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A MIXED CONTACT PROBLEM FOR THERMO-PIEZOELECTRIC MATERIALS ANALYSIS AND OPTIMAL CONTROL

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Abstract. *We consider the mathematical model describing the problem of frictionless static contact between a thermo-piezoelectric body and a rigid, electrically conductive, foundation. The body material is modeled with a nonlinear thermo-electro-elastic constitutive law. The contact is described by Signorini's conditions, which depend on temperature. First, we derive a mixed formulation of the problem. Then we use a standard results on mixed problems and Banach fixed point theorem to prove the existence and uniqueness of solution to our problem. Finaly, existence result of optimal solution is given for the boundary optimal control of the model.*

1 Introduction

In the present work, we consider the static frictionaless contact between a thermo-piezoelectric body and a conductive foundation. The body is assumed to satisfy the piezoelectric constitutive equations with additional thermal effects. The fundamentals of the piezoelectric theory were developed by [29], who deriveded the first mathematical model of a linearly elastic material which takes into account the interaction between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [21, 22, 23, 24, 26, 28], and more recently, in [8]. Recently, piezoelectric frictional contact problems, with or without the conductivity of the foundation, have been investigated in a large number of papers, see e.g., [4, 5, 20], and the references therein. In contrast, our work focuses on a frictionless contact problem for thermo-electro-elastic materials, employing a mixed variational approach.

A very large number of problems can be formulated, analyzed and numerically solved by employing mixed variational problems with Lagrange multipliers, which find applications in both analysis and mechanics. In general, they are being studied using duality, saddle points and fixed points arguments, see for instance [6, 10, 16, 9] and the references therein. Concerning the analysis of mixed variational problems associated with problems of contact with unilateral constraint, we refer to [11, 12, 13, 14, 15].

We deal with a mathematical model for the static process of frictionless contact between a thermo-piezoelectric body and an electrically conductive foundation, under the assumption of small deformations. Here, the material's behavior is modeled by a nonlinear thermo-electro-elastic constitutive law, and the contact is described by a regularized electrical conductivity condition. Our main objective in this work is to prove the existence and uniqueness of a solution with a method different from the one that existed. The goal is to reduce the steps to arrive at the desired result, and extends our analysis to the mixed problem case and examines an underlying optimal control problem. Beyond the mathematical interest, details on real applications of results as those obtained in this paper, can be found in [19, 25, 17, 14, 26]. The second aim is to examine optimal control problem described by our frictional contact model, i.e., we are looking for the *external data ι^*, considered as a control variable, such that the solution of our problem is as close as possible to the desired value, in the sens of a given cost function Υ. In application, it could represent for example the minimization of friction so that the economic loss caused by its relate phenomenas (frictional heating, frictional wear, frictional softening and damage on contact interface ...) is reduced or controlled.*

The paper is structured as follows. In the section 2, we introduce the notation, list the assumptions on the problem's data, derive the mixed variational formulation of the problem, and present our main result, stated in Theorem 3.1. The proof of this theorem is provided in the section 3, carried out in several steps and based on the Banach fixed-point theorem. Finally, the section 5 is devoted to the analysis of optimal control for our model.

2 Problem description

We consider a piezoelectric body occupying a domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ with regular boundary Γ (for instance a Lipschitz continuous boundary). We suppose Γ is divided into three measurable disjoint parts Γ_D, Γ_N and Γ_C on the one hand and $\Gamma_D \cup \Gamma_N$ is partitioned into two measurable parts Γ_a and Γ_b on the other hand such that Γ_D and Γ_a have nonzero measures. The body is supposed to be stress free at a free temperature and the temperature variations, accompanying the deformations, produce changes in the material parameters which are considered as depending on temperature. We assume that the body is clamped on Γ_D and it is subjected to a volume force f_0 in Ω, a surface traction f_N on Γ_N, a volume electric charge ϕ_0 on Ω, a surface electric charge ϕ_b on Γ_b and heat source q_0 on Ω. The electric potential and the variation of temperature are assumed to be zero, respectively on Γ_a and $\Gamma_D \cup \Gamma_N$. Moreover, an unilateral contact between the body and a thermally conductive and rigid foundation, may occur on Γ_C.

Here and below, we do not indicate the dependence of various functions on the spatial variable $x \in \bar{\Omega}$, the indices i, j, k, l take values in $\{1, .., d\}$, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $u_{i,j} = \partial u_i/\partial x_j$. Let S^d be the space of second order symmetric tensors on $\mathbb{R}^d$, while " $\cdot$ " and $\|\cdot\|$ denote the inner product and the Euclidean norm on $\mathbb{R}^d$ and S^d. We recall that

$$\forall\, u,\, v \in \mathbb{R}^d,\;\; u \cdot v = u_i v_i\;,\;\; \|v\| = (v \cdot v)^{1/2}.$$

$$\forall\, \sigma,\, \tau \in S^d,\;\; \sigma \cdot \tau = \sigma_{ij}\tau_{ij}\;,\;\; \|\tau\| = (\tau \cdot \tau)^{1/2}.$$

Moreover, if ν represents the unit exterior normal on Γ, then the normal and the tangential components of the displacement v and the stress σ on Γ are

$$v_\nu = v \cdot \nu \;\;,\;\; v_\tau = v - v_\nu\,\nu \quad \textit{and} \quad \sigma_\nu = \sigma\,\nu \cdot \nu \;\;,\;\; \sigma_\tau = \sigma\,\nu - \sigma_\nu\,\nu\,.$$

The contact process is static, then the classical formulation of our problem is as follows.

Problem (P)**.** Find a displacement field $u : \Omega \to R^d$, a stress field $\sigma : \Omega \to S$, an electric potentiel $\varphi : \Omega \to R$, an electric displacement field $D : \Omega \to R^d$, a temperature field $\theta : \Omega \to R$ and the heat flux $q : \Omega \to R^d$ such that

$$\sigma = \mathfrak{F}\,\varepsilon(u) - \mathcal{E}^* E(\varphi) - \mathcal{M}\,\theta \quad \textit{in}\ \Omega, \tag{2.1}$$

$$D = \mathcal{E}\,\varepsilon(u) + \beta\,E(\varphi) + \mathcal{P}\,\theta \quad \textit{in}\ \Omega, \tag{2.2}$$

$$q = -\mathcal{K}\,\nabla\theta \quad \textit{in}\ \Omega, \tag{2.3}$$

$$Div\sigma + f_0 = 0 \quad \textit{in}\ \Omega, \tag{2.4}$$

$$\operatorname{div} D = \phi_0 \quad \textit{in}\ \Omega, \tag{2.5}$$

$$\operatorname{div} q = q_0 \quad \textit{in}\ \Omega, \tag{2.6}$$

$$u = 0 \quad \textit{on}\ \Gamma_D, \tag{2.7}$$

$$\sigma\nu = f_N \quad on\,\Gamma_N, \tag{2.8}$$
$$q_\nu(u,\varphi,\theta) \le 0,\ (\theta-\theta_F) \le 0,\ q_\nu(u,\varphi,\theta)(\theta-\theta_F) = 0 \quad on\,\Gamma_C, \tag{2.9}$$
$$\sigma_\nu(u,\varphi,\theta) \le 0,\ (u_\nu - g) \le 0,\ \sigma_\nu(u,\varphi,\theta)(u_\nu - g) = 0 \quad on\,\Gamma_C, \tag{2.10}$$
$$\sigma_\tau = 0 \quad on\,\Gamma_C, \tag{2.11}$$
$$\varphi = 0 \quad on\,\Gamma_a, \tag{2.12}$$
$$D\cdot\nu = \phi_b \quad on\,\Gamma_b, \tag{2.13}$$
$$D\cdot\nu = \psi(\sigma_\nu(u,\varphi,\theta))\phi_L(\varphi-\varphi_F) \quad on\,\Gamma_C, \tag{2.14}$$
$$\theta = 0 \quad on\,\Gamma_D\cup\Gamma_N. \tag{2.15}$$

Notice that the stress tensor $\sigma = (\sigma_{ij})$, the electric displacement field $D = (D_i)$ are described by the thermo-electro-elastic constitutive law (2.1)-(2.2), see [24] for more details. The heat flux field $q = (q_i)$ is defined through the thermal conductivity tensor $\mathcal{K} = (k_{ij})$ by the Fourier law of heat conduction (2.3). Here $\varepsilon(u) = (\varepsilon_{ij}(u)) = (\frac{1}{2}(u_{i,j}+u_{j,i}))$ is the linearized strain tensor, $E(\varphi) = -(\varphi_{,i})$ is the electric field, $\mathfrak{F} = (\mathfrak{F}_{ijkl})$ is the nonlinear elasticity operator, $\beta = (\beta_{ij})$ is the electric permittivity tensor, $\mathcal{E} = (e_{ijk})$ is the piezoelectric tensor, $\mathcal{M} = (m_{ij})$ is the thermal expansion tensor, $\mathcal{P} = (p_i)$ is the pyroelectric tensor and $\mathcal{E}^*$ represent the transpose of $\mathcal{E}$. The equations (2.4)-(2.6) represent the equilibrium equations for the stress, the electric displacement and the heat flux fields where $\operatorname{Div}\sigma = (\sigma_{ij,j})$ *and* $\operatorname{div}\varpi = (\varpi_{i,i})$ *denote the divergence operator, respectively for tensor and vector valued functions. The relations (2.7)-(2.8), (2.12)-(2.13) and (2.14)-(2.15) are the mechanical, the electrical and the thermal boundary conditions. The conditions (2.9)-(2.10) represent the Signorini's law and (2.11) means that the contact is frictionless. Here, the functions ψ and ϕ_L used in (2.13) and (2.15) represent, respectively, the truncation function and a given positive function*

$$\phi_L(s) = \begin{cases} -L & if\ s < -L \\ s & if\ -L \le s \le L \\ L & if\ s > L \end{cases} \quad and \quad \psi(r) = \begin{cases} 0 & if\ r < 0 \\ k\,\delta r & if\ 0 \le r \le 1/\delta\,, \\ k & if\ r > 1/\delta \end{cases}$$

where $L > 0$ is a sufficiently large constant, $\delta > 0$ is a small given parameter and $k \ge 0$ is the electrical conductivity coefficient.

3 Mixed variational formulation and main result

In order to get the variational formulation of our problem, we first introduce the following spaces

$$H \;= \{u = (u_i)\,,\ u_i \in L^2(\Omega)\} \quad , \quad \mathcal{H} \;= \{\sigma = (\sigma_{ij})\,,\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}$$
$$H_1 = \{u = (u_i)\,,\ u_i \in H^1(\Omega)\} \quad , \quad \mathcal{H}_1 = \{\sigma = (\sigma_{ij}) \in \mathcal{H}\,,\ \operatorname{Div}\sigma \in H\}.$$

These are real Hilbert spaces for the following inner products

$$(u,v)_H = \int_\Omega u_i\,v_i\,dx \ \ , \ \ (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\,\tau_{ij}\,dx$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}$$
$$(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma,\operatorname{Div}\tau)_H.$$

Let $\gamma : H_1 \to H_\Gamma = H^{1/2}(\Gamma)^d$ be the trace map on Γ and $\langle\cdot,\cdot\rangle_{X',X}$ denotes the duality pairing between a space X and its dual X'. In the sequel, we denote the trace map γv of every $v \in H_1$, again by v. Then, for every $\sigma \in \mathcal{H}_1$, there exists $\sigma\nu \in H'_\Gamma = H^{-1/2}(\Gamma)^d$ satisfying the following Green formula

$$\langle\sigma\nu,\gamma v\rangle_{H'_\Gamma,H_\Gamma} = (\sigma,\varepsilon(v))_{\mathcal{H}} + (\operatorname{Div}\sigma, v)_H, \quad \forall\, v \in H_1. \tag{3.1}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle_{H'_\Gamma, H_\Gamma} = \int_\Gamma \sigma\nu \cdot \gamma v \, da, \quad \forall v \in H_1. \tag{3.2}$$

In addition, we consider the following closed subspace of H_1, defined by

$$V = \{ v \in H_1 , \quad v = 0 \;\; on \;\; \Gamma_D \} .$$

Since $meas(\Gamma_D) > 0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k \, \|v\|_{H_1} , \quad \forall v \in V . \tag{3.3}$$

Where the constant $c_k > 0$ depends only on Ω and Γ_D. Therefore, V endowed with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}.$$

Moreover, its canonical associated norm $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$, is equivalent to the usual norm $\|\cdot\|_{H_1}$ on V.
In addition, it follows from the Sobolev trace theorem that there exists a constant $c_0 > 0$ which depends only on Ω, Γ_C and Γ_D such that

$$\|v\|_{L^2(\Gamma_C)^d} \leqslant c_0 \, \|v\|_V , \quad \forall v \in V . \tag{3.4}$$

We also consider the following closed subspaces of $H^1(\Omega)$, given by

$$Q = \{\eta \in H^1(\Omega) , \;\; \eta = 0 \;\; on \;\; \Gamma_D \cup \Gamma_N\},$$
$$W = \{\psi \in H^1(\Omega) , \;\; \psi = 0 \;\; on \;\; \Gamma_a\}.$$

Over Q and W, we consider the following inner products and associated norms

$$(\theta, \eta)_Q = (\theta, \eta)_{H^1(\Omega)} \quad , \quad \|\eta\|_Q = \|\eta\|_{H^1(\Omega)} , \quad \forall \theta, \eta \in Q$$
$$(\varphi, \psi)_W = (\varphi, \psi)_{H^1(\Omega)} \quad , \quad \|\psi\|_W = \|\psi\|_{H^1(\Omega)}, \quad \forall \varphi, \psi \in W.$$

Since Γ_D and Γ_a are on nonzero measure, it follows from the Friedrichs-Poincaré inequalities that the spaces $(Q, \|\cdot\|_Q)$ and $(W, \|\cdot\|_W)$ are real Hilbert. Moreover, using the Sobolev trace theorem, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\|\eta\|_{L^2(\Gamma_C)} \leqslant c_1 \, \|\eta\|_Q , \quad \forall \eta \in Q \tag{3.5}$$

$$\|\xi\|_{L^2(\Gamma_C)} \leqslant c_2 \, \|\xi|_W , \quad \forall \xi \in W. \tag{3.6}$$

For a regular vector fields $q, D \in \{\varpi \in H, \operatorname{div} \varpi \in L^2(\Omega)\}$, the below Green formulas hold

$$(q, \nabla\eta)_{L^2(\Omega)^d} + (\operatorname{div} q, \eta)_{L^2(\Omega)} = \int_\Gamma q \cdot \nu \, \eta \, da , \quad \forall \eta \in H^1(\Omega) \tag{3.7}$$

$$(D, \nabla\xi)_{L^2(\Omega)^d} + (\operatorname{div} D, \nabla\xi)_{L^2(\Omega)} = \int_\Gamma D \cdot n \, \xi \, da , \quad \forall \xi \in H^1(\Omega). \tag{3.8}$$

In the sequel we denote respectively by $\langle \cdot, \cdot\rangle_{\Gamma_C}$, $\langle\langle \cdot, \cdot\rangle\rangle_{\Gamma_C}$ the duality pairing between $(H^{1/2}(\Gamma_C))^d$ and $(H^{-1/2}(\Gamma_C))^d$, $H^{1/2}(\Gamma_C)$ and $H^{-1/2}(\Gamma_C)$.
Moreover, we consider the sets K_1, K_2, Λ_1 and Λ_2 defined by

$$K_1 = \{v \in V , \;\; v_\nu - g \leqslant 0 \;\; on \;\; \Gamma_C\}, \; K_2 = \{\eta \in Q , \;\; \eta - \theta_F \leqslant 0 \;\; on \;\; \Gamma_C\}, \tag{3.9}$$

$$\Lambda_1 = \{\mu \in H^{-1/2}(\Gamma_C) \mid \langle \mu, \gamma v\rangle_{\Gamma_C} \geqslant 0, \quad v \in V\},$$

$$\Lambda_2 = \{\mu \in H^{-1/2}(\Gamma_C) \mid \langle\langle \mu, \gamma\eta\rangle\rangle_{\Gamma_C} \geqslant 0, \quad \eta \in Q\}. \tag{3.10}$$

Next, in order to study the problem (P), we need some hypotheses on the data's problem. We assume

$\underline{(H_1)}$ *: The elasticity operator* $\mathfrak{F} : \Omega \times S^d \to S^d$ *satisfies*

$$\|\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2)\| \leqslant M_{\mathfrak{F}} \|\xi_1 - \xi_2\|$$

$$(\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2))(\xi_1 - \xi_2) \geqslant m_{\mathfrak{F}} \|\xi_1 - \xi_2\|^2, \quad \forall \xi_1, \xi_2 \in S^d .$$

$\underline{(H_2)}$ *: The piezoelectric tensor* $\mathcal{E} = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d$ *is partial symmetric and continuous*

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega) .$$

We recall here that the transpose tensor $\mathcal{E}^* = (e^*_{ijk})$ *is given by* $e^*_{ijk} = e_{kij}$ *and we have*

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v , \quad \forall \sigma \in S^d , \forall v \in \mathbb{R}^d . \tag{3.11}$$

The thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *is symmetric and continuous*

$$m_{ij} = m_{ji} \in L^\infty(\Omega) .$$

The pyroelectric vector field $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ *is continuous*

$$p_i \in L^\infty(\Omega) .$$

Notice that the two conditions above, allows us to define $\mathcal{M}^* = \sup_{ij} \|m_{ij}\|_{L^\infty(\Omega)}$ *and* $\mathcal{P}^* = \sup_{i} \|p_i\|_{L^\infty(\Omega)}$.

$\underline{(H_3)}$ *: The electric permittivity* $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *is symmetric, continuous and definite positive*

$$\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$$

$$\beta_{ij}\, b_i\, b_j \ \geqslant\ m_\beta \, \|b\|^2 , \quad \forall\, b = (b_i) \in \mathbb{R}^d .$$

The thermal conductivity $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *is symmetric, continuous and definite positive*

$$k_{ij} = k_{ji} \in L^\infty(\Omega)$$

$$k_{ij}\, z_i\, z_j \ \geqslant\ m_{\mathcal{K}} \, \|z\|^2 , \quad \forall\, z = (z_i) \in \mathbb{R}^d.$$

$\underline{(H_4)}$ *: The surface electrical conductivity function* $\psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$*, satisfies*

(a) $(\exists\, L_\psi > 0)$, $(\forall\, r_1 , r_2 \in \mathbb{R})$, $|\psi(\cdot, r_1) - \psi(\cdot, r_2)| < L_\psi \, |r_1 - r_2|$ *a.e. on* Γ_C ,

(b) *The mapping* $x \mapsto \psi(x,u)$ *is measurable on* Γ_C ,

(c) *For all* $r \in \mathbb{R}$*, the mapping* $x \mapsto \psi(x,r)$ *is* M_ψ*-bounded a.e. on* Γ_C ,

(d) $x \mapsto \psi(x,u) = 0$ *for all* $u \leq 0$.

$\underline{(H_5)}$ *: The forces, traction, charges, heat source densities and the foundation's temperature satisfy*

$$f_0 \in L^2(\Omega)^d, \qquad \phi_0 \, , \ q_0 \in L^2(\Omega),$$

$$f_N \in L^2(\Gamma_N)^d, \qquad \phi_b \in L^2(\Gamma_b),$$

$$g \, , \ \theta_F \in L^2(\Gamma_C).$$

Also we assume that there exists $\tilde{v} \in V$ and $\tilde{\eta} \in Q$ such that

$$\tilde{v}_\nu = \tilde{v} \cdot \nu = 1 \quad a.e \text{ on } \quad \Gamma_C, \tag{3.12}$$

$$\tilde{\eta}_\nu = \tilde{\eta} \cdot \nu = 1 \quad a.e \text{ on } \quad \Gamma_C. \tag{3.13}$$

By Riesz's representation theorem, we define $f \in V,\ q_e \in W$ and $\Theta \in Q$ by

$$(f, v)_V = \int_\Omega f_0\, v\, dx + \int_{\Gamma_N} f_N\, v\, da\,, \quad \forall\, v \in V, \tag{3.14}$$

$$(q_e, \xi)_W = \int_\Omega \phi_0\, \xi\, dx - \int_{\Gamma_b} \phi_b\, \xi\, da\,, \quad \forall\, \xi \in W, \tag{3.15}$$

$$(\Theta, \eta)_{L^2(\Omega)} = \int_\Omega q_0\, \eta\, dx\,, \quad \forall\, \eta \in Q. \tag{3.16}$$

Also we define the functional $l : \Lambda_1 \times W \times W$ by

$$l(\lambda, \varphi, \xi) = \int_{\Gamma_C} \psi(\lambda)\phi_L(\varphi - \varphi_F)\, \xi\, da, \tag{3.17}$$

and the bilinear forms $b_1 : V \times M_1 \to \mathbb{R}$ and $b_1 : Q \times M_2 \to \mathbb{R}$ by

$$b_1(v, \mu) = \langle \mu, v \rangle_{\Gamma_C}, \qquad b_2(\eta, \mu) = \langle\langle \mu, \eta \rangle\rangle_{\Gamma_C}. \tag{3.18}$$

According to this notations, we can state the mixed variational formulations of problem (P), in the terms of displacement, electric potential and temperature. To this end we assume that $(u, \varphi, \theta, \sigma, D, q)$ are a regular functions which satisfy (2.1)-(2.15) and let $v \in V$, $\xi \in W$, $\eta \in Q$, $\mu_1 \in \Lambda_1$ and $\mu_2 \in \Lambda_2$ using Green's formulas (3.1) and (3.7)-(3.8) we find

$$\begin{aligned}
&(\sigma, \varepsilon(v))_{\mathcal{H}} = (f_0, v)_H + \langle \sigma\nu, \gamma v \rangle_{H'_\Gamma, H_\Gamma},\\
&(q, \nabla\eta)_H = -(\operatorname{div} q, \eta)_{L^2(\Omega)} + \int_\Gamma q \cdot \nu\, \eta\, da,\\
&(D, \nabla\xi)_H = -(\operatorname{div} D, \xi)_{L^2(\Omega)} + \int_\Gamma D \cdot n\, \xi\, da.
\end{aligned}$$

Using (2.7)-(2.8), (2.12)-(2.13), (2.14)-(2.15) and (3.12)-(3.14), we obtain

$$\begin{aligned}
&(\sigma, \varepsilon(v))_{\mathcal{H}} = (f, v)_v + \langle \sigma\nu, \gamma v \rangle_{\Gamma_C},\\
&-(q, \nabla\eta)_H + \int_\Gamma q \cdot \nu\, \eta\, da = (\Theta, \eta)_{L^2(\Omega)},\\
&-(D, \nabla\xi)_H + \int_{\Gamma_C} \psi(\sigma_\nu(u, \varphi, \theta))\phi_L(\varphi - \varphi_F)\, \xi\, da = (q_e, \xi)_W.
\end{aligned}$$

Since $\sigma_\tau = 0$ on Γ_C, then from the previous inequalities we get

$$(\sigma, \varepsilon(v))_{\mathcal{H}} = (f, v)_v + \langle \sigma_\nu, \gamma v_\nu \rangle_{\Gamma_C}, \tag{3.19}$$

$$-(q, \nabla\eta)_H + \int_\Gamma q \cdot \nu\, \eta\, da = (\Theta, \eta)_{L^2(\Omega)}, \tag{3.20}$$

$$-(D, \nabla\xi)_H + \int_{\Gamma_C} \psi(\sigma_\nu(u, \varphi, \theta))\phi_L(\varphi - \varphi_F)\, \xi\, da = (q_e, \xi)_W. \tag{3.21}$$

And we define the Lagrange multipliers λ_1 and λ_2

$$\langle \lambda_1, \gamma v \rangle_{\Gamma_C} = -\int_{\Gamma_C} \sigma_\nu v_\nu\, da, \tag{3.22}$$

$$\langle\langle \lambda_2, \eta \rangle\rangle_{\Gamma_C} = \int_{\Gamma_C} q \cdot \nu\eta\, da, \tag{3.23}$$

using (3.15)-(3.16) *and* (3.17)-(3.21) *we find*

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + b_1(\lambda_1, v) = (f, v)_v, \tag{3.24}$$

$$-(q, \nabla\eta)_H + b_2(\lambda_2, \eta) = (\Theta, \eta)_{L^2(\Omega)}, \tag{3.25}$$

$$-(D, \nabla\xi)_H + l(\lambda_1, \varphi - \varphi_F, \xi) = (q_e, \xi)_W. \tag{3.26}$$

Moreover, taking to the account (2.9)-(2.10) *and* (3.9)-(3.10) *we deduce that*

$$\lambda_1 \in \Lambda_1, \quad b_1(u, \lambda_1) = b_1(g\tilde{v}, \lambda_1), \quad b_1(u, \mu) \le b_1(g\tilde{v}, \mu) \quad \forall \mu \in \Lambda_1, \tag{3.27}$$

$$\lambda_2 \in \Lambda_2, \quad b_2(\theta, \lambda_2) = b_2(\theta_F\tilde{\eta}, \lambda_2), \quad b_2(\theta, \mu) \le b_2(\theta_F\tilde{\eta}, \mu) \quad \forall \mu \in \Lambda_2. \tag{3.28}$$

Using and (2.1)-(2.3) *and* (3.24)-(3.28) *we find the following mixed problem*

Problem (PM)**.** Find a displacement $u \in V$, an electric potential $\varphi \in W$, a temperature $\theta \in Q$, $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$ such that

$$(\mathfrak{F}\,\varepsilon(u), \varepsilon(v))_H + (\mathcal{E}^*\,\nabla\varphi, \varepsilon(v))_H - (\mathcal{M}\,\theta, \varepsilon(v))_H + b_1(v, \lambda_1) = (f, v)_V \quad \forall v \in V, \tag{3.29}$$

$$-(\mathcal{E}\,\varepsilon(u), \nabla\xi)_H + (\beta\,\nabla\varphi, \nabla\xi)_H - (\mathcal{P}\,\theta, \nabla\xi)_H + l_1(\lambda_1, \varphi - \varphi_F, \xi) \tag{3.30}$$

$$= (q_e, \xi)_W \quad \forall \xi \in W,$$

$$(\mathcal{K}\,\nabla\theta, \nabla\eta)_H + b_2(\eta, \lambda_2) = (\Theta, \eta)_{L^2(\Omega)} \qquad \forall \eta \in Q, \tag{3.31}$$

$$b_1(u, \mu_1 - \lambda_1) \le b_1(g\tilde{v}, \mu_1 - \lambda_1)\,, \quad \forall\, \mu_1 \in \Lambda_1, \tag{3.32}$$

$$b_2(\theta, \mu_2 - \lambda_2) \le b_2(\theta_F\tilde{\eta}, \mu_2 - \lambda_2)\,, \quad \forall\, \mu_2 \in \Lambda_2. \tag{3.33}$$

Our main existence result that we state now and prove in the next sections, is as follows.

Theorem 3.1. *Assume that* (H_1)-(H_5) *is hold. If there exists a positive constant* L^* *such that*

$$\max(\mathcal{M}^*, \mathcal{P}^*) + L_\psi L + M_\psi \leqslant L^*\,,$$

Then, the problem (PM) *has a unique solution.*

4 Proof of main result

The proof of Theorem 3.1 will be done in several steps, and it is based on the following abstract result.

Theorem 4.1. *Let* X *and* Y *be two Hilbert spaces,* $A : X \to X$ *a nonlinear operator,* $b : X \times Y \to \mathbb{R}$ *a bilinear form and* $j : X \to \mathbb{R}$ *a functional. Assume that there exist positive constants* m_A, L_A, M_b, L_j *and* α *such that*

$$(Ay - Ax, x - y)_X \ge m_A \|y - x\|_X^2, \;\; \forall\, x, y \in X,$$

$$\|Ay - Ax\| \le L_A \|y - x\|_X, \;\; \forall\, x, y \in X,$$

and

$$|j(x) - j(y)| \leqslant L_j \|y - x\|_X, \;\; \forall\, x, y \in X,$$

$$|b(x, \mu)| \le M_b \|y\|_X \|\mu\|_Y, \;\; \forall\, x \in X,\, \mu \in Y,$$

$$\inf_{\substack{\mu \in Y \\ \mu \neq 0_Y}} \sup_{\substack{x \in X \\ x \neq 0_X}} \frac{b(y, \mu)}{\|x\|_X \|\mu\|_Y} \ge \alpha.$$

In addition, let Λ *be a closed, convex, unbounded subset of* Y *containing* 0_Y*. Then, for given* $\tilde{f}$, $\tilde{h} \in X$, *there exists* $(x, \lambda) \in X \times \Lambda$ *unique in first argument such that*

$$(Ax, y - x)_X + j(y) - j(x) + b(y - x, \lambda) \geqslant (\tilde{f}, y - x)_X, \;\forall\, y \in X,$$

$$b(x, \mu - \lambda) \le b(\tilde{h}, \mu - \lambda), \;\forall\, \mu \in \Lambda.$$

Remark 4.2. If the functional j is differentiable, then the above problem has a unique solution $(x, \lambda) \in X \times \Lambda$.

We note here that the proof of the Theorem 4.1 is based on the saddle-point theory and Banach's fixed-point theorem (see [18] for details). Next, Let $z = (z_1, z_2, z_3)$ be given, we define the following function.

$$l_z(\xi) = \int_{\Gamma_C} z_3 \xi \, da \quad \forall \xi \in W. \tag{4.1}$$

Then, we consider the following mixed variational problem.

Problem (PM_z)**.** Find a displacement $u_z \in V$, an electric potential $\varphi_z \in W$, a temperature $\theta_z \in Q$, $\lambda_{1z} \in \Lambda_1$ and $\lambda_{2z} \in \Lambda_2$ such that

$$(\mathfrak{F}\,\varepsilon(u_z), \varepsilon(v))_H + (\mathcal{E}^*\,\nabla\varphi_z, \varepsilon(v))_H + b_1(v, \lambda_{1z}) = (f, v)_V + (z_1, \varepsilon(v))_H \qquad \forall v \in V, \tag{4.2}$$

$$-(\mathcal{E}\,\varepsilon(u_z), \nabla\xi)_H + (\beta\,\nabla\varphi_z, \nabla\xi)_H = (q, \xi)_W + (z_2, \nabla\xi)_H - l_z(\xi) \qquad \forall \xi \in W, \tag{4.3}$$

$$(\mathcal{K}\,\nabla\theta_z, \nabla\eta)_H + b_2(\eta, \lambda_{2z}) = (\Theta, \eta)_{L^2(\Omega)} \qquad \forall \eta \in Q, \tag{4.4}$$

$$b_1(u_z, \mu_1 - \lambda_{1z}) \le b_1(g\tilde{v}, \mu_1 - \lambda_{1z}), \quad \forall \mu_1 \in \Lambda_1, \tag{4.5}$$

$$b_2(\theta_z, \mu_2 - \lambda_{2z}) \le b_2(\theta_F \tilde{\eta}, \mu_2 - \lambda_{2z}), \quad \forall \mu_2 \in \Lambda_2. \tag{4.6}$$

Let $X = V \times W \times Q$ and $\Lambda = \Lambda_1 \times \Lambda_2$ two spaces, the space X endowed with the inner product

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W + (\theta, \eta)_Q\,, \tag{4.7}$$

and her associated Euclidean norm $\|\cdot\|_X$. Next, we define the operators $A : X \to X$ and $b : X \times \Lambda \to \mathbb{R}$ given by

$$\begin{aligned}(Ax, y)_X = \; & (\mathfrak{F}\,\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\mathcal{K}\,\nabla\theta, \nabla\eta)_H + (\beta\nabla\varphi, \nabla\xi)_H \\ & + (\mathcal{E}^*\,\nabla\varphi, \varepsilon(v))_H - (\mathcal{E}\varepsilon(u), \nabla\xi)_H\,,\end{aligned} \tag{4.8}$$

$$b(x, \lambda) = b_1(u, \lambda_1) + b_2(u, \lambda_2). \tag{4.9}$$

and the element F_z of X such that for all $y = (v, \xi, \eta) \in X$ we have

$$(F_z, y)_X = (f, v)_V + (z_1, \varepsilon(v))_H + (q, \xi)_W + (z_2, \nabla\xi)_H - l_z(\xi) + (\Theta, \eta)_{L^2(\Omega)}. \tag{4.10}$$

Under all these considerations, obviously we have the following lemma.

Lemma 4.3. The problem (PM_z) is equivalent to the following problem

$$\left\{\begin{array}{l} \textit{Find } x_z \in X \textit{ and } \lambda_z \in \Lambda \textit{ such that} \\ \\ (Ax_z, y)_X + b(y, \lambda_z) = (F_z, y)_X \\ b(x_z, \mu - \lambda_z) \leqslant b(h, \mu - \lambda_z)\,, \quad (\forall\, y \in X,\ \mu \in \Lambda)\,. \end{array}\right. \tag{4.11}$$

Where $h \in X$ is such that $b(h, \lambda) = b_1(g\tilde{v}, \lambda_1) + b_2(\theta_F \tilde{\eta}, \lambda_2)$.

Using the previous lemma, we obtain the following existence and uniqueness result of (PM_z).

Lemma 4.4. *1.For every z of $V \times W \times L^2(\Gamma_C)$, the problem (PM_z) has a unique solution*

$$x_z = (u_z, \varphi_z, \theta_z) \in V \times W \times Q, \quad \lambda_z = (\lambda_{1z}, \lambda_{2z}) \in \Lambda_1 \times \Lambda_2.$$

2. x_z depends continuously on z, where (x_z, λ_z) is the unique solution of the problem (PM_z).

The proof of this lemma is based on abstract result derived in [18].

Now, we start by investigating the proprieties of the operators A and b given in (4.8)-(4.9).

Lemma 4.5. *The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.*

Proof. *We consider two elements $x_1 = (u_1, \varphi_1, \theta_1)$ and $x_2 = (u_2, \varphi_2, \theta_2)$ of X. Using (4.1) and recalling $(\mathcal{E}^* \nabla\varphi, \varepsilon(u))_H = (\mathcal{E}\varepsilon(u), \nabla\varphi)_H$, we find*

$$\begin{aligned}&(Ax_1 - Ax_2, x_1 - x_2)_X \\ &= (\mathfrak{F}\,\varepsilon(u_1) - \mathfrak{F}\,\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2))_{\mathcal{H}} + (\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_H \\ &\quad + (\mathcal{K}\,\nabla\theta_1 - \Delta t\,\mathcal{K}\,\nabla\theta_2, \nabla\theta_1 - \nabla\theta_2)_H.\end{aligned}$$

Combining with (H_1) and (H_3), there exists $m > 0$ depending on $\mathfrak{F}$, β, $\mathcal{K}$, Ω, Γ_D, Γ_N, Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant m \left(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2 + \|\theta_1 - \theta_2\|_Q^2 \right),$$

and using (4.10) yields

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geqslant m\, \|x_1 - x_2\|_X^2\,. \tag{4.12}$$

In the same way, assumptions (H_1) and (H_3) imply that there exists $c_3 > 0$ such that

$$\begin{aligned}(Ax_1 - Ax_2, y)_X \leqslant c_4 \Big(&\|u_1 - u_2\|_V\, \|v\|_V + \|u_1 - u_2\|_V\, \|\xi\|_W + \|\theta_1 - \theta_2\|_Q\, \|\eta\|_Q \\ &+ \|\varphi_1 - \varphi_2\|_W\, \|\xi\|_W + \|\varphi_1 - \varphi_2\|_W\, \|v\|_V \Big).\end{aligned}$$

Choosing $y = Ax_1 - Ax_2$ and $M = 5\,c_3$, it follows from (4.10) that

$$\|Ax_1 - Ax_2\|_X \leqslant M\, \|x_1 - x_2\|_X, \tag{4.13}$$

and thus the lemma 4.5 is established.

□

Lemma 4.6. *The form $b : X \times \Lambda \to \mathbb{R}$ is a continuous bilinear form satisfying the inf-sup property.*

Proof. *It is well known as in [14] that the forms b_1 and b_2 are bilinear continuous forms satisfying the inf-sup property. Thus, there exist α_1 and α_2 such that*

$$\alpha_1 \|\mu_1\|_{-1/2,\Gamma_C} \leqslant \sup_{v \in V,\, v \neq 0_V} \frac{b_1(v, \mu_1)}{\|v\|_V}, \quad \alpha_2 \|\mu_2\|_{-1/2,\Gamma_C} \leqslant \sup_{\eta \in Q,\, \eta \neq 0_Q} \frac{b_2(\eta, \mu_2)}{\|\eta\|_Q}. \tag{4.14}$$

denoting $y_1 = (v, 0, 0) \in X$ and $y_2 = (0, 0, \eta) \in X$ we have

$$\begin{aligned}\alpha_1 \|\mu_1\|_{-1/2,\Gamma_C} &\leqslant \sup_{v \in V,\, v \neq 0_V} \frac{b_1(v, \mu_1)}{\|v\|_V} = \sup_{y_1 \in X,\, y_1 \neq 0_X} \frac{b(y_1, \mu)}{\|y_1\|_X} \leqslant \sup_{y \in X,\, y \neq 0_X} \frac{b(y, \mu)}{\|y\|_X}, \\ \alpha_2 \|\mu_2\|_{-1/2,\Gamma_C} &\leqslant \sup_{\eta \in Q,\, \eta \neq 0_Q} \frac{b_2(\eta, \mu_2)}{\|\eta\|_Q} = \sup_{y_2 \in X,\, y \neq 0_X} \frac{b(y_2, \mu)}{\|y_2\|_X} \leqslant \sup_{y \in X,\, y \neq 0_X} \frac{b(y, \mu)}{\|y\|_X}.\end{aligned}$$

Therfore we find that there exists $\alpha = min\{\alpha_1, \alpha_2\}/2\sqrt{2}$ such that

$$\alpha \|\mu\|_{-1/2,\Gamma_C} \leqslant \sup_{y \in X,\, y \neq 0_X} \frac{b(y, \mu)}{\|y\|_X}, \tag{4.15}$$

and thus the lemma 4.6 is established.

□

Lemma 4.7. *x_z depends continuously on z, where (x_z, λ_z) is the unique solution of the problem (PM_z).*

Proof. *By setting* $y = x_z - x_{z'}$, $y = x_{z'} - x_z$ *consequently in* (4.11) *and adding them we obtain*

$$(Ax_z - Ax_{z'}, x_z - x_{z'})_X + b(x_z - x_{z'}, \lambda_z - \lambda_{z'}) = (F_z - F_{z'}, x_z - x_{z'})_X. \tag{4.16}$$

On another hand by setting $\mu = \lambda_z$, $\mu = \lambda_{z'}$ *consequently in* (4.11) *and adding them we obtain that* $b(x_z - x_{z'}, \lambda_z - \lambda_{z'}) \geqslant 0$. *Then by using* (4.12) *and* (4.16) *we obtain*

$$\|x_z - x_{z'}\|_X^2 \leqslant \|F_z - F_{z'}\|_X \|x_z - x_{z'}\|_X.$$

Then

$$\|x_z - x_{z'}\|_X \leqslant \frac{1}{m}\|F_z - F_{z'}\|_X. \tag{4.17}$$

Let's majorate $\|F_z - F_{z'}\|_X$.

By using (4.10) *we obtain that for all* $y \in X$

$$(F_z - F_{z'}, y)_X = (z_1 - z_1', \varepsilon(v))_H + (z_2 - z_2', \nabla\xi)_H + l_{z'}(\xi) - l_z(\xi).$$

Then

$$|(F_z - F_{z'}, y)_X| \leqslant \|z_1 - z_1'\|_V \|v\|_V + \|z_2 - z_2'\|_W \|\xi\|_W + |l_{z'}(\xi) - l_z(\xi)|.$$

By using (4.1) *and* (3.6), *we find*

$$|(F_z - F_{z'}, y)_X| \leqslant \|z_1 - z_1'\|_V \|v\|_V + \|z_2 - z_2'\|_W \|\xi\|_W + c_2\|z_3 - z_3'\|_{L^2(\Gamma_C)} \|\xi\|_W.$$

Then there exists $c_4 > 0$ *such that*

$$|(F_z - F_{z'}, y)_X| \leqslant c_4 \|z - z'\|_{V\times W\times L^2(\Gamma_C)} \|y\|_X.$$

Thus

$$\|F_z - F_{z'}\|_X \leqslant c_4 \|z - z'\|_{V\times W\times L^2(\Gamma_C)}. \tag{4.18}$$

By using (4.17) *and* (4.18) *we obtain that there exists* $c_5 > 0$ *such that*

$$\|x_z - x_{z'}\|_X \leqslant c_5 \|z - z'\|_{V\times W\times L^2(\Gamma_C)}. \tag{4.19}$$

Then the lemma is proved.

□

Now in the last we use the lemmas 4.3-4.7, to conclude that the problem (PM_z) *has a unique solution* $(x_z, \lambda_z) \in X \times \Lambda$.

<u>Step 2.</u> We consider the following operator

$$T : V \times W \times L^2(\Gamma_C) \to V \times W \times L^2(\Gamma_C)$$

such that for all $z = (z_1, z_2, z_3) \in V \times W \times L^2(\Gamma_C)$, *we have*

$$Tz = \Big[\mathcal{M}\,\theta_z\,; \mathcal{P}\,\theta_z\,; \psi(\lambda_{1z})\,\phi_L(\varphi_z - \varphi_F)\Big].$$

Let's majorate the quantity $I = \big\|Tz - Tz'\big\|_{V\times W\times L^2(\Gamma_C)}$

$$\begin{aligned} I &\leqslant \mathcal{M}^*\big\|\theta_z - \theta_{z'}\big\|_Q + \mathcal{P}^*\big\|\varphi_z - \varphi_{z'}\big\|_W + \big\|\psi(\lambda_{1z})\,\phi_L(\varphi_z - \varphi_F) - \psi(\lambda_{1z'})\,\phi_L(\varphi_{z'} - \varphi_F)\big\|_{L^2(\Gamma_C)} \\ &\leqslant \mathcal{M}^*\big\|\theta_z - \theta_{z'}\big\|_Q + \mathcal{P}^*\big\|\varphi_z - \varphi_{z'}\big\|_W + \big\|(\psi(\lambda_{1z}) - \psi(\lambda_{1z'}))\,\phi_L(\varphi_z - \varphi_F)\big\|_{L^2(\Gamma_C)} \\ &\quad + \big\|\psi(\lambda_{1z'})\,(\phi_L(\varphi_z - \varphi_F) - \phi_L(\varphi_{z'} - \varphi_F))\big\|_{L^2(\Gamma_C)}. \end{aligned}$$

Using (H_4) *we find that*

$$I \leqslant \mathcal{M}^*\|\theta_z - \theta_{z'}\|_Q + \mathcal{P}^*\|\varphi_z - \varphi_{z'}\|_W + LL_\psi\|\lambda_{1z} - \lambda_{1z'}\|_{-1/2,\Gamma_C} + c_2M_\psi\|\varphi_z - \varphi_{z'}\|_W.$$

By using (4.7)*, we find*

$$I \leqslant \mathcal{M}^*\|x_z - x_{z'}\|_X + \mathcal{P}^*\|x_z - x_{z'}\|_X + LL_\psi\|\lambda_z - \lambda_{z'}\|_{-1/2,\Gamma_C} + c_2M_\psi\|x_z - x_{z'}\|_X. \tag{4.20}$$

On another hand by using the inf-sup property we find

$$\begin{aligned}\alpha\|\lambda_z - \lambda_{z'}\|_{-1/2,\Gamma_C} &\leqslant \sup_{y\in X,\, y\neq 0_X} \frac{b(y, \lambda_z - \lambda_{z'})}{\|y\|_X} \\ &\leqslant \sup_{y\in X,\, y\neq 0_X} \frac{(F_z - F_{z'}, y)_X - (Ax_z - Ax_{z'}, y)_X}{\|y\|_X}\end{aligned}$$

by using (4.13) *we find*

$$\alpha\|\lambda_z - \lambda_{z'}\|_{-1/2,\Gamma_C} \leqslant \|F_z - F_{z'}\|_X + M\|x_z - x_{z'}\|_X.$$

Then by using (4.18)-(4.20)*, we find that there exists* $c_6 > 0$ *such that*

$$\|Tz - Tz'\|_{V\times W\times L^2(\Gamma_C)} \leqslant c_6(\max(\mathcal{M}^*, \mathcal{P}^*) + M_\psi + LL_\psi)\|z - z'\|_{V\times W\times L^2(\Gamma_C)}. \tag{4.21}$$

We pose $L^* = \dfrac{1}{c_6}$*, hence if* $\max(\mathcal{M}^*, \mathcal{P}^*) + L_\psi L + M_\psi \leqslant L^*$*, we deduce that* T *is a contraction operator. Then, it comes from Banach fixed point theorem that* T *has a unique fixed point* z^* $(Tz^* = z^*)$*. Therefore,* $(x_{z^*}, \lambda_{z^*}) = (u_{z^*}, \varphi_{z^*}, \theta_{z^*}, \lambda_{z^*} = (\lambda_{1z^*}, \lambda_{2z^*})) \in X \times \Lambda$ *is the unique solution of the problem* (PM)*.*

5 Optimal control of the mixed problem

Here, we present an optimal control for the contact model described by the mixed variational problem (PM)*. The control variable of problem* (PM) *is denoted by*

$$\iota = (f_0, \phi_0, q_0, f_N, \phi_b, g, \theta_F, \varphi_F) \in \Xi_d. \tag{5.1}$$

With

$$\Xi_d = (L^2(\Omega))^d \times (L^2(\Omega))^2 \times L^2(\Gamma_N) \times L^2(\Gamma_b) \times (L^2(\Gamma_N))^3. \tag{5.2}$$

Following the result in theorem 3.1, we conclude that for every $\iota \in \Xi_d$*, the problem* (PM) *has unique solution which depends on* ι*, and denoted by* $(u(\iota), \varphi(\iota), \theta(\iota), \lambda_1(\iota), \lambda_2(\iota))$*. Our control problem is then formulated by : given nonempty subset* Ξ_d^{ad} *of* Ξ_d *representing the set of admissible controls, and an objetive function*

$$\Upsilon : \Xi_d \times V \times W \times Q \times \Lambda_1 \times \Lambda_2 \to \mathbf{R}, \quad \Upsilon = \Upsilon(\iota, u(\iota), \varphi(\iota), \theta(\iota), \lambda_1(\iota), \lambda_2(\iota)).$$

We are interested for the control $\iota^* \in \Xi_d^{ad}$ *and* $(u(\iota^*), \varphi(\iota^*), \theta(\iota^*), \lambda_1(\iota^*), \lambda_2(\iota^*))$ *such that*

$$\Upsilon(\iota^*, u(\iota^*), \varphi(\iota^*), \theta(\iota^*), \lambda_1(\iota^*), \lambda_2(\iota^*)) = \inf_{\iota\in\Xi_d^{ad}} \Upsilon(\iota, u(\iota), \varphi(\iota), \theta(\iota), \lambda_1(\iota), \lambda_2(\iota)). \tag{5.3}$$

To prove the existence of optimal solutios for (5.3)*, we begin by the following Lemma.*

Lemma 5.1. *Let* $\{\iota_n = (f_{0n}, \phi_{0n}, q_{0n}, f_{Nn}, \phi_{bn}, g_n, \theta_{Fn}, \varphi_F) \subset \Xi_d\}$ *be a sequence which converges weakly to* $\iota = (f_0, \phi_0, q_0, f_N, \phi_b, g, \theta_F, \varphi_F)$ *in* Ξ_d*. Then there exists* $c_6 > 0$*, such that*

$$\|u^n\|_V + \|\varphi^n\|_W + \|\theta^n\|_Q + \|\lambda_1^n\|_{H'_{\Gamma_C}} + \|\lambda_2^n\|_{H'_{\Gamma_C}} \leqslant c_6. \tag{5.4}$$

Where $(u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n)$ *is solution of problem* (PM) *corresponding to* ι_n*.*

Proof. Let $\{\iota_n = (f_{0n}, \phi_{0n}, q_{0n}, f_{Nn}, \phi_{bn}, g_n, \theta_{Fn}, \varphi_F) \subset \Xi_d\}$ be a sequence which converges weakly to $\iota = (f_0, \phi_0, q_0, f_N, \phi_b, g, \theta_F, \varphi_F)$ in Ξ_d. Then $(u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n)$ is solution of problem (PM) corresponding to ι_n, then we have

$$\begin{aligned} &(\mathcal{K}\nabla\theta^n, \nabla\eta)_H + b_2(\eta, \Lambda_2^n) = (\Theta^n, \eta)_{L^2(\Omega)}, \quad \forall \eta \in Q, \\ &b_2(\theta^n, \mu_2 - \Lambda_2^n) \leqslant b_2(\theta_{fn}\tilde{\eta}, \mu_2 - \Lambda_2^n), \quad \forall \mu_2 \in \Lambda_2. \end{aligned} \tag{5.5}$$

Using 4.15 and 5.5, we have

$$\begin{aligned} \alpha_2 \|\lambda_2^n\|_{H'_{\Gamma_C}} &\leqslant \sup_{\eta \in Q,\, \eta \neq 0_Q} \frac{b_2(\eta, \lambda_2^n)}{\|\eta\|_Q} \\ &\leqslant \sup_{\eta \in Q,\, \eta \neq 0_Q} \frac{(\Theta^n, \eta)_{L^2(\Omega)} - (\mathcal{K}\nabla\theta^n, \nabla\eta)_H}{\|\eta\|_Q} \\ &\leqslant \|\Theta^n\|_{L^2(\Omega)} + \|\mathcal{K}\|_{L^\infty(\Omega)} \|\theta^n.\|_Q \end{aligned} \tag{5.6}$$

We take $\eta = \theta^n$ in 5.5, we also find

$$(\mathcal{K}\nabla\theta^n, \nabla\theta^n)_H + b_2(\theta^n, \Lambda_2^n) = (\Theta^n, \theta^n)_{L^2(\Omega)}.$$

Then

$$(\mathcal{K}\nabla\theta^n, \nabla\theta^n)_H = (\Theta^n, \theta^n)_{L^2(\Omega)} - b_2(\theta^n, \Lambda_2^n).$$

By using the proprieties of $\mathcal{K}$ and b_2, we find

$$\begin{aligned} m_{\mathcal{K}} \|\theta^n\|_Q^2 &\leqslant (\Theta^n, \theta^n)_{L^2(\Omega)} - b_2(\theta^n, \Lambda_2^n) \\ &\quad \|\Theta^n\|_{L^2(\Omega)} \|\theta^n\|_Q + c_7 \|\theta^n\|_Q \|\Lambda_2^n\|_{H'_{\Gamma_C}}. \end{aligned}$$

Then from 5.6, we find

$$\begin{aligned} m_{\mathcal{K}} \|\theta^n\|_Q^2 &\leqslant \|\Theta^n\|_{L^2(\Omega)} \|\theta^n\|_Q + \frac{c_7}{\alpha_2} \|\theta^n\|_Q (\|\Theta^n\|_{L^2(\Omega)} + \|\mathcal{K}\|_{L^\infty(\Omega)} \|\theta^n\|_Q) \\ &\leqslant (1 + \frac{c_7}{\alpha_2}) \|\Theta^n\|_{L^2(\Omega)} \|\theta^n\|_Q + \frac{c_7}{\alpha_2} \|\mathcal{K}\|_{L^\infty(\Omega)} \|\theta^n\|_Q^2. \end{aligned}$$

Then,

$$(m_{\mathcal{K}} - \frac{c_7}{\alpha_2} \|\mathcal{K}\|_{L^\infty(\Omega)}) \|\theta^n\|_Q^2 \leqslant (1 + \frac{c_7}{\alpha_2}) \|\Theta^n\|_{L^2(\Omega)} \|\theta^n\|_Q.$$

Using young inequality with ϵ, we find

$$(m_{\mathcal{K}} - \frac{c_7}{\alpha_2} \|\mathcal{K}\|_{L^\infty(\Omega)}) \|\theta^n\|_Q^2 \leqslant \frac{(1 + \frac{c_7}{\alpha_2})}{2\epsilon} \|\Theta^n\|_{L^2(\Omega)}^2 + (1 + \frac{c_7}{\alpha_2}) \epsilon \|\theta^n\|_Q^2.$$

Then we choose ϵ, such that $\epsilon < \frac{\mathcal{K}}{1 + \frac{c_7}{\alpha_2}}$, then $m_{\mathcal{K}} - \frac{c_7}{\alpha_2} \|\mathcal{K}\|_{L^\infty(\Omega)} - (1 + \frac{c_7}{\alpha_2})\epsilon > 0$, we have

$$\|\theta^n\|_Q + \|\Lambda_2^n\|_{H'_{\Gamma_C}} \leqslant c_8. \tag{5.7}$$

Where $c_8 > 0$.
Also, we have $(u^n, \varphi^n, \lambda_1^n)$ satisfies

$$\begin{aligned} &(\mathfrak{F}\,\varepsilon(u^n), \varepsilon(v))_H + (\mathcal{E}^*\,\nabla\varphi^n, \varepsilon(v))_H - (\mathcal{M}\,\theta^n, \varepsilon(v))_H + b_1(v, \lambda_1^n) \\ &= (f_n, v)_V \quad \forall v \in V, \end{aligned} \tag{5.8}$$

$$\begin{aligned} &(\mathcal{E}\,\varepsilon(u^n), \nabla\xi)_H - (\beta\,\nabla\varphi^n, \nabla\xi)_H + (\mathcal{P}\,\theta^n, \nabla\xi)_H - l_1(\lambda_1^n, \varphi^n - \varphi_{Fn}, \xi) \\ &= -(q_{en}, \xi)_W \; \forall \xi \in W, \end{aligned} \tag{5.9}$$

$$b_1(u^n, \mu_1 - \lambda_1^n) \leq b_1(g_n\tilde{v}, \mu_1 - \lambda_1^n), \quad \forall \mu_1 \in \Lambda_1^n, \tag{5.10}$$

Using the same method as (4.8)-4.11, then we find that the problem 5.8-5.10 is equivalent to the following problem

$$\begin{cases} (Ax^n, y)_X + b(y, \tilde{\lambda}_1^n) + \tilde{l}(\tilde{\lambda}_1^n, x^n - x_f^n, y) - (\mathcal{M}\,\theta^n, \varepsilon(v))_H - (\mathcal{P}\,\theta^n, \nabla\xi)_H \\ = (F_n, y)_X, \quad \forall y \in X, \\ b(x^n, \mu - \tilde{\lambda}_1^n) \leqslant b(h, \mu - \tilde{\lambda}_1^n), \quad \forall \mu \in \mathbf{\Lambda}_1. \end{cases} \tag{5.11}$$

Where $X = V \times W$ and the operators A, b, $\tilde{l}$ and F_n are defined by :

$$\begin{aligned} &(Ax, y)_X = (\mathfrak{F}\,\varepsilon(u), \varepsilon(v))_H + (\beta\,\nabla\varphi, \nabla\xi)_H + (\mathcal{E}^*\,\nabla\varphi, \varepsilon(v))_H - (\mathcal{E}\,\varepsilon(u), \nabla\xi)_H, \\ &b(x, \lambda) = b_1(u, \lambda_1), \\ &(F_n, y)_X = (f_n, v)_v + (q_n, \xi)_W, \\ &\tilde{l}(\lambda, x, y) = l(\lambda_1, \varphi, \xi), \forall y = (v, \xi, \eta) \in X. \\ &b(h, \lambda) = b_1(g\tilde{v}, \lambda_1), \quad \forall \lambda = (\lambda_1, \lambda_2) \in \mathbf{\Lambda}. \end{aligned} \tag{5.12}$$

Using the same method for proving 5.7, we can easly show that there exits $c_9 > 0$, such that

$$\|x^n\|_X + \|\lambda_1^n\|_{H'^d_{\Gamma_C}} \leqslant c_9. \tag{5.13}$$

Finally by 5.7 and 5.13, the (5.1) is then proved. □

In the sequel, we consider $\{\iota_n = (f_{0n}, \phi_{0n}, q_{0n}, f_{Nn}, \phi_{bn}, g_n, \theta_{Fn}, \varphi_F) \subset \Xi_d\}$ *be a sequence which converges weakly to* $\iota = (f_0, \phi_0, q_0, f_N, \phi_b, g, \theta_F, \varphi_F)$ *in* Ξ_d*. We denote* $(u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n)$ *and* $(u, \varphi, \theta, \lambda_1, \lambda_2)$ *are the unique solution of problem* (PM) *corresponding to* ι_n *and* ι*, respectively. From 5.4 passing to a subsequence if necessary; we have*

$$u^n \rightharpoonup \tilde{u} \quad \text{in} \quad V, \tag{5.14}$$

$$\varphi^n \rightharpoonup \tilde{\varphi} \quad \text{in} \quad W, \tag{5.15}$$

$$\theta^n \rightharpoonup \tilde{\theta} \quad \text{in} \quad Q, \tag{5.16}$$

$$\lambda_1^n \rightharpoonup \tilde{\lambda}_1 \quad \text{in} \quad H'^d_{\Gamma_C}, \tag{5.17}$$

$$\lambda_1^n \rightharpoonup \tilde{\lambda}_1 \quad \text{in} \quad H'^d_{\Gamma_C}, \tag{5.18}$$

Using in the fact trace operator is compact, we can conclude from 5.14-5.18 that

$$u^n \to \tilde{u} \quad \text{in} \quad (L^2(\Gamma_C))^d, \tag{5.19}$$

$$\varphi^n \to \tilde{\varphi} \quad \text{in} \quad L^2(\Gamma_C), \tag{5.20}$$

$$\theta^n \to \tilde{\theta} \quad \text{in} \quad L^2(\Gamma_C). \tag{5.21}$$

By the linearity of the operators involved in the following expressions, it comes from the strong convergences 5.19-5.21 that

$$(\mathfrak{F}\,\varepsilon(u^n), \varepsilon(v))_{\mathcal{H}} \to (\mathfrak{F}\,\varepsilon(\tilde{u}), \varepsilon(v))_{\mathcal{H}}, \tag{5.22}$$

$$(\mathcal{E}^*\,\nabla\varphi^n, \varepsilon(v))_{\mathcal{H}} \to (\mathcal{E}^*\,\nabla\tilde{\varphi}, \varepsilon(v))_{\mathcal{H}} \tag{5.23}$$

$$(\mathcal{M}\,\theta^n, \varepsilon(v))_{\mathcal{H}} \to (\mathcal{M}\,\tilde{\theta}, \varepsilon(v))_{\mathcal{H}} \tag{5.24}$$

$$(\mathcal{E}\,\varepsilon(u^n), \nabla\xi)_H \to (\mathcal{E}\,\varepsilon(\tilde{u}), \nabla\xi)_H \tag{5.25}$$

$$(\beta\,\nabla\varphi^n, \nabla\xi)_H \to (\beta\,\nabla\tilde{\varphi}, \nabla\xi)_H \tag{5.26}$$

$$(\mathcal{P}\,\theta^n, \nabla\xi)_H \to (\mathcal{P}\,\tilde{\theta}, \nabla\xi)_H \tag{5.27}$$

$$(\mathcal{K}\,\nabla\theta^n, \nabla\eta)_H \to (\mathcal{K}\,\nabla\tilde{\theta}, \nabla\eta)_H \tag{5.28}$$

$$l_1(\lambda_1^n, \varphi^n - \varphi_{Fn}, \xi) \to l_1(\tilde{\lambda}_1, \tilde{\varphi} - \varphi_F, \xi) \tag{5.29}$$

$$b_1(v, \lambda_1^n) \to b_1(v, \tilde{\lambda}_1) \tag{5.30}$$

$$b_1(u^n, \mu_1 - \lambda_1^n) \to b_1(\tilde{u}, \mu_1 - \tilde{\lambda}_1) \tag{5.31}$$

$$b_2(\theta^n, \mu_2 - \lambda_2^n) \to b_2(\tilde{\theta}, \mu_2 - \tilde{\lambda}_2) \tag{5.32}$$

$$b_2(v, \lambda_2^n) \to b_2(v, \tilde{\lambda}_2) \tag{5.33}$$

$$(f_n, v)_V \to (f, v)_V \tag{5.34}$$

$$(q_{en}, \xi)_W \to (q_e, \xi)_W \tag{5.35}$$

$$(\Theta^n, \eta)_{L^2(\Omega)} \to (\Theta, \eta)_{L^2(\Omega)}. \tag{5.36}$$

Now, we have the following result.

Theorem 5.2. *Assume that the ussumptions of Theorem 3.1 hold. Then, the map* $(\iota; u(\iota), \varphi(\iota), \theta(\iota), \lambda_1(\iota), \lambda_2(\iota))$ *is upper semicontinous.*

Proof. Kepping in mind 5.5 and 5.8-5.10, we pass to the limit by using the convergence results 5.22-5.36. Then we prove that $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{\lambda}_1, \tilde{\lambda}_2$ is solution of problem (PM), and by uniquenesses of solution, we have $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{\lambda}_1, \tilde{\lambda}_2 = (u, \varphi, \theta, \lambda_1, \lambda_2$. Then the prove of 5.2 is completed. □

Next, we have the following optimal control result for the problem 5.3.

Theorem 5.3. *Under ussumptions of Theorem 3.1.* Ξ_d^{ad} *is a weakly compact subset of* Ξ_d *and* Υ *is a lower semicontinous function. Then, problem 5.3 has an optimal solution.*

Proof. Let $\{(\iota_n; u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n)\}$ be a minimizing sequence for the problem 5.3, such that, $\iota_n \in \Xi_d^{ad}$, $\{(u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n)\}$ is the solution of the mixed problem PM corresponding to ι_n and

$$\lim_{n\to\infty} \Upsilon(\iota_n; u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n) = \inf_{\iota\in\Xi_d^{ad}} \Upsilon(\iota; u(\iota), \varphi(\iota), \theta(\iota), \lambda_1(\iota), \lambda_2(\iota)) = \varkappa \in [-\infty, +\infty). \tag{5.37}$$

Using the compactness of Ξ_d^{ad}, we take a subsequence of $\{\iota_n\}$, also denoted $\{\iota_n\}$, such that $\iota_n \rightharpoonup \iota^*$ in Ξ_d and $\iota^* \in \Xi_d^{ad}$. Therefore, by theorem 5.2, we have

$$(u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n) \rightharpoonup (u^*, \varphi^*, \theta^*, \lambda_1^*, \lambda_2^*). \tag{5.38}$$

Where $(u^*, \varphi^*, \theta^*, \lambda_1^*, \lambda_2^*)$ is the solution of problem (PM) corresponding to ι^*. Also, with the lower semicontinuity of the objective function Υ, we have

$$\varkappa \leqslant \Upsilon(\iota^*; u^*, \varphi^*, \theta^*, \lambda_1^*, \lambda_2^*) \leqslant \lim_{n\to\infty} \Upsilon(\iota_n; u^n, \varphi^n, \theta^n, \lambda_1^n, \lambda_2^n) = \varkappa. \tag{5.39}$$

Then the proof of Theorem 5.3 is completed. □

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