$. If moreover, f is bijective, then f is said to be an *isomorphism of Boolean algebras*. We start our investigation with the following lemma.

Lemma 2.1. *Let R and S be commutative unitary rings and let $f : B(R) \to B(S)$ be a mapping. Then the following hold true:*

(i) f is a (unital) ring homomorphism if and only if f is a homomorphism of Boolean algebras.

(ii) f is a ring isomorphism if and only if f is an isomorphism of Boolean algebras.

Proof. (i) For the "only if" part, we only need to prove that $f(x \vee y) = f(x) \vee f(y)$ for any $x, y \in B(R)$. To this end, let $x, y \in B(R)$. As $x \vee y = x \oplus y \oplus xy$, we get:

$$\begin{aligned} f(x \vee y) &= f(x \oplus y \oplus xy) \\ &= f(x) \oplus f(y) \oplus f(xy) \text{ (because } f \text{ preserves } \oplus \text{)} \\ &= f(x) \oplus f(y) \oplus f(x)f(y) \text{ (because } f \text{ preserves multiplication)} \\ &= f(x) \vee f(y). \end{aligned}$$

For the "if" part, let $x, y \in B(R)$. Our task is to show that $f(x \oplus y) = f(x) \oplus f(y)$. Note that $x \vee y = (x \oplus y) \vee xy$. Thus, we obtain:

$$\begin{aligned} f(x \vee y) &= f((x \oplus y) \vee xy) \\ &= f(x \oplus y) \vee f(xy) \text{ (because } f \text{ preserves } \vee \text{)} \\ &= f(x \oplus y) + f(xy) - f(x \oplus y)f(xy) \\ &= f(x \oplus y) + f(xy) - f((x \oplus y)xy) \text{ (because } f \text{ preserves multiplication)} \\ &= f(x \oplus y) + f(xy) - f(0_R) \text{ (because } (x \oplus y)xy = 0_R\text{)} \\ &= f(x \oplus y) + f(xy). \end{aligned}$$

Therefore,

$$\begin{aligned} f(x \oplus y) &= f(x \vee y) - f(xy) \\ &= f(x) \vee f(y) - f(xy) \text{ (because } f \text{ preserves } \vee \text{)} \\ &= f(x) + f(y) - f(x)f(y) - f(xy) \\ &= f(x) + f(y) - 2f(x)f(y) \text{ (because } f(xy) = f(x)f(y)\text{)} \\ &= f(x) \oplus f(y). \end{aligned}$$

(ii) Follows immediately from assertion (i). □

Next, we establish the following result. We will make use of it in the proof of our main result.

Lemma 2.2. *Let R be a commutative unitary ring with nilradical N. Then the following hold true:*

(i) Spec(R) *and* Spec(R/N) *are homeomorphic.*

(ii) $B(R)$ and $B(R/N)$ are isomorphic.

Proof. (i) Consider the canonical surjection $\pi : R \to R/N$ and the induced map Spec(π) : Spec$(R/N) \to$ Spec$(R), Q \mapsto \pi^{-1}(Q)$. It is well known that Spec(π) is continuous. Moreover, as π is surjective, then Spec(π) is injective. Now, let $P \in$ Spec(R). As $P \supseteq N$ and Spec$(\pi)(P/N) = \pi^{-1}(P/N) = P$, it follows that Spec$(\pi)$ is surjective. Finally, for any $a + N \in R/N$, we have:

$$\begin{aligned} P \in \text{Spec}(\pi)(D_{R/N}(a+N)) &\Leftrightarrow \pi(P) \in D_{R/N}(a+N) \\ &\Leftrightarrow a + N \notin P/N \\ &\Leftrightarrow a \notin P \\ &\Leftrightarrow P \in D_R(a). \end{aligned}$$

Hence, $\mathrm{Spec}(\pi)(D_{R/N}(a+N)) = D_R(a)$. This proves that $\mathrm{Spec}(\pi)$ is open. Therefore $\mathrm{Spec}(\pi)$ is a homeomorphism from $\mathrm{Spec}(R/N)$ onto $\mathrm{Spec}(R)$.
(ii) Let $B(\pi) : B(R) \to B(R/N)$ given by $B(\pi)(a) = \pi(a)$ for any $a \in R$. Clearly, $B(\pi)$ is well-defined. In addition, $B(\pi)$ is a ring homomorphism from $B(R)$ into $B(R/N)$. The fact that $B(\pi)$ is bijective follows from [3, Proposition 7.17]. □

Now, we are in position to state our titular result.

Theorem 2.3. *Let R and S be commutative unitary rings. Then the following hold true:*

(i) If $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ are homeomorphic topological spaces, then $B(R)$ and $B(S)$ are isomorphic rings.

(ii) If moreover R and S are zero dimensional rings and $B(R)$ and $B(S)$ are isomorphic, then $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ are homeomorphic.

Proof. (i) Let $\theta : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ be an homeomorphism. For any $a \in B(R)$, $D_R(a)$ is a clopen subset (that is, both open and closed) of $\mathrm{Spec}(R)$. In fact $D_R(a) = V_R(1-a)$ because a is idempotent. Thus, $\theta(D_R(a))$ is a clopen subset of $\mathrm{Spec}(S)$ because θ is a homeomorphism. Thus, by using [4, Lemma 10.21.3], there exists a unique $a' \in B(S)$ such that $\theta(D_R(a)) = D_S(a')$. Let us consider, $f : B(R) \to B(S)$ given by $f(a) = a'$. We claim that f is a ring isomorphism. To see this, let $a, b \in B(R)$ and set $f(a) = a'$ and $f(b) = b'$. By definition of f, we have $\theta(D_R(a)) = D_S(a')$ and $\theta(D_R(b)) = D_S(b')$. Let $P \in D_S(a' \oplus b')$. Then $P \in \mathrm{Spec}(S)$ and $a' \oplus b' \notin P$. Let Q be the unique prime ideal of R such that $P = \theta(Q)$. As $a' \oplus b' \notin P$, then $1 - (a' \oplus b') \in P$. Thus, $a'b' = a'b'(1 - (a' \oplus b')) \in P$. This yields at least one of a' or b' is in P. Assume for a moment that $a' \in P$. In this case $P \in V_S(a') = \theta(V_R(a))$. Hence, $Q \in V_R(a)$; or equivalently, $a \in Q$. Suppose that $b \in Q$. Then, $Q \in V_R(b)$ and so $P \in V_S(b')$. This implies $b' \in P$ and so $a' \oplus b'$ would be in P, which is a contradiction. It follows that $b \notin Q$ and consequently, $a \oplus b \notin Q$. Therefore, $Q \in D_R(a \oplus b)$ and hence $P \in \theta(D_R(a \oplus b))$. The case where $b' \in P$ can be handled similarly. Thus, we have proved the following inclusion relation:

$$D_S(a' \oplus b') \subseteq \theta(D_R(a \oplus b)). \tag{2.1}$$

Conversely, let $P \in \theta(D_R(a \oplus b))$. Then $P = \theta(Q)$, where Q is a prime ideal of R. Since $Q \in D_R(a \oplus b)$, then $a \oplus b \notin Q$ and so $1 - (a \oplus b) \in Q$. This implies $ab = ab(1 - (a \oplus b)) \in Q$. Hence, either $a \in Q$ or $b \in Q$. As above and as a and b play symmetric roles, we can suppose that $a \in Q$. Thus, $Q \in V_R(a)$. So $P \in V_S(a')$. This means that $a' \in P$. If $a' \oplus b' \in P$, then we get $b' \in P$. Thus, $P \in V_S(b')$ and so $Q \in V_R(b)$. This implies $b \in Q$ and hence $a \oplus b \in Q$, a contradiction. Therefore, $a' \oplus b' \notin P$; or equivalently, $P \in D_S(a' \oplus b')$. Hence, we have proved the inclusion relation:

$$\theta(D_R(a \oplus b)) \subseteq D_S(a' \oplus b'). \tag{2.2}$$

Combining (2.1) and (2.2), we get $\theta(D_R(a \oplus b)) = D_S(a' \oplus b')$. This means that $f(a \oplus b) = a' \oplus b' = f(a) \oplus f(b)$. On the other hand, as $D_R(a) \cap D_R(b) = D_R(ab)$ and $D_S(a') \cap D_S(b') = D_S(a'b')$, it follows that:

$$\begin{aligned}\theta(D_R(ab)) &= \theta(D_R(a) \cap D_R(b)) \\ &= \theta(D_R(a)) \cap \theta(D_R(b)) \\ &= D_S(a') \cap D_S(b') \\ &= D_S(a'b').\end{aligned}$$

Hence, $f(ab) = a'b' = f(a)f(b)$. We have shown until now that f is a ring homomorphism. It remains to show that f is bijective. To this end, let $a, b \in B(R)$ such that $f(a) = f(b)$. Then $D_S(f(a)) = D_S(f(b))$. Hence, $D_R(a) = D_R(b)$. This implies $V_R(aR) = V_R(bR)$; or equivalently, $\sqrt{aR} = \sqrt{bR}$. As $a \in \sqrt{bR}$, then there exist an integer $n \geq 1$ and $\alpha \in R$ such that $a^n = b\alpha$. But, a is idempotent and so $a = b\alpha$. Similarly, there exists $\beta \in R$ such that $b = a\beta$. Thus, $ab = a(a\beta) = a\beta = b$ and $ab = (b\alpha)b = b\alpha = a$. Therefore, $a = b$. Hence, we have

demonstrated that f is one to one. Now, let $u \in B(S)$ and let $U' = D_S(u)$. This is a clopen subset of Spec(S). As θ is a homeomorphism, there exists a clopen subset U of Spec(R) such that $U' = \theta(U)$. But, by using [4, Lemma 10.21.3], there exists a unique $v \in B(R)$ such that $U = D_R(v)$. Thus, $f(v) = u$. This shows that f is onto.
(ii) Let N and N' be the nilradicals of R and S respectively. As $B(R) \cong B(S)$, it follows from Lemma 2.2 that $B(R/N) \cong B(S/N')$. But as R and S are zero dimensional, it follows that R/N and S/N' are von Neumann regular rings. So Spec(R/N) and Spec(S/N') are homeomorphic by virtue of [10, Corollary 2.8]. According to Lemma 2.2, Spec(R) and Spec(S) are also homeomorphic. This completes the proof. □

Remark 2.4. The assumption "R and S are zero dimensional rings" is essential in Theorem 2.3. Indeed, let R be a field and let S be the ring of integers. Clearly, $B(R) = \{0_R, 1_R\}$ and $B(S) = \{0, 1\}$. Thus, $B(R)$ and $B(S)$ are isomorphic rings. However, there is no bijection from Spec(R) onto Spec(S) since $|\text{Spec}(R)| = 1$ and $|\text{Spec}(S)| = \infty$.

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