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ON THE NUMBER OF FINITE GROUPOIDS WHICH ADMIT REGULAR CAYLEY GRAPHS.

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Abstract Let $\mathcal{G}$ be a finite groupoid, and we denote $\mathcal{T}$ as a subset of $\mathcal{G}$. Then we can find a Cayley graph $Cay(\mathcal{G},\mathcal{T})$ related to $\mathcal{T}$. This paper investigates necessary and sufficient conditions for $Cay(\mathcal{G},\mathcal{T})$ to be k-regular. This characterization helps us to count the number of groupoids of cardinality n which has at least one subset $\mathcal{T}$ such that $Cay(\mathcal{G},\mathcal{T})$ is k-regular which is denoted by M_n^k. Our investigation gives an upper bound for the number M_n^k.

1 Introduction

Suppose $\mathcal{K}$ is a semigroup and $\mathcal{T}$ is a nonempty subset of $\mathcal{K}$. The Cayley graph $Cay(\mathcal{K},\mathcal{T})$ of $\mathcal{K}$ related to $\mathcal{T}$ can be defined as a graph having a vertex set $\mathcal{K}$ and an edge set $E(Cay(\mathcal{K},\mathcal{T}))$ consisting of ordered pairs (x,y) such that $tx = y$ for some $t \in \mathcal{T}$. Graph constructions out of algebraic structures are of great interest and play vital roles in building strong connections between algebra and graph theory which can be found in [14, 15, 16, 18, 19]. The Cayley graphs of semigroups are first introduced by Bohdan Zelinka[1]. They are of great interest among algebraic graph theorists and combinatorialists all over the globe. In particular [2, 3, 4].
This paper defines Cayley graphs of groupoids and investigates the regularity criterion for such graphs. Regularity in graphs is exciting and has a lot of applications in network theory and other branches of graph theory[5]. This paper mainly focuses on the regularity of Cayley graphs of groupoid $\mathcal{G}$ and the necessary and sufficient conditions for a $Cay(\mathcal{G},\mathcal{T})$ to be k-regular. An n vertex graph is known to be k-regular if and only if nk is even[6]. We use the maximum of Stirling numbers given by V.V Menon [7] and the maximum of binomial coefficients[8] to get a more explicit upper bound. We introduce k-out-regularity for finding upper bounds on the number of finite groupoids that admit k-regular Cayley graphs.

2 Preliminaries

Definition 2.1. [9, 10] The tuple $(\mathcal{G},*)$ is said to be a groupoid if $\mathcal{G}$ is a nonempty set and $*$ is a binary operation on $\mathcal{G}$. If $\mathcal{G}$ is associative, then $\mathcal{G}$ is said to be a semigroup.

Definition 2.2. [9, 10, 6] Consider $\mathcal{G}$ a finite groupoid with n elements and $\mathcal{T}$ as a subset of $\mathcal{G}$. Then Cayley graph related to $\mathcal{T}$ is denoted by $Cay(\mathcal{G},\mathcal{T})$ as a graph with vertex set $\mathcal{G}$ and the set of edges is $E(Cay(\mathcal{G},\mathcal{T})) := \{(x,y) : y = tx \text{ for some } t \in \mathcal{T}\}$. The subset $\mathcal{T}$ is called the connection set associated with $Cay(\mathcal{G},\mathcal{T})$.

Definition 2.3. [6] A graph M with vertex set V and an edge set E is considered regular if for each vertex $v \in V$, $deg(v) = k$ for some $k \in \mathbb{N}$. In this case, we call the graph to be k-regular.

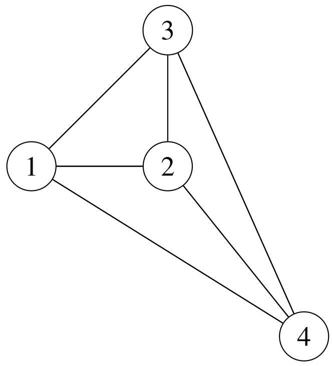

Figure 1. A 3-regular graph.

Definition 2.4. [6] A directed graph $\mathcal{D}$ is k-regular if, for each vertex $v \in V$, the number of outgoing edges must equal the number of incoming edges, equal to k. Let $in - deg(v)$ denote the number of incoming edges to the vertex v, and $out - deg(v)$ denote the number of outgoing edges from the vertex v. Then $in - deg(v) = out - deg(v) = k$ for a k-regular directed graph. Note that the Cayley graph is a directed graph. Here we illustrate an example of a 2-regular directed graph.

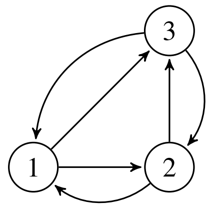

Figure 2. A 2-regular directed graph.

Definition 2.5. [11] The Stirling numbers of the second kind $S(n, k)$ count the number of ways to partition a set of n elements into k non-empty, unlabeled subsets. A recursive formula for $S(n, k)$ id given by

$$S(n, k) = S(n-1, k-1) + k\,S(n-1, k), \quad k \leq n. \tag{2.1}$$

Notice that $S(n, n) = S(n, 1) = 1$ for any positive integer n.

Definition 2.6. [9, 12] An equivalence relation R on a set $\mathcal{G}$ is a relation satisfying the following three conditions,
1. xRx for each $x \in \mathcal{G}$.
2. xRy if and only if yRx for $x, y \in \mathcal{G}$
3. xRy and yRz imply xRz for x, y and $z \in \mathcal{G}$.
The collection of equivalence classes of $\mathcal{G}$ under R is denoted by $\mathcal{G}/R := \{[x] : x \in \mathcal{G}\}$. It is clear that $\mathcal{G}/R$ forms a partition of $\mathcal{G}$

Definition 2.7. [8] For any positive integer n and any positive integer $k \leq n$, $\binom{n}{k}$ denotes the number of ways to choose a k element subset from an n element set.

Theorem 2.8. [8] *For any positive integer n and any positive integer $k \leq n$,*

$$\max\{\binom{n}{k} : k \in \{0, 1, 2, ..., n\}\} = \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}. \tag{2.2}$$

Theorem 2.9. [8] *For any positive integer n and any positive integer $k \leq n$. Then*

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k. \tag{2.3}$$

Theorem 2.10. [11, 13] *Stirling's approximation is an approximation for factorials. For any positive integer n,*

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{2.4}$$

Here the sign $\sim$ means the two functions are asymptotic; their ratio approaches 1 as n tends to infinity.

Theorem 2.11. [7] *The maximum of $S(n,k),\ 1 \leq k \leq n$ is given by,*

$$\begin{aligned}\max\{\frac{1}{n}log(S(n,k)\sqrt{2\pi})\} &= \beta - 1 + \beta^{-1} - \frac{\beta+2}{2n} + \frac{\beta^2}{24n^2(\beta+1)}(12\beta^2+22\beta+13)\\ &+ \frac{\beta^2}{24n^3(\beta+1)^2}\{(\beta+1)^2(16\beta^4+80\beta^3+147\beta^2+100\beta+19)\\ &+ (\beta+\frac{3}{2})(12\beta^3-5\beta^2-58\beta-\frac{79}{2})\} + O(\frac{\beta^6}{n^4}).\end{aligned} \tag{2.5}$$

where β is the solution of the equation $\beta e^{\beta} = n$.

3 Main results

Definition 3.1. A directed graph $\mathcal{D}$ is said to be k-out-regular if, for each vertex v in the vertex set of $\mathcal{D}$, $out - deg(v) = k$.

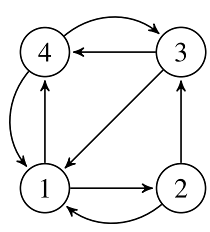

Figure 3. A 2-out-regular graph

Remark 3.2. k-regularity implies k-out-regularity. But from this example, it is clear that k-out-regularity does not imply k-regularity.

Remark 3.3. A k-out-regular graph need not contain a k-regular spanning subgraph. For example take the figure 3. Here $in - deg(2) < out - deg(2)$.

Definition 3.4. A Cayley graph $Cay(\mathcal{G},\mathcal{T})$ related to a subset $\mathcal{T}$ of $\mathcal{G}$ is said to be k-out-regular if, for each vertex v in $\mathcal{G}$, $out - deg(v) = k$.

Definition 3.5. Let n be any positive integer and k be a positive integer such that $k \leq n$. Let $P_k^{\mathcal{T}}$ denote the collection of all partitions of $\mathcal{T}$ into k blocks. Then it is known that the cardinality of $P_k^{\mathcal{T}}$ is $S(n,k)$ if $|\mathcal{T}| = n$.

Definition 3.6. A groupoid is said to admit a Cayley graph with a given property if a subset $\mathcal{T}$ of $\mathcal{G}$ exists such that $Cay(\mathcal{G},\mathcal{T})$ is a graph with the given property.

Definition 3.7. Let n be any positive integer, and k be a positive integer such that $k \leq n$ and nk is even. Let M denote the set of all groupoids of order n, which admits a k-regular Cayley graph and H denote the set of all groupoids of order n, which admits a k-out-regular Cayley graph. Let M_k^n denote the number of groupoids of order n, which admits a k-regular Cayley graph, and H_k^n denote the number of groupoids, which admits a k-out-regular Cayley graph.

Theorem 3.8. *Let n and k be positive integers such that $k \leq n$. Then*

$$M_k^n \leq H_k^n. \tag{3.1}$$

Proof. Let M be the set of all groupoids of order n, which admits a k-regular Cayley graph and H be the set of all groupoids of order n, which admits a k-out-regular Cayley graph. Take $\mathcal{G} \in M$. Then there exists a subset $\mathcal{T}$ of $\mathcal{G}$ such that $Cay(\mathcal{G}, \mathcal{T})$ is k-regular. That is for each $v \in G$, $out - deg(v) = in - deg(v) = k$. In particular, $out - deg(v) = k$. This implies $Cay(\mathcal{G}, \mathcal{T})$ is k-out-regular. That is, $\mathcal{G}$ admits a k-out-regular graph. This implies $\mathcal{G} \in H$. Since by the definition 3.7, $|M|$ = M_k^n and $|H|$= H_k^n, We have $M_k^n \leq H_k^n$. □

Definition 3.9. Let $\mathcal{G}$ be a groupoid of order n, and $\mathcal{T}$ be a subset of $\mathcal{G}$ of order m. Then $B_k^n(m)$ denotes the set of all groupoids, which admits a k-out-regular Cayley graph with a connection set of cardinality m. And let $|B_k^n(m)| = L_k^n(m)$.

Theorem 3.10. *Let n, m, and k be positive integers such that $k \leq m \leq n$. Then the following are true.*

$$H = \bigcup_{k \leq m \leq n} B_k^n(m) \tag{3.2}$$

$$H_k^n \leq \sum_{k \leq m \leq n} L_k^n(m). \tag{3.3}$$

Proof. Let $h \in H$, then h admits a k-out-regular Cayley graph. A subset $\mathcal{T}$ of $\mathcal{G}$ exists such that $Cay(\mathcal{G}, \mathcal{T})$ is k-out-regular and $|\mathcal{T}| = m$. Since $m \leq n$, we have $h \in B_k^n(m)$ for some m. That is $h \in \bigcup_{k \leq m \leq n} B_k^n(m)$ This implies $H \subseteq \bigcup_{k \leq m \leq n} B_k^n(m)$. Now, $B_k^n(m)$ is a collection of groupoids admitting k-out-regular Cayley graphs; the other inclusion is immediate. From this, it is evident that $|H| = \left| \bigcup_{k \leq m \leq n} B_k^n(m) \right|$. That is $H_k^n \leq \sum_{k \leq m \leq n} L_k^n(m)$. □

Corollary 3.11. *Let n, m, and k be positive integers such that $k \leq m \leq n$. Then*

$$M_k^n \leq \sum_{k \leq m \leq n} L_k^n(m). \tag{3.4}$$

Proof. This immediately follows from Theorem 3.8. □

Definition 3.12. Let $\mathcal{G}$ be a groupoid and $\mathcal{T}$ be a subset of $\mathcal{G}$. For each $x \in \mathcal{G}$, define a binary relation $R_x^{\mathcal{T}}$ on $\mathcal{T}$ in such a way that $t_1 R_x^{\mathcal{T}} t_2$ if and only if $t_1 x = t_2 x$.

Lemma 3.13. *The relation $R_x^{\mathcal{T}}$ defined on $\mathcal{T}$ is an equivalence relation.*

Proof. Let $t, t_1, t_2, t_3 \in \mathcal{T}$. We have $tx = tx$ which implies $tR_x^{\mathcal{T}} t$. That is $R_x^{\mathcal{T}}$ reflexive. If $t_1 x = t_2 x$ then $t_2 x = t_1 x.R_x^{\mathcal{T}}$ is symmetric. Now if $t_1 x = t_2 x$ and $t_2 x = t_3 x$ then $t_1 x = t_3 x$. $R_x^{\mathcal{T}}$ is transitive. Then $\mathcal{T}/R_x^{\mathcal{T}}$ denote the set of all equivalence classes of $R_x^{\mathcal{T}}$. □

Lemma 3.14. *The number of distinct equivalence classes of $R_x^{\mathcal{T}}$ equals the number of outgoing edges from x in $Cay(\mathcal{G}, \mathcal{T})$. Moreover, each equivalence class represents a distinct edge from x.*

Proof. Let $E_x = \{(x, y_1), (x, y_2), (x, y_3), ..., (x, y_k)\}$ be the set of all distinct outgoing edges from x in $Cay(\mathcal{G}, \mathcal{T})$. Then there exists $t_1, t_2, t_3, ..., t_k$ where $t_i \neq t_j$ for $i \neq j$ such that $y_1 = t_1 x, y_2 = t_2 x, y_3 = t_3 x, ..., y_k = t_k x$. This implies $t_1, t_2, t_3, ..., t_k$ are not $R_x^{\mathcal{T}}$ related. That is $t_1 R_x^{\mathcal{T}}, t_2 R_x^{\mathcal{T}}, t_3 R_x^{\mathcal{T}}, ..., t_k R_x^{\mathcal{T}}$ are distinct equivalence classes of $R_x^{\mathcal{T}}$. Now define a map τ from $\mathcal{T}/R_x^{\mathcal{T}}$ to E_x such that the element $tR_x^{\mathcal{T}}$ maps to (x, tx). This map is well-defined. For if $t_1 R_x^{\mathcal{T}} = t_2 R_x^{\mathcal{T}}$ then $t_1 R_x^{\mathcal{T}} t_2$. That is $t_1 x = t_2 x$ implies $(x, t_1 x) = (x, t_2 x)$.Now we have to prove that τ is indeed a bijection. τ is injective, $(x, t_1 x) = (x, t_2 x) \Rightarrow t_1 x = t_2 x \Rightarrow t_1 R_x^{\mathcal{T}} t_2 \Rightarrow t_1 R_x^{\mathcal{T}} = t_2 R_x^{\mathcal{T}}$. τ is surjective, let $(x, y) \in E_x$ then $y = tx$ for some $t \in \mathcal{T}$. That is $t \in tR_x^{\mathcal{T}} \Rightarrow \tau$ maps $tR_x^{\mathcal{T}}$ to (x, y). This proves τ is bijection and each $tR_x^{\mathcal{T}}$ corresponds to an edge (x, tx). □

Theorem 3.15. *Let $\mathcal{G}$ be a groupoid and $\mathcal{T}$ be a subset of $\mathcal{G}$. Then $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular if and only if for each $x \in \mathcal{G}$, $|\mathcal{T}/R_x^{\mathcal{T}}| = k$.*

Proof. For the necessary part, let us assume that $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular.
Enumerate $\mathcal{G} := \{x_1, x_2, x_3, ..., x_n\}$ and $T := \{y_1, y_2, y_3, ..., y_m\}$, $m \leq n$.
For every $x \in \mathcal{G}$, there exists $g_1, g_2, g_3, ..., g_n \in \mathcal{G}$ such that (x, g_i) is an edge in $Cay(\mathcal{G},\mathcal{T})$ for $i = 1, 2, 3, ..., k$. If $k = 3$, This situation can be illustrated in the following figure.

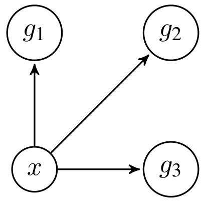

Figure 4. Out going edges from x.

Thus for each $x \in \mathcal{G}$, we have $g_1x, g_2x, g_3x, ..., g_kx$ are all distinct and $g_1, g_2, g_3, ..., g_k$ represent distinct equivalence classes of the relation $R_x^{\mathcal{T}}$. We have $g_1R_x^{\mathcal{T}}, g_2R_x^{\mathcal{T}}, g_3R_x^{\mathcal{T}}, ..., g_kR_x^{\mathcal{T}}$ are all distinct. That is $|\mathcal{T}/R_x^{\mathcal{T}}| = k$.
For sufficient part, assume that for every $x \in \mathcal{G}$, $|\mathcal{T}/R_x^{\mathcal{T}}| = k$. We will show that $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular. Let $x \in \mathcal{G} \times \mathcal{G}$ and $\mathcal{T}/R_x^{\mathcal{T}} := \{g_1R_x^{\mathcal{T}}, g_2R_x^{\mathcal{T}}, g_3R_x^{\mathcal{T}}, ..., g_kR_x^{\mathcal{T}}\}$, $g_iR_x^{\mathcal{T}} \neq g_jR_x^{\mathcal{T}}$ for $i \neq j$. By lemmas 3.13 and 3.14 each $g_iR_x^{\mathcal{T}}$ is an edge going from x. That is, x has distinct neighbors. This proves that $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular. □

Theorem 3.16. *Let $\mathcal{G}$ be a finite set with n elements, and $\mathcal{T}$ be a subset of $\mathcal{G}$ with m elements. Then a binary operation exists on $\mathcal{G}$ such that $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular for $k \leq m \leq n$.*

Proof. Assume that $\mathcal{G}$ be the set of n elements $x_1, x_2, x_3, ..., x_n$. Theorem 3.15 enables us to construct the desired binary operation. We have shown that $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular if and only if for each $x \in \mathcal{G}$, $|\mathcal{T}/R_x^{\mathcal{T}}| = k$. We must construct a Cayley graph of $\mathcal{G}$ with connection set $\mathcal{T}$. Since $R_x^{\mathcal{T}}$ is an equivalence relation on $\mathcal{T}$, $\mathcal{T}/R_x^{\mathcal{T}}$ forms a partition of $\mathcal{T}$ into k blocks. Now choose $P_1, P_2, P_3, ..., P_n$ from $P_k^{\mathcal{T}}$ = set of all partitions of $\mathcal{T}$ into k blocks. Here P_is need not be distinct. Assign P_i to x_i in such a way that $\mathcal{T}/R_{x_i}^{\mathcal{T}} = P_i = \{t_{i1}R_{x_i}^{\mathcal{T}}, t_{i2}R_{x_i}^{\mathcal{T}}, t_{i3}R_{x_i}^{\mathcal{T}}, ..., t_{ik}R_{x_i}^{\mathcal{T}}\}$. Then $|\mathcal{T}/R_{x_i}^{\mathcal{T}}| = k$. Now note that for any element $t \in t_{i1}R_{x_i}^{\mathcal{T}}$, we have $tx_i = t_{i1} = r_{i1}$ (say). Similarly, for any $t \in t_{ij}R_{x_i}^{\mathcal{T}}$, $1 \leq j \leq k$, we have $tx_i = t_{ij}x_i = r_{ij}$. It is clear that to define a binary operation on $\mathcal{G}$, we have to assign values to n^2 elements of $\mathcal{G} \times \mathcal{G}$ from $\mathcal{G}$. By this construction, we can assign values to nm elements out of n^2 elements of $\mathcal{G} \times \mathcal{G}$.

For each i, $1 \leq i \leq n$, $\sum_{j=1}^{k} |t_{ij}R_{x_i}^{\mathcal{T}}| = m$. Note that $r_{ij_1} \neq r_{ij_2}$ for $j_1 \neq j_2$. So a proper assignment of values from $\mathcal{G}$ to these nm elements followed by any possible assignment to the rest of the $n^2 - nm$ elements of $\mathcal{G} \times \mathcal{G}$ from $\mathcal{G}$ constructs a binary operation on $\mathcal{G}$ in which $Cay(\mathcal{G},\mathcal{T})$ is k-out-regular. □

Example 3.17. Consider the set $A = \{1, 2, 3, 4\}$ Let us construct a binary operation so that A becomes a groupoid that could admit a 2-out-regular groupoid with the connection set $T = \{1, 2, 3\}$. Here we have $k = 2$. Then we have to select $P_1, P_2, P_3 \in P_2^T$.
Take $P_1 = \{\{1\}, \{2, 3\}\}$, $P_2 = \{\{2\}, \{1, 3\}\}$, $P_3 = \{\{3\}, \{1, 2\}\}$.
Now for each element in A we have to assign one P_i, $i = 1, 2, 3$.
Let us view such an assignment by the following diagram.
$1 \to P_1 = \{\{1\}, \{2, 3\}\} \to \{1, 2\}$
$2 \to P_2 = \{\{2\}, \{1, 3\}\} \to \{1, 2\}$
$3 \to P_3 = \{\{3\}, \{1, 2\}\} \to \{1, 2\}$
$4 \to P_2 = \{\{2\}, \{1, 3\}\} \to \{1, 2\}$
This diagram indicates the following fixed mappings of elements of $A \times A \to A$.
$(1, 1) \to 1$, $(2, 1) \to 2$, $(3, 1) \to 2$
$(2, 2) \to 1$, $(1, 2) \to 2$, $(3, 2) \to 2$
$(3, 3) \to 1$, $(1, 3) \to 1$, $(2, 3) \to 2$

$(2,4) \to 1$, $(1,4) \to 1$, $(3,4) \to 2$
Here we have completed the 12 mappings fixed by the above diagram. Now we must assign values for the remaining four elements of $A \times A$. It can be done by any choice of elements from T. Here we assign 1 to each of the four elements remaining in $A \times A$. $(4,4) \to 1$, $(4,2) \to 1$, $(4,1) \to 1$ and $(4,3) \to 1$. These 16 mappings together give us a binary operation on $A \times A$. Now let us consider $Cay(A,T)$.

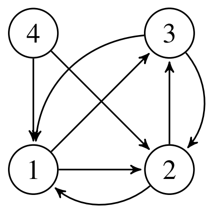

Figure 5. $Cay(A,T)$

Here the graph is 2-out-regular. We have constructed a binary operation on A such that A is groupoid where $Cay(A,T)$ is 2-out-regular.

Theorem 3.18. *The following is true for $k \leq m \leq n$.*

$$L_k^n(m) = \binom{n}{m}\left(S(m,k)\binom{n-1}{k}k!\right)^n n^{n^2-nm} \tag{3.5}$$

$$M_k^n \leq (n-1)^{nk}\left(\frac{\sqrt{8}\pi k}{n-2}\right)^{\frac{n}{2}} n^{n^2-nk}(n-k+1)e^{n\mu} \tag{3.6}$$

Where μ is a constant that depends on n.

Proof. Given a subset $\mathcal{T}$ of $\mathcal{G}$ with m elements, using Theorem 3.16, we will count the number of groupoids that can be formed, which admits a k-out-regular Cayley graph. Corresponding to each m-element subset, we find a groupoid that admits a k-out-regular Cayley graph. We have $\binom{n}{m}$ possibilities to select an m-element subset from $\mathcal{G}$. After selecting one such subset, we can assign $P_1, P_2, P_3, ..., P_n \in P_k^{\mathcal{T}}$ to each $x_1, x_2, x_3, ..., x_n \in G$. This can be done in $S(m,k)^n$ ways by the product rule. Since $|P_k^{\mathcal{T}}| = S(m,k)$. The following tabular diagram illustrates the above procedure. Here we must assign values to r_{ij} such that $r_{ij_1} \neq r_{ij_2}$ for $j_1 \neq j_2$. This means every

x_1	$\to$	r_{11}	r_{12}	r_{13}	...	$r_{1(k-1)}$	r_{1k}	$\to$	P_1
x_2	$\to$	r_{21}	r_{22}	r_{23}	...	$r_{2(k-1)}$	r_{2k}	$\to$	P_2
x_3	$\to$	r_{31}	r_{32}	r_{33}	...	$r_{3(k-1)}$	r_{3k}	$\to$	P_3
.	.	.	.	.	...	.	.	.	.
.	.	.	.	.	...	.	.	.	.
.	.	.	.	.	...	.	.	.	.
x_n	$\to$	r_{n1}	r_{n2}	r_{n3}	...	$r_{n(k-1)}$	r_{nk}	$\to$	P_n

Figure 6. Value assignment table.

row must contain different entries, but rows need not be distinct. To avoid loops, we assume that $r_{ij} \neq x_i$. So each row has $\binom{n-1}{k}k!$ possibilities. And there are $\left(\binom{n-1}{k}k!\right)^n$ to fill the entire two-dimensional array given in the Figure 6. This procedure assigns values to nm elements of $\mathcal{G} \times \mathcal{G}$. Now there are n^{n^2-nm} ways to fill the rest of the $n^2 - nm$ elements of $\mathcal{G} \times \mathcal{G}$. Then applying the product rule, we have

$$L_k^n(m) = \binom{n}{m}\left(S(m,k)\binom{n-1}{k}k!\right)^n n^{n^2-nm}. \tag{3.7}$$

Hence we proved 3.5.
Now by corollary 3.11

$$M_k^n \leq \sum_{k \leq m \leq n} L_k^n(m) \leq \sum_{k \leq m \leq n} \left(\binom{n}{m} \left(S(m,k) \binom{n-1}{k} k! \right)^n n^{n^2-nm} \right) \tag{3.8}$$

Here $m \geq k \Rightarrow nm \geq nk \Rightarrow n^{n^2-nm} \leq n^{n^2-nk}$. Then

$$M_k^n \leq \left(\binom{n-1}{k} k! \right)^n n^{n^2-nk} \sum_{k \leq m \leq n} \binom{n}{m} S(m,k)^n. \tag{3.9}$$

Now by Theorem 2.8, we have

$$M_k^n \leq \left(\binom{n-1}{k} k! \right)^n n^{n^2-nk} \binom{n}{\lfloor \frac{n}{2} \rfloor} \sum_{k \leq m \leq n} S(m,k)^n. \tag{3.10}$$

For large n and k, by Stirlings approximation $k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$.
By Theorem 2.9 $\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \left(\frac{ne}{\lfloor \frac{n}{2} \rfloor}\right)^{\lfloor \frac{n}{2} \rfloor}$. Also $\binom{n-1}{k} \leq \left(\frac{(n-1)e}{k}\right)^k$ and $\left\lfloor \frac{n}{2} \right\rfloor \geq \frac{n}{2} - 1$.
Then $\left(\frac{ne}{\lfloor \frac{n}{2} \rfloor}\right)^{\lfloor \frac{n}{2} \rfloor} \leq \left(\frac{2}{n-2}\right)^{\frac{n}{2}}$. By Theorem 2.11, we have a maximum of $S(m,k)$ for $k \leq m$.
Let $\mu_m = \max\{\frac{1}{m} log(S(n,k)\sqrt{2\pi}) : 1 \leq k \leq m\}$. Then

$$\begin{aligned} \frac{1}{m} log(S(m,k)\sqrt{2\pi}) \leq \mu_m &\Rightarrow log(S(m,k)\sqrt{2\pi}) \leq m\mu_m \\ &\Rightarrow S(m,k)\sqrt{2\pi} \leq e^{m\mu_m} \\ &\Rightarrow S(m,k) \leq \frac{e^{m\mu_m}}{\sqrt{2\pi}} \end{aligned} \tag{3.11}$$

Let $\mu = \max\{m\mu_m : k \leq m \leq n\}$. Then

$$\sum_{k \leq m \leq n} S(m,k)^n \leq \sum_{k \leq m \leq n} \left(\frac{e^{nm\mu_m}}{\sqrt{2\pi}^n} \right) \leq \frac{(n-k+1)e^{n\mu}}{(2\pi)^{\frac{n}{2}}}. \tag{3.12}$$

Combining these results, we have

$$M_k^n \leq (n-1)^{nk} \left(\frac{\sqrt{8}\pi k}{n-2} \right)^{\frac{n}{2}} n^{n^2-nk}(n-k+1)e^{n\mu}.$$

This justifies the truth of the inequality 3.6. □

4 Conclusion remarks

This paper investigated and obtained results regarding the weak upper bounds for the number of finite groupoids that admit regular Cayley graphs. These bounds play vital roles in networks associated with algebraic structures. Therefore, the results of this work are variant, and significant. Obtaining stronger bounds will be an interesting problem for further study.

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