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Compatible Leibniz Algebras

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Abstract *In this article, we study in the context of compatible Leibniz (***C-Lei***) algebras, some notions such as matched pairs, Manin triples and Leibniz bialgebras. In a particular case, we show that these notions are equivalent and present some examples of non-Lie C-Lei algebras.*

1 Introduction

The notion of Leibniz algebra [4, 5, 8] is a notion which generalises that of Lie algebra [5, 12], which is found in several mathematical fields, notably in differential geometry and field theory [2, 7]. The study of the compatibility of Leibniz algebras then applies to Lie algebras. A Leibniz algebra is a vector space g *together with a bilinear map* $[,]_g : g \oplus g \to g$ *such that*

$$\forall\, x, y, z \in g,\ \ [x, [y, z]_g]_g = [[x, y]_g, z]_g + [y, [x, z]_g]_g \ \ (\ Leibniz\ identity\ ). \tag{1.1}$$

Two Leibniz algebras $(g, [,]_{g1})$ *and* $(g, [,]_{g2})$ *are compatible, if and only if*

$$\forall k_1, k_2 \in \mathbb{K},\ \ (g, k_1[,]_{g1} + k_2[,]_{g2})$$

are also Leibniz algebras. In this case, we denote the compatibility of the two algebras $(g, [,]_{g1})$ *and* $(g, [,]_{g2})$ *by* $(g, [,]_{g1}, [,]_{g2})$*. Compatibility is equivalent to the following property:*

$$\begin{aligned}\forall x, y, z \in g,\ \ &[y, [x, z]_{g2}]_{g1} + [[x, y]_{g2}, z]_{g1} - [x, [y, z]_{g2}]_{g1} \\ &+ [y, [x, z]_{g1}]_{g2} + [[x, y]_{g1}, z]_{g2} - [x, [y, z]_{g1}]_{g2} = 0.\end{aligned} \tag{1.2}$$

*Note that if both algebras are Lie algebras, we find the definition of compatible Lie (***C-Lie***) algebra introduced by I.Z.Golubchik and V.V.Sokolov [6]. A compatible Leibniz sub-algebra of algebra* $(g, [,]_{g1}, [,]_{g2})$ *is a subspace of* g*, which is a Leibniz sub-algebra of the algebra* $(g, k_1[,]_{g1} + k_2[,]_{g2})$*, for any* $k_1, k_2 \in \mathbb{K}$*. This document is structured as follows: In part* 2*, we illustrate the notion of compatibility with an example. Part* 3 *is devoted to the notion representation of the C-Lei algebra, and some results follow this study. This section also looks at the matched pairs of C-Lei algebras. In part* 4*, we show that any Manin triple of C-Lei algebras is isomorphic to a standard Manin triple (***Theorem 3.6***) i.e. a triplet of C-Lei algebras* $(g \oplus g^*, g, g^*)$ *with* $g \oplus g^*$ *equipped of form* $<,>$*, defined by:*

$$< x + \xi, y + \eta >=< \xi, y > - < \eta, x > \quad x, y \in g, \quad \xi, \eta \in g^*. \tag{1.3}$$

g *and* g^* *are isotropic sub-algebras of* $g \oplus g^*$ *and* $<,>$ *is invariant for the structure of* $g \oplus g^*$*. Part* 5 *introduces Lie admissible C-Lei algebras, followed by a result on this subject. In part* 6*, we study compatible Leibniz bialgebras and show the equivalence between the notions mentioned above (***Theorem 5.4***). Finally, in the last part, we give examples of C-Lei algebras based on the classification of non-Lie Leibniz algebras given in [11].*

Example 1.1. Consider two bilinear operations on a vector space g over $\mathbb{C}$ of basis $\{x, y, z\}$, defined by:

$$[z,z]_1 = x, \quad [y,y]_1 = x,$$
$$[z,z]_2 = x, \quad [y,y]_2 = -x, \quad [y,z]_2 = x.$$

Let us show that $(g, [,]_1, [,]_2)$ is a C-Lei algebra. First, we have, for all $l \in g$, $[x,l]_1 = [x,l]_2 = [l,x]_1 = [l,x]_2 = 0$. Let $u, v, w \in g$. Let's put

$$u = u_1 x + u_2 y + u_3 z,$$
$$v = v_1 x + v_2 y + v_3 z,$$
$$w = w_1 x + w_2 y + w_3 z.$$

By straightforward computations, we have

$$[u,v]_1 = (u_2v_2 + u_3v_3)x := \alpha^1_{uv}x, \qquad [u,v]_2 = (-u_2v_2 + u_2v_3 + u_3v_3)x := \alpha^2_{uv}x,$$
$$[v,w]_1 = (v_2w_2 + v_3w_3)x := \alpha^1_{vw}x, \qquad [v,w]_2 = (-v_2w_2 + v_2w_3 + v_3w_3)x := \alpha^2_{vw}x,$$
$$[u,w]_1 = (-u_2w_2 + u_3w_3)x := \alpha^1_{uw}x, \quad [u,w]_2 = (-u_2w_2 + u_2w_3 + u_3w_3)x := \alpha^2_{uw}x.$$

Hence

$$[[u,v]_1,w]_1 + [v,[u,w]_1]_1 - [u,[v,w]_1]_1 = \alpha^1_{uv}[x,w]_1 + \alpha^1_{uw}[v,x]_1 - \alpha^1_{vw}[u,x]_1 = 0,$$
$$[[u,v]_2,w]_2 + [v,[u,w]_2]_2 - [u,[v,w]_2]_2 = \alpha^2_{uv}[x,w]_2 + \alpha^2_{uw}[v,x]_2 - \alpha^2_{vw}[u,x]_2 = 0.$$

Moreover, for all $k_1, k_2 \in \mathbb{C}$, the bracket $[u,v] = k_1[u,v]_1 + k_2[u,v]_2$ verifies (1.1). Indeed,

$$\begin{aligned}&[[u,v],w] + [v,[u,w]] - [u,[v,w]] = k_1^2([[u,v]_1,w]_1 + [v,[u,w]_1]_1 - [u,[v,w]_1]_1)\\ &+ k_2^2([[u,v]_2,w]_2 + [v,[u,w]_2]_2 - [u,[v,w]_2]_2) - k_1k_2([[u,v]_1,w]_2 + [v,[u,w]_1]_2\\ &- [u,[v,w]_1]_2 + [[u,v]_2,w]_1 + [v,[u,w]_2]_1 - [u,[v,w]_2]_1).\end{aligned}$$

Because (1.1), the components of k_1^2 and k_2^2 are zero. For the component of k_1k_2, we have

$$[[u,v]_1,w]_2 = \alpha^1_{uv}[x,w]_2 = 0, \quad [[u,v]_2,w]_1 = \alpha^2_{uv}[x,w]_1 = 0,$$
$$[u,[v,w]_1]_2 = \alpha^1_{vw}[u,x]_2 = 0, \quad [u,[v,w]_2]_1 = \alpha^2_{vw}[u,x]_1 = 0,$$
$$[v,[u,w]_1]_2 = \alpha^1_{uw}[v,x]_2 = 0, \quad [v,[u,w]_2]_1 = \alpha^2_{uw}[v,x]_1 = 0.$$

Example 1.2. Any C-Lie algebra is a C-Lei algebra.

2 Representation and Matched Pair of C-Lei algebras

Definition 2.1. [13] A representation of a Leibniz algebra $(g, [,]_g)$ is a triplet (V, ρ^L, ρ^R), or (ρ^L, ρ^R), where V is a vector space and $\rho^L, \rho^R : g \to gl(V)$ are linear maps, such that for all $x, y \in g$:

$$\rho^L([x,y]_g) = [\rho^L(x), \rho^L(y)]_{gl(V)}, \tag{2.1}$$
$$\rho^R([x,y]_g) = [\rho^L(x), \rho^R(y)]_{gl(V)}, \tag{2.2}$$
$$\rho^R(x) \circ \rho^L(y) = -\rho^R(x) \circ \rho^R(y). \tag{2.3}$$

Definition 2.2. A representation of C-Lei algebra, $(g, [,]_{g1}, [,]_{g2})$, on a vector space V is defined by four linear maps $\rho_1^L, \rho_1^R, \rho_2^L, \rho_2^R : g \to gl(V)$ such that for all k_1, $k_2 \in \mathbb{K}$, the following triple $(V, k_1\rho_1^L + k_2\rho_2^L, k_1\rho_1^R + k_2\rho_2^R)$ is a representation of Leibniz algebra $(g, k_1[,]_{g1} + k_2[,]_{g2})$. We denote it by $(V, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$.

Remark 2.3. If (V, ρ, μ) is a representation of C-Lie algebra, $(g, [,]_{g1}, [,]_{g2})$ [10], then the triple $(V, (\rho, -\rho), (\mu, -\mu))$ is a representation of g as a C-Lei algebra.

Proposition 2.4. *With the notation in the Definition 2.2, $(V, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ is a representation of $(g, [,]_{g1}, [,]_{g2})$, if and only if triplets (V, ρ_1^L, ρ_1^R) and (V, ρ_2^L, ρ_2^R) are representations of $(g, [,]_{g1})$ and $(g, [,]_{g2})$ respectively and for all $x, y \in g$*

$$\rho_1^L([x,y]_{g2}) + \rho_2^L([x,y]_{g1}) = [\rho_1^L(x), \rho_2^L(y)]_{gl(V)} + [\rho_2^L(x), \rho_1^L(y)]_{gl(V)}, \tag{2.4}$$

$$\rho_1^R([x,y]_{g2}) + \rho_2^R([x,y]_{g1}) = [\rho_1^L(x), \rho_2^R(y)]_{gl(V)} + [\rho_2^L(x), \rho_1^R(y)]_{gl(V)}, \tag{2.5}$$

$$\rho_1^R(y)\rho_2^L(x) + \rho_2^R(y)\rho_1^L(x) = -\rho_1^R(y)\rho_2^R(x) - \rho_2^R(y)\rho_1^R(x). \tag{2.6}$$

Proof. $(\Longrightarrow)$ (V, ρ_1^L, ρ_1^R) and (V, ρ_2^L, ρ_2^R) are representation of $(g, [,]_{g1})$ and $(g, [,]_{g2})$ respectively, they correspond to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$. Show (2.4). For $(k_1, k_2) = (1, 1)$, $(V, \rho_1^L + \rho_2^L, \rho_1^R + \rho_2^R)$ is a representation of $(g, [,]_{g1} + [,]_{g2})$. By (2.1), we have

$$\begin{aligned}
&\rho_1^L([x,y]_{g2}) + \rho_2^L([x,y]_{g1}) + \rho_1^L(x)\rho_1^L(y) - \rho_1^L(y)\rho_1^L(x) + \rho_2^L(x)\rho_2^L(y) - \rho_2^L(y)\rho_2^L(x) \\
&\quad = \rho_1^L(x)\rho_2^L(y) - \rho_2^L(y)\rho_1^L(x) + \rho_2^L(x)\rho_1^L(y) - \rho_1^L(y)\rho_2^L(x) + \rho_1^L(x)\rho_1^L(y) \\
&\quad\quad + \rho_2^L(x)\rho_2^L(y) - \rho_1^L(y)\rho_1^L(x) - \rho_2^L(y)\rho_2^L(x)
\end{aligned}$$

After simplification, we have the result. By (2.2), we have

$$\begin{aligned}
&\rho_1^R([x,y]_{g1}) + \rho_1^R([x,y]_{g2}) + \rho_2^R([x,y]_{g1}) + \rho_2^R([x,y]_{g2}) = \rho_1^R([x,y]_{g2}) + \rho_2^R([x,y]_{g1}) \\
&+ [\rho_1^L(x), \rho_1^R(y)]_{gl(V)} + [\rho_2^L(x), \rho_2^R(y)]_{gl(V)} \\
&[\rho_1^L(x) + \rho_2^L(x), \rho_1^R(y) + \rho_2^R(y)]_{gl(V)} = \rho_1^L(x)\rho_1^R(y) - \rho_1^R(y)\rho_1^L(x) + \rho_1^L(x)\rho_2^R(y) \\
&- \rho_2^R(y)\rho_1^L(x) + \rho_2^L(x)\rho_1^R(y) - \rho_1^R(y)\rho_2^L(x) + \rho_2^L(x)\rho_2^R(y) - \rho_2^R(y)\rho_2^L(x).
\end{aligned}$$

The right-hand sides of the above two equalities are equal. After simplification, we get condition (2.5). Let us now show (2.6). By (2.3),

$$(\rho_1^R(y) + \rho_2^R(y))(\rho_1^L(x) + \rho_2^L(x)) = -(\rho_1^R(y) + \rho_2^R(y))(\rho_1^R(x) + \rho_2^R(x)).$$

Therefore, we have

$$\rho_1^R(y)\rho_2^L(x) + \rho_2^R(y)\rho_1^L(x) = -\rho_1^R(y)\rho_2^R(x) - \rho_2^R(y)\rho_1^R(x).$$

$(\Longleftarrow)$ A simple calculation shows that $(V, k_1\rho_1^L + k_2\rho_2^L, k_1\rho_1^R + k_2\rho_2^R)$ satisfies conditions (2.1), (2.2) and (2.3). □

Let $(g, [,]_{gi})$ a algebra. We define the following maps: L^i, $R^i : g \to gl(g)$ and L^{i*}, $R^{i*} : g \to gl(g^*)$ by posing $L^i(x)y = [x,y]_{gi}$, $R^i(x)y = [y,x]_{gi}$, $< L^{i*}(x)\xi, y >= - < \xi, [x,y]_{gi} >$, and $< R^{i*}(x)\xi, y >= - < \xi, [y,x]_{gi} >$, for all $x, y \in g$ $\xi \in g^*$. If there is a algebra structure $[,]_{g^{*i}}$ on the dual space g^*, multiplications will be noted $\mathcal{L}^i$ and $\mathcal{R}^i$.

Corollary 2.5. *Let $(g, [,]_{g1}, [,]_{g2})$ be a C-Lei algebra. Then the triplet $(g, (L^1, R^1), (L^2, R^2))$ is a representation of $(g, [,]_{g1}, [,]_{g2})$. It is called the regular representation.*

Proof. We know that (g, L^1, R^1) is a representation of $(g, [,]_{g1})$ and that (g, L^2, R^2) is a representation of $(g, [,]_{g2})$. Let us now show that conditions (2.4), (2.5) and (2.6) are verified. According to the compatibility of g, we have for (2.4), (2.5) and (2.6) respectively

- $L^1([x,y]_{g2})(z) + L^2([x,y]_{g1})(z)$
  $= [x, [y,z]_{g1}]_{g2} - [y, [x,z]_{g1}]_{g2} + [x, [y,z]_{g2}]_{g1} - [y, [x,z]_{g2}]_{g1},$
  $= L^2(x)L^1(y)(z) - L^1(y)L^2(x)(z) + L^1(x)L^2(y)(z) - L^2(y)L^1(x)(z)$
  $= [L^1(x), L^2(y)]_{gl(g)}(z) + [L^2(x), L^1(y)]_{gl(g)}(z)$

- $R^1([x,y]_{g2})(z) + R^2([x,y]_{g1})(z)$
  $= [x,[z,y]_{g1}]_{g2} - [[x,z]_{g1},y]_{g2} + [x,[z,y]_{g2}]_{g1} - [[x,z]_{g2},y]_{g1}$
  $= L^2(x)R^1(y)(z) - R^2(y)L^1(x)(z) + L^1(x)R^2(y)(z) - R^1(y)L^2(x)(z)$
  $= [L^2(x),R^1(y)]_{gl(g)}(z) + [L^1(x),R^2(y)]_{gl(g)}(z).$
- $R^1(y)L^2(x)(z) + R^2(y)L^1(x)(z) + R^1(y)R^2(x)(z) + R^2(y)R^1(x)(z)$
  $= [[x,z]_{g2},y]_{g1} + [[x,z]_{g1},y]_{g2} + [[z,x]_{g2},y]_{g1} + [[z,x]_{g1},y]_{g2}$
  $= [x,[z,y]_{g1}]_{g2} - [z,[x,y]_{g1}]_{g2} + [x,[z,y]_{g2}]_{g1} - [z,[x,y]_{g2}]_{g1}$
  $+ [[z,x]_{g2},y]_{g1} + [[z,x]_{g1},y]_{g2} = 0.$

The result is in. □

Proposition 2.6. *Let* $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ *be a representation of* $(g,[,]_{g1},[,]_{g2})$. *Then the triple* $(V^*,(\rho_1^{L*},-\rho_1^{L*}-\rho_1^{R*}),(\rho_2^{L*},-\rho_2^{L*}-\rho_2^{R*}))$ *is a representation of* $(g,[,]_{g1},[,]_{g2})$. *It is called the dual representation of* $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$.

Proof. $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a representation of $(g,[,]_{g1},[,]_{g2})$, by Lemma 2.10 [13], we have $(V^*,\rho_1^{L*},-\rho_1^{L*}-\rho_1^{R*})$ and $(V^*,\rho_2^{L*},-\rho_2^{L*}-\rho_2^{R*})$ are representations of $(g,[,]_{g1})$ and $(g,[,]_{g2})$ respectively. Let's check the condition (2.4). By compatibility of g, we have

- $< (\rho_1^{L*}([x,y]_{g2}) + \rho_2^{L*}([x,y]_{g1}))\xi, z >$
  $= -< \xi, (\rho_1^L([x,y]_{g2})z + (\rho_2^L([x,y]_{g1})z >$
  $= -< \xi, [[x,y]_{g2},z]_{g1} + [[x,y]_{g1},z]_{g2} >$
  $= -< [x,[y,z]_{g1}]_{g2} - [y,[x,z]_{g1}]_{g2} + [x,[y,z]_{g2}]_{g1} - [y,[x,z]_{g2}]_{g1} >$
  $= -< \xi, (\rho_2^L(x)\rho_1^L(y) - \rho_2^L(y)\rho_1^L(x) + \rho_1^L(x)\rho_2^L(y) - \rho_1^L(y)\rho_2^L(x))(z) >$
  $= -< (\rho_1^{L*}(y)\rho_2^{L*}(x) - \rho_1^{L*}(x)\rho_2^{L*}(y) + \rho_2^{L*}(y)\rho_1^{L*}(x) - \rho_2^{L*}(x)\rho_1^{L*}(y))\xi, z >$
  $= \quad < ([\rho_1^{L*}(x),\rho_2^{L*}(y)]_{gl(V^*)} + [\rho_2^{L*}(x),\rho_1^{L*}(y)]_{gl(V^*)})\xi, z > .$

Let's show condition (2.5). By (2.4),

- $<((-\rho_1^{L*}-\rho_1^{R*})([x,y]_{g2}) + (-\rho_2^{L*}-\rho_2^{R*})([x,y]_{g1}))\xi, z >$
  $= < \xi, (\rho_1^L([x,y]_{g2}) + \rho_1^R([x,y]_{g2}) + \rho_2^L([x,y]_{g1}) + \rho_2^R([x,y]_{g1}))(z) >$
  $= < \xi, (\rho_1^L(x)\rho_2^L(y) - \rho_2^L(y)\rho_1^L(x) + \rho_2^L(x)\rho_1^L(y) - \rho_1^L(y)\rho_2^L(x))(z) >$
  $+ < \xi, (\rho_1^L(x)\rho_2^R(y) - \rho_2^R(y)\rho_1^L(x) + \rho_2^L(x)\rho_1^R(y) - \rho_1^R(y)\rho_2^L(x))(z) > .$
  $= < (\rho_1^{L*}(y)\rho_2^{L*}(x) - \rho_1^{L*}(x)\rho_2^{L*}(y) + \rho_2^{L*}(y)\rho_1^{L*}(x) - \rho_2^{L*}(x)\rho_1^{L*}(y))\xi, z >$
  $+ < (\rho_2^{R*}(y)\rho_1^{L*}(x) - \rho_1^{L*}(x)\rho_2^{R*}(y) + \rho_1^{R*}(y)\rho_2^{L*}(x) - \rho_2^{L*}(x)\rho_1^{R*}(y))\xi, z > .$

On the other hand

$$< (\rho_1^{L*}(x)(-\rho_2^{L*}(y) - \rho_2^{R*}(y)) - (-\rho_2^{L*}(y) - \rho_2^{R*}(y))\rho_1^{L*}(x)$$
$$+ \rho_2^{L*}(x)(-\rho_1^{L*}(y) - \rho_1^{R*}(y)) - (-\rho_1^{L*}(y) - \rho_1^{R*}(y))\rho_2^{L*}(x))\xi, z >$$
$$= < (-\rho_1^{L*}(x)\rho_2^{L*}(y) - \rho_1^{L*}(x)\rho_2^{R*}(y) + \rho_2^{L*}(y)\rho_1^{L*}(x) + \rho_2^{R*}(y)\rho_1^{L*}(x)$$
$$- \rho_2^{L*}(x)\rho_1^{L*}(y) - \rho_2^{L*}(x)\rho_1^{R*}(y) + \rho_1^{L*}(y)\rho_2^{L*}(x) + \rho_1^{R*}(y)\rho_2^{L*}(x))\xi, z > .$$

Hence the equality. Let us now show condition (2.6). Note that $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a representation, therefore

$$\rho_2^R(x)\rho_1^L(y) + \rho_2^R(x)\rho_1^R(y) + \rho_1^R(x)\rho_2^L(y) + \rho_1^R(x)\rho_2^R(y) = 0,$$

thus, we have

$$\rho_1^{L*}(y)\rho_2^{R*}(x) + \rho_1^{R*}(y)\rho_2^{R*}(x) + \rho_2^{L*}(y)\rho_1^{R*}(x) + \rho_2^{R*}(y)\rho_1^{R*}(x) = 0.$$

We can deduce that

$$\begin{aligned}
&(-\rho_2^{L*}(y)-\rho_2^{R*}(y))\rho_1^{L*}(x)+(-\rho_1^{L*}(y)-\rho_1^{R*}(y))\rho_2^{L*}(x)\\
&\quad+(\rho_1^{L*}(y)+\rho_1^{R*}(y))(\rho_2^{L*}(x)+\rho_2^{R*}(x))+(\rho_2^{L*}(y)+\rho_2^{R*}(y))(\rho_1^{L*}(x)+\rho_1^{R*}(x))\\
&=-\rho_2^{L*}(y)\rho_1^{L*}(x)-\rho_2^{R*}(y)\rho_1^{L*}(x)-\rho_1^{L*}(y)\rho_2^{L*}(x)-\rho_1^{R*}(y)\rho_2^{L*}(x)\\
&\quad+\rho_1^{L*}(y)\rho_2^{L*}(x)+\rho_1^{L*}(y)\rho_2^{R*}(x)+\rho_1^{R*}(y)\rho_2^{L*}(x)+\rho_1^{R*}(y)\rho_2^{R*}(x)\\
&\quad+\rho_2^{L*}(y)\rho_1^{L*}(x)+\rho_2^{L*}(y)\rho_1^{R*}(x)+\rho_2^{R*}(y)\rho_1^{L*}(x)+\rho_2^{R*}(y)\rho_1^{R*}(x)=0.
\end{aligned}$$

Hence the result. □

Corollary 2.7. *The triple* $(g^*,(L^{1*},-L^{1*}-R^{1*}),(L^{2*},-L^{2*}-R^{2*}))$ *is a representation of any C-Lei algebra* $(g,[,]_{g1},[,]_{g2})$.

Proposition 2.8. $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ *is a representation of* $(g,[,]_{g1},[,]_{g2})$ *if and only if the triple* $(g\oplus V,[,]_1,[,]_2)$ *is a C-Lei algebra, where* $[,]_1,[,]_2:(g\oplus V)\times(g\oplus V)\to g\oplus V$ *are defined by*

$$[x+u,y+v]_1=[x,y]_{g1}+\rho_1^L(x)v+\rho_1^R(y)u,$$
$$[x+u,y+v]_2=[x,y]_{g2}+\rho_2^L(x)v+\rho_2^R(y)u,$$

for any $x,y\in g$ *and* $u,v\in V$.

Let us first show the following lemma.

Lemma 2.9. *Let* $(g,[,]_{g1},[,]_{g2})$ *be a C-Lei algebra. Then, for all* $x,y,z\in g$

$$[[x,y]_{g1},z]_{g2}+[[x,y]_{g2},z]_{g1}=-[[y,x]_{g1},z]_{g2}-[[y,x]_{g2},z)]_{g1}.$$

Proof.

$$[[x,y]_{g2},z]_{g1}+[[x,y]_{g1},z]_{g2}=[x,[y,z]_{g2}]_{g1}-[y,[x,z]_{g2}]_{g1}+[x,[y,z]_{g1}]_{g2}-[y,[x,z]_{g1}]_{g2}$$
$$[[y,x]_{g2},z]_{g1}+[[y,x]_{g1},z]_{g2}=[y,[x,z]_{g2}]_{g1}-[x,[y,z]_{g2}]_{g1}+[y,[x,z]_{g1}]_{g2}-[x,[y,z]_{g1}]_{g2}.$$

Lemma is proven. □

Let us now prove the above proposition.

Proof. Suppose $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a representation of g. Let us show that $[,]_1$ and $[,]_2$ are Leibniz brackets. Since (V,ρ_1^L,ρ_1^R) is a representation of $(g,[,]_{g1})$, for all $x,y\in g$, $u\in V$, we have by (2.1)

$$[[x,y]_1,u]_1+[y,[x,u]_1]_1-[x,[y,u]_1]_1=\rho_1^L([x,y]_{g1})u+\rho_1^L(y)(\rho_1^L(x))u)-\rho_1^L(x)(\rho_1^L(y)u)=0.$$

By (2.2), we have

$$[[x,u]_1,y]_1+[u,[x,y]_1]_1-[x,[u,y]_1]_1=\rho_1^R(y)(\rho_1^L(x))u)+\rho_1^R([x,y]_{g1})u-\rho_1^L(x)(\rho_1^R(y)u)=0.$$

By (2.2) and (2.3), we have

$$[[u,x]_1,y]_1+[x,[u,y]_1]_1-[u,[x,y]_1]_1=\rho_1^R(y)(\rho_1^R(x))u)+\rho_1^L(x)(\rho_1^R(y)u)-\rho_1^R([x,y]_{g1})u=0.$$

Moreover, by definition of bracket $[,]_1$, we have

$$[[x,u]_1,v]_1+[u,[x,v]_1]_1-[x,[u,v]_1]_1=0,$$
$$[[u,x]_1,v]_1+[x,[u,v]_1]_1-[u,[x,v]_1]_1=0,$$
$$[[u,v]_1,x]_1+[v,[u,x]_1]_1-[u,[v,x]_1]_1=0.$$

We use the same method to study $[,]_2$. Let's study the compatibility. By condition (2.5), we have

- $[[x,u]_1,y]_2+[u,[x,y]_1]_2-[x,[u,y]_1]_2+[[x,u]_2,y]_1+[u,[x,y]_2]_1-[x,[u,y]_2]_1$
$=\rho_2^R(y)(\rho_1^L(x)u)+\rho_2^R([x,y]_{g1})u-\rho_2^L(x)(\rho_1^R(y)u)+\rho_1^R(y)(\rho_2^L(x)u)+\rho_1^R([x,y]_{g2})u$
$-\rho_1^L(x)(\rho_2^L(y)u)=0.$

By the condition (2.4), we have

- $[[x,y]_1,u]_2+[y,[x,u]_1]_2-[x,[y,u]_1]_2+[[x,y]_2,u]_1+[y,[x,u]_2]_1-[x,[y,u]_2]_1$
$=\rho_2^L([x,y]_{g1})u+\rho_2^L(y)(\rho_1^L(x)u)-\rho_2^L(x)(\rho_1^L(y)u)+\rho_1^L([x,y]_{g2})u+\rho_1^L(y)(\rho_2^L(x)u)$
$-\rho_1^L(x)(\rho_2^L(y)u)=0.$

By the condition (2.5) and (2.6), we have

- $[[u,x]_1,y]_1+[x,[u,y]_1]_1-[u,[x,y]_1]_1+[[u,x]_2,y]_1+[x,[u,y]_2]_1-[u,[x,y]_2]_1$
$=\rho_2^R(y)(\rho_1^R(x)u)+\rho_2^L(x)(\rho_1^R(x)u)-\rho_2^R([x,y]_{g1})u+\rho_1^R(y)(\rho_2^R(x)u)+\rho_1^L(x)(\rho_2^R(x)u)$
$-\rho_1^R([x,y]_{g2})u=0.$

The same procedure applies for $[,]_2$. Conversely, we suppose $[,]_1$ and $[,]_2$ define a compatible Leibniz structure on $g\oplus V$, let us show that $(V,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a representation of $(g,[,]_{g1},[,]_{g2})$. We have

$0=[[u,x]_1,y]_1+[x,[u,y]_1]_1-[u,[x,y]_1]_1=\rho_1^R(y)(\rho_1^R(x)u)+\rho_1^L(x)(\rho_1^R(y)u)-\rho_1^R([x,y]_{g1})u.$

Therefore, $\rho_1^R([x,y]_{g1})=\rho_1^R(y)\rho_1^R(x)+\rho_1^L(x)\rho_1^R(y)$. In the same way

$0=[[x,y]_{g1},u]_1+[y,[x,u]_1]_1-[x,[y,u]_1]_1=\rho_1^L([x,y]_{g1})u+\rho_1^L(y)(\rho_1^L(x)u)-\rho_1^L(x)(\rho_1^L(y)u).$

Therefore, $\rho_1^L([x,y]_{g1})=\rho_1^L(x)\rho_1^L(y)-\rho_1^L(y)\rho_1^L(x)$. Moreover, we have

$$\rho_1^R(y)\rho_1^L(x)u+\rho_1^R(y)\rho_1^R(x)u=[[x,u]_1,y]_1+[[u,x]_1,y]_1=0.$$

We can deduce, (V,ρ_1^L,ρ_1^R) is a representation of $(g,[,]_{g1})$. We do the same for (V,ρ_2^L,ρ_2^R). By the compatibility of g, we have

$$\begin{aligned}0=&[[x,y]_1,u]_2+[y,[x,u]_1]_2-[x,[y,u]_1]_2+[[x,y]_2,u]_1+[y,[x,u]_2]_1-[x,[y,u]_2]_1\\=&\rho_2^L([x,y]_{g1})u+\rho_1^L([x,y]_{g2})u+\rho_2^L(y)\rho_1^L(x)u-\rho_2^L(x)\rho_1^L(y)u+\rho_1^L(y)\rho_2^L(x)u-\rho_1^L(x)\rho_2^L(y)u.\end{aligned}$$

Thus, we obtain

$$\rho_2^L([x,y]_{g1})+\rho_1^L([x,y]_{g2})=\rho_2^L(x)\rho_1^L(y)-\rho_1^L(y)\rho_2^L(x)+\rho_1^L(x)\rho_2^L(y)-\rho_2^L(y)\rho_1^L(x).$$

By compatibility of g and Lemma 2.9, we have

- $$\begin{aligned}\rho_1^R([x,y]_{g2})u+\rho_2^R([x,y]_{g1})u&=[u,[x,y]_{g2}]_{g1}+[[x,y]_{g1},u]_{g2}\\&=[[u,x]_2,y]_1+[x,[u,y]_2]_1+[[u,x]_1,y]_2+[x,[u,y]_1]_2\\&=[x,[u,y]_2]_1+[x,[u,y]_1]_2-[[x,u]_2,y]_1-[[x,u]_1,y]_2\\&=\rho_1^L(x)\rho_2^R(y)u+\rho_2^L(x)\rho_1^R(y)u-\rho_1^R(y)\rho_2^L(x)u-\rho_2^R(y)\rho_1^L(x)u.\end{aligned}$$

Finally, applying Lemma 2.9, we have

- $$\begin{aligned}&\rho_1^R(y)\rho_2^L(x)u+\rho_2^R(y)\rho_1^L(x)u+\rho_1^R(y)\rho_2^R(x)u+\rho_2^R(y)\rho_1^R(x)u\\&\quad=[[x,u]_2,y]_1+[[x,u]_1,y]_2+[[u,x]_2,y]_1+[[u,x]_1,y]_2=0,\end{aligned}$$

which finishes the proof. □

Definition 2.10. ([1, 13]) Let $(g_1, [,]_{g_1})$ and $(g_2, [,]_{g_2})$ be two Leibniz algebras. If there exists a representation (ρ_1^L, ρ_1^R) of g_1 on g_2 and a representation (ρ_2^L, ρ_2^R) of g_2 on g_1 such that the identities

$$\rho_1^R(x)[u,v]_{g2} - [u, \rho_1^R(x)v]_{g2} + [v, \rho_1^R(x)u]_{g2} - \rho_1^R(\rho_2^L(v)x)u + \rho_1^R(\rho_2^L(u)x)v = 0, \tag{2.7}$$

$$\rho_1^L(x)[u,v]_{g2} - [\rho_1^L(x)u, v]_{g2} - [u, \rho_1^L(x)v]_{g2} - \rho_1^L(\rho_2^R(u)x)v - \rho_1^R(\rho_2^R(v)x)u = 0, \tag{2.8}$$

$$[\rho_1^L(x)u, v]_{g2} + \rho_1^L(\rho_2^R(u)x)v + [\rho_1^R(x)u, v]_{g2} + \rho_1^L(\rho_2^L(u)x)v = 0, \tag{2.9}$$

$$\rho_2^R(u)[x,y]_{g1} - [x, \rho_2^R(u)y]_{g1} + [y, \rho_2^R(u)x]_{g1} - \rho_2^R(\rho_1^L(y)u)x + \rho_2^R(\rho_1^L(x)u)y = 0, \tag{2.10}$$

$$\rho_2^L(u)[x,y]_{g1} - [\rho_2^L(u)x, y]_{g1} - [x, \rho_2^L(u)y]_{g1} - \rho_2^L(\rho_1^R(x)u)y - \rho_2^R(\rho_1^R(y)u)x = 0, \tag{2.11}$$

$$[\rho_2^L(u)x, y]_{g1} + \rho_2^L(\rho_1^R(x)u)y + [\rho_2^R(u)x, y]_{g1} + \rho_2^L(\rho_1^L(x)u)y = 0, \tag{2.12}$$

hold for all $x, y \in g_1$ and $u, v \in g_2$, then we call $(g_1, g_2; (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ a matched pair of Leibniz algebras g_1 and g_2.

Definition 2.11. Let $(g, [,]_{g1}, [,]_{g2})$ and $(h, [,]_{h1}, [,]_{h2})$ be two C-Lei algebras. Let $\rho_1^L,\ \rho_1^R,\ \rho_2^L,\ \rho_2^R :$ $g \to gl(h)$ and $\mu_1^L,\ \mu_1^R,\ \mu_2^L,\ \mu_2^R : h \to gl(g)$ be linear maps. We call that the following sextuple $(g, h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R), (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ is a matched pair of C-Lei algebras if and only if, for all $k_1, k_2 \in \mathbb{K}$

$$(g,\ h,\ (k_1\rho_1^L + k_2\rho_2^L,\ k_1\rho_1^R + k_2\rho_2^R),\ (k_1\mu_1^L + k_2\mu_2^L,\ k_1\mu_1^R + k_2\mu_2^R))$$

is a matched pair of $(g,\ k_1[,]_{g1} + k_2[,]_{g2})$ and $(h,\ k_1[,]_{h1} + k_2[,]_{h2})$.

Proposition 2.12. *Under the hypotheses of the above definition, the conditions (i) and (ii) are equivalent:*

(i) $(g, h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R), (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ *is a matched pair of algebras* $(g, [,]_{g1}, [,]_{g2})$ *and* $(h, [,]_{h1}, [,]_{h2})$.

(ii) The equations below are valid:

a) $(h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ *and* $(g, (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ *are representations of* $(g, [,]_{g1}, [,]_{g2})$ *and* $(h, [,]_{h1}, [,]_{h2})$ *respectively.*

b) $((g, [,]_{g1}), (h, [,]_{h1}), (\rho_1^L, \rho_1^R), (\mu_1^L, \mu_1^R))$ *and* $((g, [,]_{g2}), (h, [,]_{h2}), (\rho_2^L, \rho_2^R), (\mu_2^L, \mu_2^R))$ *are matched pairs of Leibniz algebras.*

c) For any $x, y \in g$ and $u, v \in h$, The following equations hold:

1) $$\rho_1^R(x)([u,v]_{h2}) - [u, \rho_1^R(x)v]_{h2} + [v, \rho_1^R(x)u]_{h2} - \rho_1^R(\mu_2^L(v)x)u + \rho_1^R(\mu_2^L(u)x)v$$
$$+ \rho_2^R(x)([u,v]_{h1}) - [u, \rho_2^R(x)v]_{h1} + [v, \rho_2^R(x)u]_{h1} - \rho_2^R(\mu_1^L(v)x)u + \rho_2^R(\mu_1^L(u)x)v = 0.$$

2) $$\rho_1^L(x)([u,v]_{h2}) - [\rho_1^L(x)u, v]_{h2} - [u, \rho_1^L(x)v]_{h2} - \rho_1^L(\mu_2^R(u)x)v - \rho_1^R(\mu_2^R(v)x)u$$
$$+ \rho_2^L(x)([u,v]_{h1}) - [\rho_2^L(x)u, v]_{h1} - [u, \rho_2^L(x)v]_{h1} - \rho_2^L(\mu_1^R(u)x)v - \rho_2^R(\mu_1^R(v)x)u = 0.$$

3) $$[\rho_1^L(x)u, v]_{h2} + \rho_1^L(\mu_2^R(u)x)v + [\rho_1^R(x)u, v]_{h2} + \rho_1^L(\mu_2^L(u)x)v + [\rho_2^L(x)u, v]_{h1}$$
$$+ \rho_2^L(\mu_1^R(u)x)v + [\rho_2^R(x)u, v]_{h1} + \rho_2^L(\mu_1^L(u)x)v = 0.$$

4) $$\mu_1^R(u)([x,y]_{g2}) - [x, \mu_1^R(u)y]_{g2} + [y, \mu_1^R(u)x]_{g2} - \mu_1^R(\rho_2^L(y)u)x + \mu_1^R(\rho_2^L(x)u)y$$
$$+ \mu_2^R(u)([x,y]_{g1}) - [x, \mu_2^R(u)y]_{g1} + [y, \mu_2^R(u)x]_{g1} - \mu_2^R(\rho_1^L(y)u)x + \mu_2^R(\rho_1^L(x)u)y = 0.$$

5) $$\mu_1^L(u)([x,y]_{g2}) - [\mu_1^L(u)x, y]_{g2} - [x, \mu_1^L(u)y]_{g2} - \mu_1^L(\rho_2^R(x)u)y - \mu_1^R(\rho_2^R(y)u)x$$
$$+ \mu_2^L(u)([x,y]_{g1}) - [\mu_2^L(u)x, y]_{g1} - [x, \mu_2^L(u)y]_{g1} - \mu_2^L(\rho_1^R(x)u)y - \mu_2^R(\rho_1^R(y)u)x = 0.$$

6) $$[\mu_1^L(u)x, y]_{g2} + \mu_1^L(\rho_2^R(x)u)y + [\mu_1^R(u)x, y]_{g2} + \mu_1^L(\rho_2^L(x)u)y + [\mu_2^L(u)x, y]_{g1}$$
$$+ \mu_2^L(\rho_1^R(x)u)y + [\mu_2^R(u)x, y]_{g1} + \mu_2^L(\rho_1^L(x)u)y = 0.$$

Proof. $(i) \Longrightarrow (ii)$ We have $(h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ and $(g, (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ are representations of C-Lei algebras $(g, [,]_{g1}, [,]_{g2})$ and $(h, [,]_{h1}, [,]_{h2})$ respectively, thus $a)$ is verified. The case $b)$ corresponds to $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$. For $c)$, we have for all $k_1, k_2 \in \mathbb{K}$ $(g, h, (k_1\rho_1^L + k_2\rho_2^L, k_1\rho_1^R + k_2\rho_2^R), (k_1\mu_1^L + k_2\mu_2^L, k_1\mu_1^R + k_2\mu_2^R))$ is a matched pair of Leibniz algebras $(g, k_1[,]_{g1} + k_2[,]_{g2})$ and $(h, k_1[,]_{h1} + k_2[,]_{h2})$. Check the six identities of (ii) $c)$ 1) to (ii) $c)$ 6). For (ii) $c)$ 1). By (2.7), for all $x \in g$, $u, v \in h$, we have

$$\begin{aligned}
\bullet\ &(k_1\rho_1^R + k_2\rho_2^R)(x)(k_1[u,v]_{h1} + k_2[u,v]_{h2}) - k_1[u, (k_1\rho_1^R + k_2\rho_2^R)(x)v]_{h1}\\
&- k_2[u, (k_1\rho_1^R + k_2\rho_2^R)(x)v]_{h2} + k_1[v, (k_1\rho_1^R + k_2\rho_2^R)(x)u]_{h1}\\
&+ k_2[v, (k_1\rho_1^R + k_2\rho_2^R)(x)u]_{h2} - (k_1\rho_1^R + k_2\rho_2^R)((k_1\mu_1^L + k_2\mu_2^L)(v))x)u\\
&+ (k_1\rho_1^R + k_2\rho_2^R)((k_1\mu_1^L + k_2\mu_2^L)(u))x)v\\
=\ & k_1^2(\rho_1^R(x)[u,v]_{h1} - [u, \rho_1^R(x)v]_{h1} + [v, \rho_1^R(x)u]_{h1} - \rho_1^R(\mu_1^L(v)x)u + \rho_1^R(\mu_1^L(u)x)v)\\
&+ k_1k_2(\rho_1^R(x)[u,v]_{h2} - [u, \rho_1^R(x)v]_{h2} + [v, \rho_1^R(x)u]_{h2} - \rho_1^R(\mu_2^L(v)x)u + \rho_1^R(\mu_2^L(u)x)v\\
&+ \rho_2^R(x)[u,v]_{h1} - [u, \rho_2^R(x)v]_{h1} + [v, \rho_2^R(x)u]_{h1} - \rho_2^R(\mu_1^L(v)x)u + \rho_2^R(\mu_1^L(u)x)v)\\
&+ k_2^2(\rho_2^R(x)[u,v]_{h2} - [u, \rho_2^R(x)v]_{h2} + [v, \rho_2^R(x)u]_{h2} - \rho_2^R(\mu_2^L(v)x)u + \rho_2^R(\mu_2^L(u)x)v) = 0.
\end{aligned}$$

For $(k_1, k_2) = (1, 0)$, $((g, [,]_{g1}), (h, [,]_{h1}), (\rho_1^L, \rho_1^R), (\mu_1^L, \mu_1^R))$ is a matched pair and for $(k_1, k_2) = (0, 1)$, $((g, [,]_{g2}), (h, [,]_{h2}), (\rho_2^L, \rho_2^R), (\mu_2^L, \mu_2^R))$ is a matched pair, by (2.7) and (2.10), (ii) $c)$ 1) is verified. For (ii) $c)$ 2), by (2.8), we have

$$\begin{aligned}
\bullet\ &(k_1\rho_1^L + k_2\rho_2^L)(x)(k_1[u,v]_{h1} + k_2[u,v]_{h2}) - k_1[(k_1\rho_1^L + k_2\rho_2^L)(x)u, v]_{h1}\\
&- k_2[(k_1\rho_1^L + k_2\rho_2^L)(x)u, v]_{h2} - k_1[u, (k_1\rho_1^L + k_2\rho_2^L)(x)v]_{h1}\\
&- k_2[u, (k_1\rho_1^L + k_2\rho_2^L)(x)v]_{h2} - (k_1\rho_1^L + k_2\rho_2^L)((k_1\mu_1^R + k_2\mu_2^R)(u)x)v\\
&- (k_1\rho_1^R + k_2\rho_2^R)((k_1\mu_1^R + k_2\mu_2^R)(u)x)v\\
=\ & k_1^2(\rho_1^L(x)[u,v]_{h1} - [\rho_1^L(x)u, v]_{h1} - [u, \rho_1^L(x)v]_{h1} - \rho_1^L(\mu_1^R(u)x)v - \rho_1^R(\mu_1^R(v)x)u)\\
&- k_1k_2(\rho_1^L(x)([u,v]_{h2}) - [\rho_1^L(x)u, v]_{h2} - [u, \rho_1^L(x)v]_{h2} - \rho_1^L(\mu_2^R(u)x)v\\
&- \rho_1^R(\mu_2^R(v)x)u + \rho_2^L(x)([u,v]_{h1}) - [\rho_2^L(x)u, v]_{h1} - [u, \rho_2^L(x)v]_{h1} - \rho_2^L(\mu_1^R(u)x)v\\
&- \rho_2^R(\mu_1^R(v)x)u) + k_2^2(\rho_2^L(x)[u,v]_{h2} - [\rho_2^L(x)u, v]_{h2} - [u, \rho_2^L(x)v]_{h2}\\
&- \rho_2^L(\mu_2^R(u)x)v - \rho_2^R(\mu_2^R(v)x)u) = 0.
\end{aligned}$$

Using conditions (2.8) and (2.11), we obtain the desired result. For (ii) $c)$ 3), by (2.9), we have

$$\begin{aligned}
\bullet\ &k_1[(k_1\rho_1^L + k_2\rho_2^L)(x)u, v]_{h1} + k_2[(k_1\rho_1^L + k_2\rho_2^L)(x)u, v]_{h2}\\
&+ (k_1\rho_1^L + k_2\rho_2^L)((k_1\mu_1^R + k_2\mu_2^R)(u)x)v + k_1[(k_1\rho_1^R + k_2\rho_2^R)(x)u, v]_{h1}\\
&+ k_2[(k_1\rho_1^R + k_2\rho_2^R)(x)u, v]_{h2} + (k_1\rho_1^L + k_2\rho_2^L)((k_1\mu_1^L + k_2\mu_2^L)(u)x))v\\
=\ & k_1^2([\rho_1^L(x)u, v]_{h1} + \rho_1^L(\mu_1^R(u)x)v + [\rho_1^R(x)u, v]_{h1} + \rho_1^L(\mu_1^L(u)x)v)\\
&+ k_1k_2([\rho_1^L(x)u, v]_{h2} + \rho_1^L(\mu_2^R(u)x)v + [\rho_1^R(x)u, v]_{h2} + \rho_1^L(\mu_2^L(u)x)v\\
&+ [\rho_2^L(x)u, v]_{h1} + \rho_2^L(\mu_1^R(u)x)v + [\rho_2^R(x)u, v]_{h1} + \rho_2^L(\mu_1^L(u)x)v\\
&+ k_2^2([\rho_2^L(x)u, v]_{h2} + \rho_2^L(\mu_2^R(u)x)v + [\rho_2^R(x)u, v]_{h2} + \rho_2^L(\mu_2^L(u)x)v) = 0.
\end{aligned}$$

We have the result by (2.9) and (2.12). For (ii) $c)$ 4), 5), 6), we act in the same way. Conversely, $(ii) \Longrightarrow (i)$ is obvious from the calculation we have just developed. □

Corollary 2.13. *Under the hypotheses of Definition 2.11, we assume that*
1) $(h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ *and* $(g, (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ *are representations of* $(g, [,]_{g1}, [,]_{g2})$ *and* $(h, [,]_{h1}, [,]_{h2})$ *respectively.*
2) *For all* $i, j = 1, 2$, $((g, [,]_{gi}), (h, [,]_{hj}), (\rho_i^L, \rho_i^R), (\mu_j^L, \mu_j^R))$ *is a matched pair of Leibniz algebras. Then* $(g, h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R), (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ *is a matched pair of C-Lei algebras* $(g, [,]_{g1}, [,]_{g2})$ *and* $(h, [,]_{h1}, [,]_{h2})$.

Proof. Let us show condition $c)$ of Proposition 2.12. By (2.7), since we have by hypothesis $((g,[,]_{g1}),(h,[,]_{h2}),(\rho_1^L,\rho_1^R),(\mu_2^L,\mu_2^R))$ is a matched pair of Leibniz algebras, $((g,[,]_{g1}),(h,[,]_{h2}),(\rho_2^L,\rho_2^R),(\mu_1^L,\mu_1^R))$ is also a matched pair of Leibniz algebras, thus

$$\rho_1^R(x)([u,v]_{h2})-[u,\rho_1^R(x)v]_{h2}+[v,\rho_1^R(x)]_{h2}-\rho_1^R(\mu_2^L(v)x)u+\rho_1^R(\mu_2^L(u)x)v=0,$$
$$\rho_2^R(x)([u,v]_{h1})-[u,\rho_2^R(x)v]_{h1}+[v,\rho_2^R(x)u]_{h1}-\rho_2^R(\mu_1^L(v)x)u+\rho_2^R(\mu_1^L(u)x)v=0.$$

We obtain $c)$ 1) by adding the two equations term by term. In the same way, we show the remaining identities. □

Proposition 2.14. *Let $(g,(\mu_1^L,\mu_1^R),(\mu_2^L,\mu_2^R))$ and $(h,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ be two representations of g and h respectively. Then $(g,h,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R),(\mu_1^L,\mu_1^R),(\mu_2^L,\mu_2^R))$ is a matched pair if and only if*

$$[x+u,y+v]_1=[x,y]_{g1}+\mu_1^R(v)x+\mu_1^L(u)y+[u,v]_{h1}+\rho_1^L(x)v+\rho_1^R(y)u,$$
$$[x+u,y+v]_2=[x,y]_{g2}+\mu_2^R(v)x+\mu_2^L(u)y+[u,v]_{h2}+\rho_2^L(x)v+\rho_2^R(y)u,$$

define a C-Lei algebra structure on $g\oplus h$.

Proof. ($\Longrightarrow$) We have (h,ρ_1^L,ρ_1^R) and (g,μ_1^L,μ_1^R) are representations of $(g,[,]_{g1})$ and $(h,[,]_{h1})$ respectively and $(g,h,(\rho_1^L,\rho_1^R),(\mu_1^L,\mu_1^R))$ is a matched pair of $(g,[,]_{g1})$ and $(h,[,]_{h1})$. We show that $[,]_1$ is a Leibniz bracket.

- $[[v,x]_1,y]_1+[x,[v,y]_1]_1-[v,[x,y]_1]_1=-\mu_1^L(v)([x,y]_{g1})+[\mu_1^L(v)x,y]_{g1}+[x,\mu_1^L(v)y]_{g1}$
$$+\mu_1^L(\rho_1^R(x)v)y+\mu_1^R(\rho_1^R(y)v)x-\rho_1^R([x,y]_{g1})v+\rho_1^L(x)(\rho_1^R(y)v)+\rho_1^R(y)(\rho_1^R(x)v).$$

By (2.2) and (2.3), we have $-\rho_1^R([x,y]_{g1})v+\rho_1^L(x)(\rho_1^R(y)v)+\rho_1^R(y)(\rho_1^R(x)v)=0$. By (2.11),

$$-\mu_1^L(v)([x,y]_{g1})+[\mu_1^L(v)x,y]_{g1}+[x,\mu_1^L(v)y]_{g1}+\mu_1^L(\rho_1^R(x)v)y+\mu_1^R(\rho_1^R(y)v)x=0,$$

hence $[[v,x]_1,y]_1+[x,[v,y]_1]_1-[v,[x,y]_1]_1=0$.

- $[[x,v]_1,y]_1+[v,[x,y]_1]_1-[x,[v,y]_1]_1=\mu_1^L(\rho_1^L(x)v)y+\rho_1^R(y)(\rho_1^L(x)v)+[\mu_1^R(v)x,y]_1$
$$+\mu_1^L(v)([x,y]_{g1})+\rho_1^R([x,y]_{g1})v-[x,\mu_1^L(v)y]_{g1}-\rho_1^L(x)(\rho_1^R(y))v-\mu_1^R(\rho_1^R(y)v)x.$$

By (2.12), $[\mu_1^R(v)x,y]_{g1}+\mu_1^L(\rho_1^L(x)v)y=-[\mu_1^L(v)x,y]_{g1}-\mu_1^L(\rho_1^R(x)v)y$, and by (2.2),

$$\rho_1^R([x,y]_{g1})v+\rho_1^R(y)(\rho_1^L(x)v)-\rho_1^L(x)(\rho_1^R(y)v)=0,$$

applying condition (2.11), then $[[x,v]_1,y]_1+[v,[x,y]_1]_1-[x,[v,y]_1]_1=0$.

- $[[x,y]_1,v]_1+[y,[x,v]_1]_1-[x,[y,v]_1]_1=\mu_1^R(v)([x,y]_{g1})-[x,\mu_1^R(v)y]_{g1}+[y,\mu_1^R(v)x]_{g1}$
$$-\mu_1^R(\rho_1^L(y)v)x+\mu_1^R(\rho_1^L(x)v)y+\rho_1^L([x,y]_{g1})v+\rho_1^L(y)(\rho_1^L(x)v)-\rho_1^L(x)(\rho_1^L(y)v).$$

By (2.10), we have

$$\mu_1^R(v)([x,y]_{g1})-[x,\mu_1^R(v)y]_{g1}+[y,\mu_1^R(v)x]_{g1}-\mu_1^R(\rho_1^L(y)v)x+\mu_1^R(\rho_1^L(x)v)y=0.$$

By (2.1), we have

$$\rho_1^L([x,y]_{g1})v+\rho_1^L(y)(\rho_1^L(x)v)-\rho_1^L(x)(\rho_1^L(y)v)=0.$$

From this, we can deduce that the following equation: $[[x,y]_1,v]_1+[y,[x,v]_1]_1-[x,[y,v]_1]_1=0$.

- $[[x,u]_1,v]_1+[u,[x,v]_1]_1-[x,[u,v]_1]_1=-\rho_1^L(x)([u,v]_{h1})+[\rho_1^L(x)u,v]_{h1}+[u,\rho_1^L(x)v]_{h1}$
$$+\rho_1^L(\mu_1^R(u)x)v+\rho_1^R(\mu_1^R(v)x)u-\mu_1^R([u,v]_{h1})x+\mu_1^R(v)(\mu_1^R(u)x)+\mu_1^L(u)(\mu_1^R(v)x)).$$

By (2.8), we have

$$-\rho_1^L(x)([u,v]_{h1})+[\rho_1^L(x)u,v]_{h1}+[u,\rho_1^L(x)v]_{h1}+\rho_1^L(\mu_1^R(u)x)v+\rho_1^R(\mu_1^R(v)x)u=0.$$

By (2.2) and (2.3), we conclude that

$$-\mu_1^R([u,v]_{h1})x+\mu_1^R(v)(\mu_1^R(u)x)-\mu_1^L(u)(\mu_1^R(v)x)=0.$$

We find that, $[[x,u]_1,v]_1+[u,[x,v]_1]_1-[x,[u,v]_1]_1=0$.

- $[[u,x]_1,v]_1+[x,[u,v]_1]_1-[u,[x,v]_1]_1$

$$=[\rho_1^R(x)u,v]_{h1}+\rho_1^L(\mu_1^L(u)x)+\mu_1^R(v)(\mu_1^L(u)x)+\rho_1^L(x)([u,v]_{h1})+\mu_1^R([u,v]_{h1})x$$
$$-[u,\rho_1^L(x)v]_{h1}-\mu_1^L(u)(\mu_1^R(v)x)-\rho_1^R(\mu_1^R(v)x)u.$$

By (2.8), the following equality is justified

$$\rho_1^L(x)([u,v]_{h1})-[u,\rho_1^L(x)v]_{h1}-\rho_1^R(\mu_1^R(v)x)u=[\rho_1^L(x)u,v]_{h1}+\rho_1^L(\mu_1^R(u)x)v.$$

By (2.9), we have

$$[\rho_1^L(x)u,v]_{h1}+\rho_1^L(\mu_1^R(u)x)v+[\rho_1^R(x)u,v]_{h1}+\rho_1^L(\mu_1^L(u)x)v=0.$$

By (2.2), we have

$$\mu_1^R([u,v]_{h1})x+\mu_1^R(v)(\mu_1^L(u)x)-\mu_1^L(u)(\mu_1^R(u)x)=0.$$

We obtain $[[u,x]_1,v]_1+[x,[u,v]_1]_1-[u,[x,v]_1]_1=0$

- $[[u,v]_1,x]_1+[v,[u,x]_1]_1-[u,[v,x]_1]_1=\rho_1^R(x)([u,v]_{h1})-[u,\rho_1^R(x)v]_{h1}+[v,\rho_1^R(x)u]_{h1}$

$$-\rho_1^R(\mu_1^L(v)x)u+\rho_1^R(\mu_1^L(u)x)v+\mu_1^L([u,v]_{h1})x+\mu_1^L(v)(\mu_1^L(u)x)-\mu_1^L(u)(\mu_1^L(v)x).$$

By (2.7), we have

$$\rho_1^R(x)([u,v]_{h1})-[u,\rho_1^R(x)v]_{h1}+[v,\rho_1^R(x)(u)]_{h1}-\rho_1^R(\mu_1^L(v)x)+\rho_1^R(\mu_1^L(u)x)v=0.$$

By (2.2), we have $\mu_1^L([u,v]_{h1})x+\mu_1^L(v)(\mu_1^L(u)x)-\mu_1^L(u)(\mu_1^L(v)x)=0$. Therefore, we have

$$[[u,v]_1,x]_1+[v,[u,x]_1]_1-[u,[v,x]_1]_1=0.$$

The bracket $[,]_2$ is studied using the same method. Let's now study the compatibility of $g\oplus h$

- $[[v,x]_1,y]_2+[x,[v,y]_1]_2-[v,[x,y]_1]_2+[v,x]_2,y]_1+[x,[v,y]_2]_1-[v,[x,y]_2]_1$

$$\begin{aligned}=\ &\mu_1^L(v)([x,y]_{g2})-[\mu_1^L(v)x,y]_{g2}-[x,\mu_1^L(v)y]_{g2}-\mu_1^L(\rho_2^R(x)v)y\\&-\mu_1^R(\rho_2^R(y)v)x+\mu_2^L(v)([x,y]_{g1})-[\mu_2^L(v)x,y]_{g1}-[x,\mu_2^L(v)y]_{g1}\\&-\mu_2^L(\rho_1^R(x)v)y-\mu_2^R(\rho_1^R(y)v)x-\rho_2^R([x,y]_{g1})v-\rho_1^R([x,y]_{g2})v\\&+\rho_2^L(x)\rho_1^R(y)+\rho_1^L(x)(\rho_2^R(y)v)+\rho_2^R(y)(\rho_1^R(x)v)+\rho_1^R(y)(\rho_2^R(x)v=0,\end{aligned}$$

because of identities (2.5), (2.6) and (ii) $c)$ 5) of Proposition 2.12.

- $[[x,v]_1,y]_2+[v,[x,y]_1]_2-[x,[v,y]_1]_2+[x,v]_2,y]_1+[v,[x,y]_2]_1-[x,[v,y]_2]_1$

$$\begin{aligned}=\ &[\mu_1^R(v)x,y]_{g2}+\mu_2^L(\rho_1^L(x)v)y+\rho_2^R(y)(\rho_1^L(x)v)+\rho_2^R([x,y]_{g1})v\\&+\mu_2^L(v)([x,y]_{g1})-[x,\mu_1^R(v)y]_{g2}-\rho_2^L(x)(\rho_1^R(y)v)-\mu_2^R(\rho_1^R(y)v)x\\&+[\mu_2^R(v)x,y]_{g1}+\rho_1^R(y)(\rho_2^L(x)v)+\mu_1^L(\rho_2^L(x)v)y+\mu_1^L(v)([x,y]_{g2})\\&+\rho_1^R([x,y]_{g2})v-[x,\mu_2^L(v)y]_{g1}-\rho_1^L(x)(\rho_2^R(y)v)-\mu_1^R(\rho_2^R(y)v)x=0,\end{aligned}$$

because (2.5), (ii) $c)$ 5) and (ii) $c)$ 6).

- $[[x,y]_1,v]_2+[y,[x,v]_1]_2-[x,[y,v]_1]_2+[x,y]_2,v]_1+[y,[x,v]_2]_1-[x,[y,v]_2]_1$

$$\begin{aligned}=\ &\rho_2^L([x,y]_{g1})v+\mu_2^R(v)([x,y]_{g1})+[y,\mu_1^R(v)x]_{g2}+\rho_2^L(y)(\rho_1^L(x)v)\\&+\mu_2^R(\rho_1^L(x)v)y-[x,\mu_1^R(v)y]_{g2}-\rho_2^L(x)(\rho_1^L(y)v)-\mu_2^R(\rho_1^L(y)v)x\\&+\rho_1^L([x,y]_{g2})v+\mu_1^R(v)([x,y]_{g2})+[y,\mu_2^R(v)x]_{g1}+\rho_1^L(y)(\rho_2^L(x)v)\\&+\mu_1^R(\rho_2^L(x)v)y-[x,\mu_2^R(v)y]_{g1}-\rho_1^L(x)(\rho_2^L(y)v)-\mu_1^R(\rho_2^L(y)v)x=0,\end{aligned}$$

because (2.4) and (ii) $c)$ 4). In the same way, we show the three remaining identities. Hence the conclusion.
$(\Longleftarrow)$ Let us show that the conditions of Proposition 2.12 are satisfied. For $x, y \in g$ and $v \in h$,

- $[[x,y]_1, v]_2 + [y,[x,v]_1]_2 - [x,[y,v]_1]_2 + [[x,y]_2, v]_1 + [y,[x,v]_2]_1 - [x,[y,v]_2]_1$

$$\begin{aligned} &= \rho_2^L([x,y]_{g1})v + \mu_2^R(v)[x,y]_{g1}) + [y, \mu_1^R(v)x]_{g2} + \rho_2^L(y)(\rho_1^L(x)v) \\ &\quad + \mu_2^R(\rho_1^L(x)v)y - [x, \mu_1^R(v)y]_{g2} - \rho_2^L(x)(\rho_1^L(y)v) - \mu_2^R(\rho_1^L(y)v)x \\ &\quad + \rho_1^L([x,y]_{g2})v + \mu_1^R(v)([x,y]_{g2}) + [y, \mu_2^R(v)x]_{g1} + \rho_1^L(y)(\rho_2^L(x)v) \\ &\quad + \mu_1^R(\rho_2^L(x)v)y - [x, \mu_2^R(v)y]_{g1} - \rho_1^L(x)(\rho_2^L(y)v) - \mu_1^R(\rho_2^L(y)v)y = 0. \quad (2.13) \end{aligned}$$

Since $(h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ is a representation of $(g, [,]_{g1}, [,]_{g2})$, by (2.4)

$$\begin{aligned} &\rho_2^L([x,y]_{g1})v + \rho_2^L(y)(\rho_1^L(x)v) - \rho_2^L(x)(\rho_1^L(y)v) \\ &\qquad + \rho_1^L([x,y]_{g2})v + \rho_1^L(y)(\rho_2^L(x)v) - \rho_1^L(x)(\rho_2^L(y)v) = 0. \end{aligned}$$

The remaining terms in (2.13) form condition (ii) $c)$ 4) of Proposition 2.12.

- $[[v,x]_1, y]_2 + [x,[v,x]_1]_2 - [v,[x,y]_1]_2 + [[v,x]_2, y]_1 + [x,[v,x]_2]_1 - [v,[x,y]_2]_1$

$$\begin{aligned} &= [\mu_1^L(v)x, y]_{g2} + \mu_2^L(\rho_1^R(x)v)y + \rho_2^R(y)(\rho_1^R(x)v) + [x, \mu_1^L(v)y]_{g2} \\ &\quad + \rho_2^L(x)(\rho_1^R(y)v) + \mu_2^R(\rho_1^R(y)v)x - \mu_2^L(v)([x,y]_{g1}) - \rho_2^R([x,y]_{g1})v \\ &\quad + [\mu_2^L(v)x, y]_{g1} + \mu_1^L(\rho_2^R(x)v)y + \rho_1^R(y)(\rho_2^R(x)v) + [x, \mu_2^L(v)y]_{g1} \\ &\quad + \rho_1^L(x)(\rho_2^R(y)v) + \mu_1^R(\rho_2^R(y)v)x - \mu_1^L(v)([x,y]_{g2}) - \rho_1^R([x,y]_{g2})v = 0. \quad (2.14) \end{aligned}$$

By Equation (2.5), we have

$$\begin{aligned} &\rho_2^L(x)(\rho_1^R(y)v) - \rho_2^R([x,y]_{g1})v + \rho_1^L(x)(\rho_2^R(y)v) - \rho_1^R([x,y]_{g2})v \\ &\qquad = \rho_2^R(x)\rho_1^L(x) + \rho_1^R(y)\rho_2^L(x). \end{aligned}$$

Therefore, by Equation (2.6), we have

$$\begin{aligned} &\rho_2^R(y)(\rho_1^R(x)v) + \rho_2^L(x)(\rho_1^R(y)v) - \rho_2^R([x,y]_{g1})v + \rho_1^R(y)(\rho_2^R(x)v) \\ &+ \rho_1^L(x)(\rho_2^R(y)v) - \rho_1^R([x,y]_{g2})v \\ &\quad = \rho_2^R(y)(\rho_1^R(x)v) + \rho_1^R(y)(\rho_2^R(x)v) + \rho_2^R(y)(\rho_1^L(x)v) + \rho_1^R(y)(\rho_2^L(x)v) = 0. \end{aligned}$$

The remaining terms in (2.14) form condition (ii) $c)$ 5) of Proposition 2.12. Furthermore, by Lemma 2.9, we have

$$[[v,x]_1, y]_2 + [[x,v]_1, y]_2 + [[v,x]_2, y]_1 + [[x,v]_2, y]_1 = 0. \quad (2.15)$$

By (2.6), we have

$$-\rho_1^R(y)(\rho_2^L(x)v) - \rho_2^R(y)(\rho_1^L(x)v) - \rho_1^R(y)(\rho_2^R(x)v) - \rho_2^R(y)(\rho_1^R(x)v) = 0. \quad (2.16)$$

Summing (2.15) and (2.16) gives

$$\begin{aligned} &[\mu_1^L(v)x + \rho_1^R(x)v, y]_2 - \rho_2^R(y)(\rho_1^R(x)v) + [\mu_1^R(v)x + \rho_1^L(x)v, y]_2 - \rho_2^R(y)(\rho_1^L(x)v) \\ &+ [\mu_2^L(v)x + \rho_2^R(x)v, y]_1 - \rho_1^R(y)(\rho_2^R(x)v) + [\mu_2^R(v)x + \rho_2^L(x)v, y]_1 - \rho_1^R(y)(\rho_2^L(x)v) = 0 \end{aligned}$$

This is written as

$$\begin{aligned} &[\mu_1^L(v)x, y]_{g2} + \mu_1^L(\rho_2^R(x)v)y + [\mu_1^R(v)x, y]_{g2} + \mu_1^L(\rho_2^L(x)v)y \\ &\qquad + [\mu_2^L(v)x, y]_{g1} + \mu_2^L(\rho_1^R(x)v)y + [\mu_2^R(v)x, y]_{g1} + \mu_2^L(\rho_1^L(x)v)y = 0. \end{aligned}$$

Hence the condition (ii) $c)$ 6) of Proposition 2.12. Equations (ii) $c)$ 1), (ii) $c)$ 2) and (ii) $c)$ 3) of Proposition 2.12 can be demonstrated in the same way. □

Proposition 2.15. *Let* $(g, [,]_{g1}, [,]_{g2})$ *and* $(h, [,]_{h1}, [,]_{h2})$ *be two C-Lei algebras. We assume*

$$1) \quad [x+u, y+v]_1 = [x,y]_{g1} + \mu_1^R(v)x + \mu_1^L(u)y + [u,v]_{h1} + \rho_1^L(x)v + \rho_1^R(y)u,$$
$$[x+u, y+v]_2 = [x,y]_{g2} + \mu_2^R(v)x + \mu_2^L(u)y + [u,v]_{h2} + \rho_2^L(x)v + \rho_2^R(y)u,$$

define a structure of C-Lei algebra on $g \oplus h$*.*

$$2) \quad h((g,[,]_{gi}),(h,[,]_{hi}),(\rho_i^L,\rho_i^R),(\mu_j^L,\mu_j^R)), \quad i,j=1,2 \textit{ is a matched pair.}$$

Then $(g, h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R), (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ *is a matched pair of* g *and* h*.*

Proof. According to Corollary 2.13, we only need to show that triples $(h, (\rho_1^L, \rho_1^R), (\rho_2^L, \rho_2^R))$ and $(g, (\mu_1^L, \mu_1^R), (\mu_2^L, \mu_2^R))$ are representations of g and h respectively. $(g, h, (\rho_1^L, \rho_1^R), (\mu_1^L, \mu_1^R))$ is a matched pair, thus (h, ρ_1^L, ρ_1^R) is a representation of algebra $(g, [,]_{g1})$. Likewise, (h, ρ_2^L, ρ_2^R) is a representation of algebra $(g, [,]_{g2})$ since $((g,[,]_{g2}),(h,[,]_{h2}),(\rho_2^L,\rho_2^R),(\mu_2^L,\mu_2^R))$ is a matched pair. Moreover, by compatibility

- $$\begin{aligned}&[[x,y]_1,v]_2 + [y,[x,v]_1]_2 - [x,[y,v]_1]_2 + [[x,y]_2,v]_1 + [y,[x,v]_2]_1 - [x,[y,v]_2]_1\\ &= \rho_2^L([x,y]_{g1})v + \mu_2^R(v)([x,y]_{g1}) + [y,\mu_1^R(v)x]_{g2} + \rho_2^L(y)(\rho_1^L(x)v)\\ &\quad + \mu_2^R(\rho_1^L(x)v)y - [x,\mu_1^R(v)y]_{g2} - \rho_2^L(x)(\rho_1^L(y)v) - \mu_2^R(\rho_1^L(y)v)x\\ &\quad + \rho_1^L([x,y]_{g2})v + \mu_1^R(v)([x,y]_{g2}) + [y,\mu_2^R(v)x]_{g1} + \rho_1^L(y)(\rho_2^L(x)v)\\ &\quad + \mu_1^R(\rho_2^L(x)v)y - [x,\mu_2^R(v)y]_{g1} - \rho_1^L(x)(\rho_2^L(y)v) - \mu_1^R(\rho_2^L(y)v)x = 0.\end{aligned}$$

By (2.10), we have

$$\mu_2^R(v)([x,y]_{g1}) + \mu_2^R(\rho_1^L(x)v)y - \mu_2^R(\rho_1^L(y)v)x + [y,\mu_2^R(v)x]_{g1} - [x,\mu_2^R(v)y]_{g1} = 0,$$
$$[y,\mu_1^R(v)x]_{g2} - [x,\mu_1^R(v)y]_{g2} + \mu_1^R(v)([x,y]_{g2}) + \mu_1^R(\rho_2^L(x)v)y - \mu_1^R(\rho_2^L(y)v)x = 0.$$

We then concluded that we have condition (2.4):

$$\begin{aligned}&\rho_2^L([x,y]_{g1})v + \rho_2^L(y)(\rho_1^L(x)v) - \rho_2^L(x))(\rho_1^L(y)v) + \rho_1^L([x,y]_{g2})v + \rho_1^L(y)(\rho_2^L(x)v)\\ &- \rho_1^L(x)(\rho_2^L(y)v) = 0.\end{aligned}$$

- $$\begin{aligned}&[[x,v]_1,y]_2 + [v,[x,y]_1]_2 - [x,[v,y]_1]_2 + [[x,v]_2,y]_1 + [v,[x,y]_2]_1 - [x,[v,y]_2]_1\\ &= [\mu_1^R(v)x,y]_{g2} + \mu_2^L(\rho_1^L(x)v)y + \rho_2^R(y)(\rho_1^L(x)v) + \rho_2^R([x,y]_{g1})v + \mu_2^L(v)([x,y]_{g1})\\ &- [x,\mu_1^L(v)y]_{g2} - \rho_2^L(x)(\rho_1^R(y)v) - \mu_2^R(\rho_1^R(y)v)x + [\mu_2^R(v)x,y]_{g1} + \rho_1^R(y)(\rho_2^L(x)v)\\ &+ \mu_1^L(\rho_2^L(x)v)y + \mu_1^L(v)([x,y]_{g2}) + \rho_1^R([x,y]_{g2})v - [x,\mu_2^L(v)y]_{g1} - \rho_1^L(x)(\rho_2^R(y)v)\\ &- \mu_1^R(\rho_2^R(y)v)x = 0.\end{aligned}$$

By (2.11), we have

$$\mu_2^L(v)([x,y]_{g1}) - \mu_2^R(\rho_1^R(y)v)x - [x,\mu_2^L(v)y]_{g1} = [\mu_2^L(v)x,y]_{g1} + \mu_2^L(\rho_1^R(x)v)y.$$

By (2.12), we have

$$[\mu_2^L(v)x,y]_{g1} + [\mu_2^R(v)x,y]_{g1} + \mu_2^L(\rho_1^R(x)v)y + \mu_2^L(\rho_1^L(x)v)y = 0.$$

By (2.11), we have

$$\mu_1^L(v)([x,y]_{g2}) - [x,\mu_1^L(v)y]_{g2} = [\mu_1^L(v)x,y]_{g2} + \mu_1^L(\rho_2^R(x)v)y + \mu_1^R(\rho_2^R(y)v)x.$$

By (2.12), we have

$$[\mu_1^R(v)x,y]_{g2} + \mu_1^L(\rho_2^L(x)v)y + [\mu_1^L(v)x,y]_{g2} + \mu_1^L(\rho_2^R(x)v)y = 0.$$

After all these steps, the condition (2.5) is verified:

$$\rho_1^R([x,y]_{g2})v + \rho_2^R([x,y]_{g1})v = \rho_1^L(x)(\rho_2^R(y)v) - \rho_2^R(y)(\rho_1^L(x)v) + \rho_2^L(x)(\rho_1^R(y)v) - \rho_1^R(y)(\rho_2^L(x)v) \tag{2.17}$$

- $[[v,x]_1,y]_2 + [x,[v,y]_1]_2 - [v,[x,y]_1]_2 + [v,x]_2,y]_1 + [x,[v,y]_2]_1 - [v,[x,y]_2]_1$

$$\begin{aligned} &= -\mu_1^L(v)([x,y]_{g2}) + [\mu_1^L(v)x,y]_{g2} + [x,\mu_1^L(v)y]_{g2} + \mu_1^L(\rho_2^R(x)v)y \\ &+ \mu_1^R(\rho_2^R(y)v)x - \mu_2^L(v)([x,y]_{g1}) + [\mu_2^L(v)x,y]_{g1} + [x,\mu_2^L(v)y]_{g1} \\ &+ \mu_2^L(\rho_1^R(x)v)y + \mu_2^R(\rho_1^R(y)v)x - \rho_2^R([x,y]_{g1})v - \rho_1^R([x,y]_{g2})v \\ &+ \rho_2^L(x)(\rho_1^R(y)u) + \rho_1^L(x)(\rho_2^R(y)v) + \rho_2^R(y)(\rho_1^R(x)v) + \rho_1^R(y)(\rho_2^R(x)v) = 0. \end{aligned}$$

Because (2.11), we have

$$[\mu_1^L(v)x,y]_{g2} + [x,\mu_1^L(v)y]_{g2} + \mu_1^L(\rho_2^R(x)v)y + \mu_1^R(\rho_2^R(y)v)x - \mu_1^L(v)([x,y]_{g2}) = 0,$$
$$[\mu_2^L(v)x,y]_{g1} + [x,\mu_2^L(v)y]_{g1} + \mu_2^L(\rho_1^R(x)v)y + \mu_2^R(\rho_1^R(y)v)x - \mu_2^L(v)([x,y]_{g1}) = 0.$$

The result is

$$\rho_2^R([x,y]_{g1})v + \rho_1^R([x,y]_{g2})v = \rho_2^L(x)(\rho_1^R(y)u) + \rho_1^L(x)(\rho_2^R(y)v) + \rho_2^R(y)(\rho_1^R(x)v) + \rho_1^R(y)(\rho_2^R(x)v). \tag{2.18}$$

Finally (2.17) and (2.18) leads to condition (2.6). The result of this is that $(h,(\rho_1^L,\rho_1^R),(\rho_2^L,\rho_2^R))$ is a representation of the C-Lei algebra. We show in the same way that $(g,(\mu_1^L,\mu_1^R),(\mu_2^L,\mu_2^R))$ is a representation of the C-Lei algebra. □

Proposition 2.16. *Let* $(g,[,]_{g1},[,]_{g2})$, $(h,[,]_{h1},[,]_{h2})$ *be two C-Lie algebras Let* $\rho_1,\rho_2 : g \to gl(h)$ *and* $\mu_1,\mu_2 : h \to gl(g)$ *be four linear maps. Then the following conditions are equivalent:*
(i) $(g,h,(\rho_1,\rho_2),(\mu_1,\mu_2))$ *is a matched pair of C-Lie algebras.*
(ii) $(g,h,(\rho_1,-\rho_1),(\rho_2,-\rho_2),(\mu_1,-\mu_1),(\mu_2,-\mu_2))$ *is a matched pair of C-Lei algebras.*

Proof. $(i) \Longrightarrow (ii)$. Show that $(g,h,(k_1\rho_1+k_2\rho_2,-k_1\rho_1-k_2\rho_2),(k_1\mu_1+k_2\mu_2,-k_1\mu_1-k_2\mu_2)$ is a matched pair of Leibniz algebras of $(g,k_1[,]_{g1}+k_2[,]_{g2})$ and $(h,k_1[,]_{h1}+k_2[,]_{h2})$ i.e.

- $(h,k_1\rho_1+k_2\rho_2,-k_1\rho_1-k_2\rho_2)$ is a representation of Leibniz algebra $(g,k_1[,]_{g1}+k_2[,]_{g2})$.
- $(g,k_1\mu_1+k_2\mu_2,-k_1\mu_1-k_2\mu_2)$ is a representation of Leibniz algebra $(h,k_1[,]_{h1}+k_2[,]_{h2})$.
- $(g,h,(k_1\rho_1+k_2\rho_2,-k_1\rho_1-k_2\rho_2),(k_1\mu_1+k_2\mu_2,-k_1\mu_1-k_2\mu_2))$ checks the conditions of (2.7), to (2.12). We have

Since $((g,k_1[,]_{g1}+k_2[,]_{g2}),(h,k_1[,]_{h1}+k_2[,]_{h2}),k_1\rho_1+k_2\rho_2,k_1\mu_1+k_2\mu_2)$ is a matched pair of Lie algebras, then (h,ρ_1,ρ_2) is a representation of C-Lie algebra. $(g,[,]_{g1},[,]_{g2})$. Such as ρ_1 and ρ_2 are representations respectively of C-Lie algebras. $(g,[,]_1)$ and $(g,[,]_2)$, we have

$$\rho_1([x,y]_{g1}) = \rho_1(x)\rho_1(y) - \rho_1(y)\rho_1(x), \quad \rho_2([x,y]_{g2}) = \rho_2(x)\rho_2(y) - \rho_2(y)\rho_2(x).$$
$$\rho_1([x,y]_{g2}) + \rho_2([x,y]_{g1}) = \rho_1(x)\rho_2(y) - \rho_2(y)\rho_1(x) + \rho_2(x)\rho_1(y) - \rho_1(y)\rho_2(x).$$

From the above, we deduced (2.1). The calculation for (2.2) and (2.3) is straightforward. In the same way, we show that $(g,k_1\mu_1+k_2\mu_2,-k_1\mu_1-k_2\mu_2)$ is a representation of Leibniz algebra $(h,k_1[,]_{h1}+k_2[,]_{h2})$. As for the last point, for $x,y \in g$, $u,v \in h$, we have

$$\begin{aligned} &(-k_1\rho_1-k_2\rho_2)(x)(k_1[u,v]_{h1} + k_2[u,v]_{h2}) - k_1[u,(-k_1\rho_1-k_2\rho_2)(x)v]_{h1} \\ &-k_2[u,(-k_1\rho_1-k_2\rho_2)(x)v]_{h2} + k_1[v,(-k_1\rho_1-k_2\rho_2)(x)u]_{h2} \\ &+k_2[v,(-k_1\rho_1-k_2\rho_2)(x)u]_{h1} - (-k_1\rho_1-k_2\rho_2)((k_1\mu_1+k_2\mu_2)(v)x)u \\ &+(-k_1\rho_1-k_2\rho_2)((k_1\mu_1+k_2\mu_2)(u)x)v \end{aligned}$$

$$= k_1^2(-\rho_1(x)([u,v]_{h1}) + [u,\rho_1(x)v]_{h1} - [v,\rho_1(x)u]_{h1} + \rho_1(\mu_1(v)x)u - \rho_1(\mu_1(u)x)v)$$
$$+k_2^2(-\rho_2(x)([u,v]_{h2}) + [u,\rho_2(x)v]_{h2} - [v,\rho_2(x)u]_{h2} + \rho_2(\mu_2(v)x)u - \rho_2(\mu_2(u)x)v)$$
$$+k_1k_2(-\rho_2(x)([u,v]_{h1}) + [u,\rho_2(x)v]_{h1} - [v,\rho_2(x)u]_{h1} - \rho_2(\mu_1(u)x)v + \rho_2(\mu_1(v)x)u$$
$$-\rho_1(x)([u,v]_{h2}) + [u,\rho_1(x)v]_{h2} - [v,\rho_1(x)u]_{h2} - \rho_1(\mu_2(u)x)v + \rho_1(\mu_2(v)x)u).$$

By Definition 2.10 and Proposition 4, condition (ii) (c) [10], we have

$$\rho_1(x)([u,v]_{h1}) - [\rho_1(x)u,v]_{h1} - [u,\rho_1(x)v]_{h1} + \rho_1(\mu_1(u)x)v - \rho_1(\mu_1(v)x)u = 0,$$
$$\rho_2(x)([u,v]_{h2}) - [\rho_2(x)u,v]_{h2} - [u,\rho_2(x)v]_{h2} + \rho_2(\mu_2(u)x)v - \rho_2(\mu_2(v)x)u = 0,$$
$$\rho_2(x)([u,v]_{h1}) - [u,\rho_2(x)v]_{h1} - [\rho_2(x)u,v]_{h1} + \rho_2(\mu_1(u)x)v - \rho_2(\mu_1(v)x)u +$$
$$\rho_1(x)([u,v]_{h2}) - [u,\rho_1(x)v]_{h2} - [\rho_1(x)u,v]_{h2} + \rho_1(\mu_2(u)x)v - \rho_1(\mu_2(v)x)u = 0.$$

Therefore, condition (2.7) is verified. The condition (2.8) is verified for the same reasons as above. The verification of (2.9) is self-evident, the conditions (2.10), (2.11) and (2.12) are treated in the same way.
$(ii) \implies (i)$ We have $(g,h,(\rho_1,-\rho_1),(\rho_2,-\rho_2),(\mu_1,-\mu_1),(\mu_2,-\mu_2))$ is a matched pair of C-Lei algebras, then $(h,(\rho_1,-\rho_1),(\rho_2,-\rho_2))$ is a representation of $(g,[,]_{g1},[,]_{g2})$, we deduce that $(h,\rho_1,-\rho_1)$ and $(h,\rho_2,-\rho_2)$ are representations of Lie algebras, respectively of $(g,[,]_{g1})$ and $(g,[,]_{g2})$, thus

$$\rho_1([x,y]_{g1}) = \rho_1(x)\rho_1(y) - \rho_1(y)\rho_1(x),$$
$$\rho_2([x,y]_{g2}) = \rho_2(x)\rho_2(y) - \rho_2(y)\rho_2(x),$$
$$\rho_1([x,y]_{g2}) + \rho_2([x,y]_{g1}) = [\rho_1(x),\rho_2(y)]_{gl(V)} + [\rho_2(x),\rho_1(y)]_{gl(V)}.$$

By Proposition 2 [10], we can deduce that (h,ρ_1,ρ_2) is a representation of C-Lie algebra. $(g,[,]_{g1},[,]_{g2})$. We show the same that (g,μ_1,μ_2) is a representation of C-Lie algebra. Moreover, we have $(h,[,]_{h1},[,]_{h2})$. $((g,[,]_{g1}),(h,[,]_{h1}),\rho_1,\mu_1)$ is a matched pair, indeed, by $ii)$ $b)$ of Proposition 2.12, we deduce that $((g,[,]_{g1}),(h,[,]_{h1}),(\rho_1,-\rho_1),(\mu_1,-\mu_1))$ is a matched pair, it checks the identities of (2.7) to (2.12). Thus, by (2.8) and (2.11), we have

$$\rho_1(x)([u,v]_{h1}) - [\rho_1(x)u,v]_{h1} - [u,\rho_1(x)]_{h1} + \rho_1(\mu_1(u)x)v - \rho_1(\mu_1(v)x)u = 0.$$
$$\mu_1(u)([x,y]_{g1}) - [\mu_1(u)x,y]_{g1} - [x,\mu_1(u)y]_{g1} + \mu_1(\rho_1(x)u)y - \mu_1(\rho_1(y)u)x = 0.$$

Definition 3 [10] is verified. In the same way, $((g,[,]_{g2}),(h,[,]_{h2}),\rho_2,\mu_2)$ is a matched pair. Finally, we have by (ii) $c)$ $2)$ and (ii) $c)$ $5)$ of Proposition 2.12

$$\rho_1(x)([u,v]_{h2}) - [\rho_1(x)u,v]_{h2} - [u,\rho_1(x)v]_{h2} + \rho_1(\mu_2(u)x)v - \rho_1(\mu_2(v)x)u$$
$$+ \rho_2(x)([u,v]_{h1}) - [\rho_2(x)u,v]_{h1} - [u,\rho_2(x)v]_{h1} + \rho_2(\mu_1(u)x)v - \rho_2(\mu_1(v)x)u = 0.$$
$$\mu_1(u)([x,y]_{h2}) - [\mu_1(u)x,y]_{h2} - [x,\mu_1(u)y]_{h2} + \mu_1(\rho_2(x)u)y - \mu_1(\rho_2(y)u)x$$
$$+ \mu_2(u)([x,y]_{h1}) - [\mu_2(u)x,y]_{h1} - [x,\mu_2(u)y]_{h1} + \mu_2(\rho_1(x)u)y - \mu_2(\rho_1(y)u)x = 0.$$

The last two equalities correspond to condition (c) of Proposition 4 [10]. □

3 Manin Triples of C-Lei algebras

Definition 3.1. A quadratic C-Lei algebra is a algebra $(g,[,]_{g1},[,]_{g2})$ equipped with a non-degenerate form $\omega \in \bigwedge^2 g^*$ such that $(g,k_1[,]_{g1}+k_2[,]_{g2},\omega)$, for all $k_1,k_2 \in \mathbb{K}$ is a quadratic [3]. It is noted by $(g,[,]_{g1},[,]_{g2},\omega)$.

Remark 3.2. $(g,[,]_{g1},[,]_{g2},\omega)$ is a quadratic if and only if $(g,[,]_{g1},\omega)$ and $(g,[,]_{g2},\omega)$ are quadratic.

Definition 3.3. A triple of C-Lei algebras

$$((a,[,]_{a1},[,]_{a2},\omega),(b,[,]_{b1},[,]_{b2}),(c,[,]_{c1},[,]_{c2})),$$

is called a Manin triple of C-Lei algebras, if for any k_1,k_2,

(1) $(a, k_1[,]_{a1} + k_2[,]_{a2}, \omega)$ is a quadratic,

(2) Algebras $(b, k_1[,]_{b1} + k_2[,]_{b2})$ and $(c, k_1[,]_{c1} + k_2[,]_{c2})$ are both isotropic sub-algebras of $(a, k_1[,]_{a1} + k_2[,]_{a2})$,

(3) $a = b \oplus c$ as vector spaces.

Remark 3.4. In the context of this definition, the structures over b and c are the induced structures, hence we adopt the notation (a, b, c, ω).

Proposition 3.5. *Let $(a, [,]_{a1}, [,]_{a2}, \omega)$ be a quadratic C-Lei algebra. Let $(b, [,]_{b1}, [,]_{b2})$ and $(c, [,]_{c1}, [,]_{c2})$ be C-Lei algebras. Then we have (a, b, c, ω) is a Manin triple of C-Lei algebras if and only if the two triples $((a, [,]_{a1}, \omega), (b, [,]_{b1}), (c, [,]_{c1}))$ and $((a, [,]_{a2}, \omega), (b, [,]_{b2}), (c, [,]_{c2}))$ are Manin triples of Leibniz algebras [13].*

Proof. $(\Longrightarrow)$. It's obvious, just take $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$.
$(\Longleftarrow)$, $(b, [,]_{b1})$ and $(c, [,]_{c1})$ are sub-algebras of $(a, [,]_{a1})$ see also $(b, [,]_{b2}), (c, [,]_{c2})$ are sub-algebras of $(a, [,]_{a2})$, then $(b, [,]_{b1}, [,]_{b2})$ and $(c, [,]_{c1}, [,]_{c2})$ are sub-algebras of $(a, [,]_{a1}, [,]_{a2})$. Moreover, if for k_1, k_2 fixed we pose $[y, z]_a = k_1[y, z]_{a1} + k_2[y, z]_{a2}$, we have

$$\begin{aligned}
\omega(x, [y, z]_a) &= \omega(x, k_1[y, z]_{a1} + k_2[y, z]_{a2}) = k_1\omega(x, [y, z]_{a1}) + k_2\omega(x, [y, z]_{a2}) \\
&= k_1\omega([x, z]_{a1} + [z, x]_{a1}, y) + k_2\omega([x, z]_{a2} + [z, x]_{a2}, y) \\
&= \omega([x, z]_a + [z, x]_a, y).
\end{aligned}$$

The proposition is proven. □

Two Manin triples of C-Lei algebras (g, g_1, g_2, ω_g) and (h, h_1, h_2, ω_h) are isomorphic if the exists an isomorphism of Leibniz algebras $\Gamma : g \to h$ such that for all $x, y \in g$, $\Gamma(g_1) = h_1$, $\Gamma(g_2) = h_2$, $\omega_g(x, y) = \omega_h(\Gamma(x), \Gamma(y))$. Let $(g, [,]_{g1}, [,]_{g2})$ be a C-Lei algebra. Suppose that there is a C-Lei algebra structure $(g^*, [,]_{g^*1}, [,]_{g^*2})$ on g^*. If there is a C-Lei algebra structure on the space $g \oplus g^*$ such that $(g \oplus g^*, g, g^*)$ is a Manin triple, with the invariant skew-symmetric bilinear form $<,>$ on $g \oplus g^*$, given by (1.1), $(g \oplus g^*, g, g^*)$ is qualified as standard Manin triple of C-Lei algebras.

Theorem 3.6. *Every Manin triple of C-Lei algebras (g, h, l, ω) is isomorphic to standard one.*

Proof. Let (g, h, l, ω) a Manin triple of C-Lei algebras. We consider the linear map $\psi : l \to h^*$, defined by $< \psi(a), x >= \omega(a, x)$ for any $x \in h$ and $a \in l$. Extending ψ to h by setting $\psi(x) = x$ for any $x \in h$. Then ψ is a linear isomorphism from $h \oplus l$ to $h \oplus h^*$. Thus, ψ induces a Manin triples structure of C-Lei algebras on $(h \oplus h^*, h, h^*)$, defined by

$$\begin{aligned}
&[x, y]_1 = [x, y]_{g1}, && [\eta, \xi]_1 = \psi([\psi^{-1}(\eta), \psi^{-1}(\xi)]_{g1}) \\
&[x, y]_2 = [x, y]_{g2}, && [\eta, \xi]_2 = \psi([\psi^{-1}(\eta), \psi^{-1}(\xi)]_{g2}) \\
&[x, \eta]_1 = \psi([x, \psi^{-1}(\eta)]_{g1}), && [\eta, x]_1 = \psi([\psi^{-1}(\eta), x]_{g1}) \\
&[x, \eta]_2 = \psi([x, \psi^{-1}(\eta)]_{g2}), && [\eta, x]_2 = \psi([\psi^{-1}(\eta), x]_{g2}),
\end{aligned}$$

for $x, y \in h$ and $\eta, \xi \in h^*$. On another side we have, for all $x, y \in h$, $a, b \in l$

- $$\begin{aligned}
< \psi(a + x), \psi(b + y) > &= < \psi(a) + x, y + \psi(b) + y > \\
&= < \psi(a), y > - < \psi(b), x > \\
&= \omega(a, y) - \omega(b, x) = \omega(a + x, b + y).
\end{aligned}$$

For the invariance condition [13], let's show

$$\begin{aligned}
&< x, [\xi, \eta]_1 >=< [x, \eta]_1 + [\eta, x]_1, \xi >, && < \xi, [x, y]_1 >=< [\xi, y]_1 + [y, \xi]_1, x > \\
&< x, [\xi, y]_1 >=< [x, y]_1 + [y, x]_1, \xi >, && < x, [y, \xi]_1 >=< [x, \xi]_1 + [\xi, x]_1, y > \\
&< x, [y, z]_1 >=< [x, z]_1 + [z, x]_1, y >, && < \xi, [\eta, \zeta]_1 >=< [\xi, \zeta]_1 + [\zeta, \xi]_1, \eta > .
\end{aligned}$$

We have

- $< x, [\xi, \eta]_1 > = \omega(x, [\psi^{-1}(\xi), \psi^{-1}(\eta)]_{g1})$
$= \omega([x, \psi^{-1}(\eta)]_{g1} + [\psi^{-1}(\eta), x]_{g1}, \psi^{-1}(\xi))$
$=< [x, \eta]_1 + [\eta, x]_1, \xi > .$
- $< \xi, [\eta, \zeta]_1 > =< \xi, \psi([\psi^{-1}(\eta), \psi^{-1}(\zeta)]_{g1}))$
$= \omega(\psi^{-1}(\xi), [\psi^{-1}(\eta), \psi^{-1}(\zeta)]_{g1})$
$= \omega([\psi^{-1}(\xi), \psi^{-1}(\zeta)]_{g1} + [\psi^{-1}(\zeta), \psi^{-1}(\xi)]_{g1}, \psi^{-1}(\eta)).$
$=< [\xi, \zeta]_1 + [\zeta, \xi]_1, \eta > .$
- $< x, [y, \xi]_1 > =< x, \psi([y, \psi^{-1}(\xi)]_{g1} >$
$= \omega(x, [y, \psi^{-1}(\xi)]_{g1})$
$= \omega([x, \psi^{-1}(\xi)]_{g1} + [\psi^{-1}(\xi), x]_{g1}, y)$
$=< [x, \xi]_1 + [\xi, x]_1, y > .$

Do the same to show the other identities and $[,]_2$. Conclude with remark 3.2. □

Proposition 3.7. *Let $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g^*1}, [,]_{g^*2})$ be two C-Lei algebras. If $(g \oplus g^*, g, g^*)$ is a standard Manin triple and if $((g, [,]_{gi}), (g^*, [,]_{g^*j}), (L^{i*}, -L^{i*} - R^{i*}), (\mathcal{L}^{j*}, -\mathcal{L}^{j*} - \mathcal{R}^{j*}))$ is a matched pair, for $i, j = 1, 2$. Next, we say*

$$(g, g^*, (L^{1*}, -L^{1*} - R^{1*}), (L^{2*}, -L^{2*} - R^{2*}), (\mathcal{L}^{1*}, -\mathcal{L}^{1*} - \mathcal{R}^{1*}), (\mathcal{L}^{2*}, -\mathcal{L}^{2*} - \mathcal{R}^{2*}))$$

is a matched pair of C-Lei algebras $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g^*1}, [,]_{g^*2})$.

Proof. Let's put

$$[x + u, y + v]_1 = [x, y]_{g1} + \rho_1^L(x)v + \rho_1^R(y)u + [u, v]_{g^*1} + \mu_1^L(u)y + \mu_1^R(v)x$$
$$[x + u, y + v]_2 = [x, y]_{g2} + \rho_2^L(x)v + \rho_2^R(y)u + [u, v]_{g^*2} + \mu_2^L(u)y + \mu_2^R(v)x,$$

for $x, y \in g$ and $u, v \in g^*$. Let $<, >$ be the skew-symmetric bilinear form on $g \oplus g^*$ defined by (1.2), we have for $\eta, \xi \in g^*$

$$< \eta, \mu_1^R(\xi)x > =< \eta, [x, \xi]_1 >=< [\eta, \xi]_{g^*1} + [\xi, \eta]_{g^*1}, x >= - < \eta, \mathcal{R}^{1*}(\xi)x + \mathcal{L}^{1*}(\xi)x > .$$

Then, $\mu_1^R = -\mathcal{L}^{1*} - \mathcal{R}^{1*}$.

$$< \eta, \mu_1^L(\xi)x > =< \eta, [\xi, x]_1 >= - < [\xi, \eta]_{g^*1}, x >= - < \mathcal{L}^1(\xi)\eta, x >=< \eta, \mathcal{L}^{1*}(\xi)x > .$$

Then, $\mu_1^L = \mathcal{L}^{1*}$.

$$< \rho_1^L(x)\eta, y > =< [x, \eta]_1, y >= - < [y, \eta]_1 + [\eta, y]_1, x >= - < \eta, [x, y]_{g1} >=< L^{1*}(x)\eta, y > .$$

Then, $\rho_1^L = L^{1*}$.

$$< \rho_1^R(x)\eta, y > =< [\eta, x]_1, y >= - < [y, x]_1 + [x, y]_1, \eta >= - < R^1(x)y + L^1(x)y, \eta >$$
$$=< y, (R^{1*} + L^{1*})(x))\eta >= - < (R^{1*} + L^{1*})(x)\eta, y > .$$

Then, $\rho_1^R = -R^{1*} - L^{1*}$. We do the same for $\rho_2^L, \rho_2^R, \mu_2^L$ and μ_2^R. By Proposition 2.15, we conclude the result. □

4 Lie Admissible C-Lei algebras

Definition 4.1. A C-Lei algebra $(g, [,]_{g1}, [,]_{g2})$ is said to be Lie admissible if for all $k_1, k_2 \in \mathbb{K}$, $(g, k_1[,]_{g1} + k_2[,]_{g2})$ is Lie admissible [9].

Lemma 4.2. *A Leibniz algebra* $(g, [,]_g)$ *is Lie admissible if and only if for all* $x, y, z \in g$,

$$\circlearrowleft_{x,y,z}[[x, y]_g, z]_g = 0.$$

Proof. Let's put $[x, y] = [x, y]_g - [y, x]_g$. We have

$$\begin{aligned}\circlearrowleft_{x,y,z}[[x, y], z] &= [x, [z, y]_g]_g - [z, [x, y]_g]_g - [[x, z]_g, y]_g \\ &+ [z, [y, x]_g]_g - [y, [z, x]_g]_g - [[z, y]_g, x]_g \\ &+ [y, [x, z]_g]_g - [x, [y, z]_g]_g - [[y, x]_g, z]_g \\ &+ [[x, y]_g, z]_g + [[y, z]_g, x]_g + [[z, x]_g, y]_g.\end{aligned}$$

By the Leibniz identity, we have

$$\circlearrowleft_{x,y,z}[[x, y], z] = 0 \Leftrightarrow \circlearrowleft_{x,y,z}[[x, y]_g, z]_g = 0.$$

Hence the result. □

Proposition 4.3. *A C-Lei algebra* $(g, [,]_{g1}, [,]_{g2})$ *is Lie admissible if and only if*

1) $(g, [,]_{g1})$ *and* $(g, [,]_{g2})$ *are Lie admissible,*

2) For all $x, y, z \in g$, $\circlearrowleft_{x,y,z}[[x, y]_{g1}, z]_{g2} + \circlearrowleft_{x,y,z}[[x, y]_{g2}, z]_{g1} = 0$.

Proof. Let $x, y, z \in g$ and $k_1, k_2 \in \mathbb{K}$. To simplify writing, we pose $[x, y]_g^{k1k2} = k_1([x, y]_{g1} - [y, x]_{g1}) + k_2([x, y]_{g2} - [y, x]_{g2})$. After development of $[x, y]_g^{k1k2}$, the components of k_1^2 and k_2^2 are zero because of Lemma 4.2 and the identity of Leibniz. By compatibility, the component of k_1k_2 is

$$\circlearrowleft_{x,y,z}[[x, y]_{g1}, z]_{g2} + \circlearrowleft_{x,y,z}[[x, y]_{g2}, z]_{g1} = 0.$$

Hence the result. □

5 Compatible Leibniz Bialgebras

Let $\triangle_1$ *and* $\triangle_2$ *be the duals of* $[,]_{g*1}$ *and* $[,]_{g*2}$ *respectively. For all* $k_1, k_2 \in \mathbb{K}$, *we pose* $\triangle = k_1 \triangle_1 + k_2\triangle_2$, $R = k_1R^1 + k_2R^2$, $L = k_1L^1 + k_2L^2$ *and* 1 *is the identity of* g.

Definition 5.1. [13] Let $(g, [,]_g)$ and $(g^*, [,]_{g^*})$ be Leibniz algebras. Then (g, g^*) is called a Leibniz bialgebra if the following conditions hold, for all $x, y \in g$

$$(a)\ \tau_{12}((R_y \otimes 1) \triangle_1 x) = (R_x \otimes 1) \triangle_1 y,$$

$$\begin{aligned}(b)\ \ \triangle_1 [x, y]_{g1} &= (1 \otimes R^1_y - L^1_y \otimes 1 - R^1_y \otimes 1 - \tau_{12} \circ (1 \otimes L^1_y) - \tau_{12} \circ (1 \otimes R^1_y)) \triangle_1 x \\ &+ (1 \otimes L^1_x + L^1_x \otimes 1 + R^1_x \otimes 1) \triangle_1 y,\end{aligned}$$

where $\tau_{12} : g \otimes g \to g \otimes g$ is the exchange operator defined by $\tau_{12}(x \otimes y) = y \otimes x$.

Definition 5.2. Let $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g*1}, [,]_{g*2})$ be C-Lei algebras. Then the couple (g, g^*) is called compatible Leibniz bialgebra if and only if for all $k_1, k_2 \in \mathbb{K}$,

$$((g, k_1[,]_{g1} + k_2[,]_{g2}), (g^*, k_1[,]_{g*1} + k_2[,]_{g*2}))$$

is a Leibniz bialgebra.

Proposition 5.3. *with the notation in the definition above, then* $((g, [,]_{g1}, [,]_{g2}), (g^*, [,]_{g*1}, [,]_{g*2}))$ *is Compatible Leibniz bialgebra if and only if*
(i) $((g, [,]_{g1}), (g^*, [,]_{g*1}))$ *and* $((g, [,]_{g2}), (g^*, [,]_{g*2}))$ *are both Leibniz bialgebras.*
(ii) For all $x, y \in g$

1) $(R^1(x) \otimes 1) \triangle_2 y + (R^2(x) \otimes 1) \triangle_1 y = \tau_{12}((R^1(y) \otimes 1) \triangle_2 x) + \tau_{12}((R^2(y) \otimes 1) \triangle_1 x)$,

2) $\triangle_1 [x, y]_{g2} + \triangle_2[x, y]_{g1}$

$$\begin{aligned}&= (1 \otimes R^2(y) - L^2(y) \otimes 1 - R^2(y) \otimes 1 - \tau_{12} \circ (1 \otimes L^2(y)) - \tau_{12} \circ (1 \otimes R^2(y))) \triangle_1 x \\ &+ (1 \otimes L^2(x) + L^2(x) \otimes 1 + R^2(x) \otimes 1) \triangle_1 y \\ &+ (1 \otimes R^1(y) - L^1(y) \otimes 1 - R^1(y) \otimes 1 - \tau_{12} \circ (1 \otimes L^1(y)) - \tau_{12} \circ (1 \otimes R^1(y))) \triangle_2 x \\ &+ (1 \otimes L^1(x) + L^1(x) \otimes 1 + R^1(x) \otimes 1) \triangle_2 y.\end{aligned}$$

Proof. ($\Longrightarrow$) Condition (i) corresponds to the cases $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$. Let us show the condition (ii). For $(k_1, k_2) = (1, 1)$, we have $((g, [,]_{g1} + [,]_{g2}), (g^*, [,]_{g*1} + [,]_{g*2}))$ is Leibniz bialgebra. For all $x, y \in g$, we have

$$((R^1(x) + R^2(x)) \otimes 1)(\triangle_1 y + \triangle_2 y) = \tau_{12}(((R^1(y) + R^2(y)) \otimes 1)(\triangle_1 x + \triangle_2 x).$$

Using condition (i), we obtain, after simplification, equation (ii) 1). Let us now show equation (ii) 2)

$$\begin{aligned}
&(\triangle_1 + \triangle_2)([x, y]_{g1} + [x, y]_{g2}) \\
&= (1 \otimes (R^1(y) + R^2(y)) - (L^1(y) + L^2(y)) \otimes 1 - (R^1(y) + R^2(y)) \otimes 1 \\
&- \tau_{12} \circ (1 \otimes L^1(y) + 1 \otimes L^2 1(y)) - \tau_{12} \circ (1 \otimes R^1(y) + 1 \otimes R^2(y)))(\triangle_1 x + \triangle_2 x) \\
&+ (1 \otimes L^1(x) + 1 \otimes L^2(x) + L^1(x) \otimes 1 + L^2(x) \otimes 1 + R^1(x) \otimes 1 + R^2(x) \otimes 1)(\triangle_1 y + \triangle_2 y).
\end{aligned}$$

After development, we obtain the result.

($\Longleftarrow$) On the one hand, for k_1, $k_2 \in \mathbb{K}$, we have

$$\begin{aligned}
&(R(x) \otimes 1) \triangle y = (k_1 R^1(x) \otimes 1 + k_2 R^2(x) \otimes 1)(k_1 \triangle_1 y + k_2 \triangle_2 y) \\
&= k_1^2 (R^1(x) \otimes 1) \triangle_1 y + k_2^2 (R^2(x) \otimes 1) \triangle_2 y + k_1 k_2 ((R^1(x) \otimes 1) \triangle_2 y + (R^2(x) \otimes 1) \triangle_1 y.
\end{aligned}$$

On another side

$$\begin{aligned}
&\tau_{12}((k_1 R^1(y) \otimes 1 + k_2 R^2(y) \otimes 1)(k_1 \triangle_1 x + k_2 \triangle_2 x)) \\
&\quad = k_1^2 \tau_{12}((R^1(y) \otimes 1) \triangle_1 x) + k_2^2 \tau_{12}((R^2(y) \otimes 1) \triangle_2 x) \\
&\quad + k_1 k_2 \tau_{12}((R^1(y) \otimes 1) \triangle_2 x + (R^2(y) \otimes 1) \triangle_1 x).
\end{aligned}$$

From conditions (i) and (ii) 1), we have the equality of the two parts. Finally, we have

$$\triangle (k_1 [x, y]_{g1} + k_2 [x, y]_{g2}) = k_1^2 \triangle_1 [x, y]_{g1} + k_2^2 \triangle_2 [x, y]_{g2} + k_1 k_2 (\triangle_1 [x, y]_{g2} + \triangle_2 [x, y]_{g1})$$

Furthermore, expand the calculation

$$\begin{aligned}
&(1 \otimes R(y) - L(y) \otimes 1 - R(y) \otimes 1 - \tau_{12}(1 \otimes L(y)) - \tau_{12}(1 \otimes R(y))) \triangle x \\
&+ (1 \otimes L(x) + L(x) \otimes 1 + R(x) \otimes 1) \triangle y.
\end{aligned}$$

Furthermore, expand the calculation, then use (ii) 2) to find the desired equality. □

Theorem 5.4. *with the notation in the Definition 5.2, the following conditions are equivalent:*
(i) $((g, [,]_{g1}, [,]_{g2}), (g^*, [,]_{g*1}, [,]_{g*2}))$ *is a compatible Leibniz bialgebra,*
(ii) $(g, g^*, (L^{1*}, -L^{1*} - R^{1*}), (L^{2*}, -L^{2*} - R^{2*}), (\mathcal{L}^{1*}, -\mathcal{L}^{1*} - \mathcal{R}^{1*}), (\mathcal{L}^{2*}, -\mathcal{L}^{2*} - \mathcal{R}^{2*}))$ *is a matched pair of C-Lei algebras,*
(iii) $(g \oplus g^*, g, g^*)$ *is a standard Manin triple of C-Lei algebras.*

Proof. $(i) \Longleftrightarrow (ii)$ Let us show that Proposition 2.12 is equivalent to Proposition 5.2. The left hand side of Equation (2.12) of Definition 2.10 act on an arbitrary element $v \in g^*$, we have

$$\begin{aligned}
&< [\mathcal{L}^{1*}_u x, y]_{g1}, v >- < \mathcal{L}^{1*}_{L^{1*}_x u} y, v > - < \mathcal{L}^{1*}_{R^{1*}_x u} y, v > - < [\mathcal{L}^{1*}_u x, y]_{g1}, v > \\
&- < [\mathcal{R}^{1*}_u x, y]_{g1}, v > + < \mathcal{L}^{1*}_{L^{1*}_x u} y, v > \\
&\qquad =< y, [R^{1*}_x u, v]_{g*1} > - < R^1_y (\mathcal{R}^{1*}_u x), v > \\
&\qquad =< y, [R^{1*}_x u, v]_{g*1} > - < x, [R^{1*}_y v, u]_{g*1} > \\
&\qquad = - < (R^1_x \otimes 1) \triangle_1 y, u \otimes v > + < \tau_{12}((R^1_y \otimes 1) \triangle_1 x), u \otimes v > .
\end{aligned}$$

Equation (2.12) is equivalent to

$$\tau_{12}((R^1_y \otimes 1) \triangle_1 x) = (R^1_x \otimes 1) \triangle_1 y. \quad (5.1)$$

The left hand side of equation (2.11) act on an arbitrary element $v \in g^*$, we have

$$< \mathcal{L}_u^{1*}([x,y]_{g1}), v > - < [\mathcal{L}_u^{1*}x, y]_{g1}, v > - < [x, \mathcal{L}_u^{1*}y]_{g1}, v > + < \mathcal{L}^{1*}_{L_x^{1*}u}y, v >$$
$$+ < \mathcal{L}^{1*}_{R_x^{1*}u}y, v > - < \mathcal{L}^{1*}_{L_y^{1*}u}x, v > \quad - < \mathcal{R}^{1*}_{L_y^{1*}u}x, v > - < \mathcal{L}^{1*}_{R_y^{1*}u}x, v > - < \mathcal{R}^{1*}_{R_y^{1*}u}x, v > .$$

It is equivalent to

$$< - \triangle_1 ([x,y]_{g1}) + (1 \otimes R_y^1) \triangle_1 x + (1 \otimes L_x^1) \triangle_1 y + (L_x^1 \otimes 1) \triangle_1 y, u \otimes v >$$
$$+ < (R_x^1 \otimes 1) \triangle_1 y - (L_y^1 \otimes 1) \triangle_1 x - \tau_{12}((1 \otimes L_y^1) \triangle_1 x) - (R_y^1 \otimes 1) \triangle_1 x, u \otimes v >$$
$$- < \tau_{12}((1 \otimes R_y^1) \triangle_1 x), u \otimes v > .$$

From this we deduce that equation (2.11) is equivalent to

$$\begin{aligned} \triangle_1[x,y]_{g1} =& (1 \otimes R_y^1 - L_y^1 \otimes 1 - R_y^1 \otimes 1 - \tau_{12} \circ (1 \otimes L_y^1) - \tau_{12} \circ (1 \otimes R_y^1)) \triangle_1 x \\ &+ (1 \otimes L_x^1 + L_x^1 \otimes 1 + R_x^1 \otimes 1) \triangle_1 y. \end{aligned} \tag{5.2}$$

We deduce that (5.1) and (5.2) $\Longleftrightarrow$ (2.11) and (2.12). Let us now show that the following implication is true: (5.1) and (5.2) $\Longrightarrow$ (2.10), (2.9), (2.8) and (2.7). The left hand side of equation (2.10) act on an arbitrary element $v \in g^*$, we have

$$- < \mathcal{L}_u^{1*}[x,y]_{g1}, v > - < \mathcal{R}_u^{1*}[x,y]_{g1}, v > + < [x, \mathcal{L}_u^{1*}y]_{g1}, v > + < [x, \mathcal{R}_u^{1*}y]_{g1}, v >$$
$$- < [y, \mathcal{L}_u^{1*}x]_{g1}, v > - < [y, \mathcal{R}_u^{1*}x]_{g1}, v > + < \mathcal{L}^{1*}_{L_y^{1*}u}x, v > + < \mathcal{R}^{1*}_{L_y^{1*}u}x, v >$$
$$- < \mathcal{L}^{1*}_{L_x^{1*}u}y, v > - < \mathcal{R}^{1*}_{L_x^{1*}u}y, v >$$
$$=< \triangle_1([x,y]_{g1}) + \tau_{12} \triangle_1 ([x,y]_{g1}) - (1 \otimes L_x^1) \triangle_1 y - \tau_{12}((L_x^1 \otimes 1) \triangle_1 y), u \otimes v >$$
$$+ < (1 \otimes L_y^1) \triangle_1 x + \tau_{12}((L_y^1 \otimes 1) \triangle_1 x) + \tau_{12}((L_y^1 \otimes 1) \triangle_1 x) + \tau_{12}((1 \otimes L_y^1) \triangle_1 x), u \otimes v >$$
$$- < (L_x^1 \otimes 1) \triangle_1 y + \tau_{12}((1 \otimes L_x^1) \triangle_1 y), u \otimes v > .$$

Thus, (2.10) is equivalent to

$$\begin{aligned} &\triangle_1([x,y]_{g1}) + \tau_{12}(\triangle_1[x,y]_{g1}) \\ &\quad = - (1 \otimes L_y^1 + \tau_{12} \circ (L_y^1 \otimes 1) + L_y^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^1)) \triangle_1 x \\ &\quad\quad + (1 \otimes L_x^1 + \tau_{12} \circ (L_x^1 \otimes 1) + L_x^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_x^1)) \triangle_1 y. \end{aligned} \tag{5.3}$$

On the other hand, according to (5.2), we have

$$\begin{aligned} &\triangle_1([x,y]_{g1}) + \tau_{12}(\triangle_1[x,y]_{g1}) \\ &\quad = - (1 \otimes L_y^1 + \tau_{12} \circ (L_y^1 \otimes 1) + L_y^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^1)) \triangle_1 x \\ &\quad\quad + (1 \otimes L_x^1 + \tau_{12} \circ (L_x^1 \otimes 1) + L_x^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_x^1)) \triangle_1 y \\ &\quad\quad - (R_y^1 \otimes 1) \triangle_1 x + \tau_{12} \circ (R_x^1 \otimes 1) \triangle_1 y + (R_x^1 \otimes 1) \triangle_1 y - \tau_{12} \circ (R_y^1 \otimes 1) \triangle_1 x. \end{aligned}$$

By (5.1), we find (5.3). We have just shown that (5.1) and (5.2) $\Longrightarrow$ (2.10). For the other implications, we follow the same method. Thus $(g, g^*, (L^{1*}, -L^{1*} - R^{1*}), (\mathcal{L}^{1*}, -\mathcal{L}^{1*} - \mathcal{R}^{1*}))$ is a matched pair if and only if $((g, [,]_{g1}), (g^*, [,]_{g*1}))$ is a Leibniz bialgebra. In the same way, we have $(g, g^*, (L^{2*}, -L^{2*} - R^{2*}), (\mathcal{L}^{2*}, -\mathcal{L}^{2*} - \mathcal{R}^{2*}))$ is a matched pair if and only if $((g, [,]_{g2}), (g^*, [,]_{g*2}))$ is a Leibniz bialgebra. The left hand side of equation (ii) $c)$ 6) act on an arbitrary element $v \in g^*$, we have

$$< [\mathcal{L}_u^{1*}x, y]_{g2}, v > - < \mathcal{L}^{1*}_{L_x^{2*}u}y, v > - < \mathcal{L}^{1*}_{R_x^{2*}u}y, v > - < [\mathcal{L}_u^{1*}x, y]_{g2}, v >$$
$$- < [\mathcal{R}_u^{1*}x, y]_{g2}, v > + < \mathcal{L}^{1*}_{L_x^{2*}u}y, v > + < [\mathcal{L}_u^{2*}x, y]_{g1}, v > - < \mathcal{L}^{2*}_{L_x^{1*}u}y, v >$$
$$- < \mathcal{L}^{2*}_{R_x^{1*}u}y, v > - < [\mathcal{L}_u^{2*}x, y]_{g1}, v > - < [\mathcal{R}_u^{2*}x, y]_{g1}, v > + < \mathcal{L}^{2*}_{L_x^{1*}u}y, v >$$
$$= - < \mathcal{L}^{1*}_{R_x^{2*}u}y, v > - < [\mathcal{R}_u^{1*}x, y]_{g2}, v > - < \mathcal{L}^{2*}_{R_x^{1*}u}y, v > - < [\mathcal{R}_u^{2*}x, y]_{g1}, v > .$$

A simple calculation gives

$$-<\mathcal{L}^{1*}_{R^{2*}_x u}y, v>=-<\triangle_1 y, R^{2*}_x u\otimes v>=<((R^2_x\otimes 1)\triangle_1 y, u\otimes v>.$$
$$-<[\mathcal{R}^{1*}_u x, y]_{g2}, v>=-<\triangle_1 x, R^{2*}_y v\otimes u>=<\tau_{12}((R^2_y\otimes 1)\triangle_1 x), u\otimes v>.$$

By swapping the indices 1 and 2 in the above equations, we obtain

$$-<\mathcal{L}^{2*}_{R^{1*}_x u}y, v>=<(R^1_x\otimes 1)\triangle_2 y, u\otimes v>,$$
$$-<[\mathcal{R}^{2*}_u x, y]_{g1}>=<\tau_{12}((R^1_y\otimes 1)\triangle_2 x), u\otimes v>.$$

Equation (ii) $c)$ 6) is then equivalent to

$$(R^1_x\otimes 1)\triangle_2 y+(R^2_x\otimes 1)\triangle_1 y=\tau_{12}((R^1_y\otimes 1)\triangle_2 x)+\tau_{12}((R^2_y\otimes 1)\triangle_1 x). \tag{5.4}$$

Similarly for condition (ii) $c)$ 5), The left hand side act on an arbitrary element $v\in g^*$, we have

$$\begin{aligned}&<\mathcal{L}^{1*}_u[x,y]_{g2}, v>-<[\mathcal{L}^{1*}_u x, y]_{g2}, v>-<[x,\mathcal{L}^{1*}_u y]_{g2}, v>+<\mathcal{L}^{1*}_{L^{2*}_x u}y, v>\\&+<\mathcal{L}^{1*}_{R^{2*}_x u}y, v>-<\mathcal{L}^{1*}_{L^{2*}_y u}x, v>-<\mathcal{L}^{1*}_{R^{2*}_y u}x, v>-<\mathcal{R}^{1*}_{L^{2*}_y u}x, v>-<\mathcal{R}^{1*}_{R^{2*}_y u}x, v>\\&+<\mathcal{L}^{2*}_u[x,y]_{g1}, v>-<[\mathcal{L}^{2*}_u x, y]_{g1}, v>-<[x,\mathcal{L}^{2*}_u y]_{g1}, v>+<\mathcal{L}^{2*}_{L^{1*}_x u}y, v>\\&+<\mathcal{L}^{2*}_{R^{1*}_x u}y, v>-<\mathcal{L}^{2*}_{L^{1*}_y u}x, v>-<\mathcal{L}^{2*}_{R^{1*}_y u}x, v>-<\mathcal{R}^{2*}_{L^{1*}_y u}x, v>-<\mathcal{R}^{2*}_{R^{1*}_y u}x, v>.\end{aligned}$$

To calculate this sum, we divide it into two parts. The first part contains the sum of the first nine terms, which we calculate. We deduce the second part by permuting the indices of the first. We have

$$\begin{aligned}&<\mathcal{L}^{1*}_u[x,y]_{g2}, v>-<[\mathcal{L}^{1*}_u x, y]_{g2}, v>-<[x,\mathcal{L}^{1*}_u y]_{g2}, v>+<\mathcal{L}^{1*}_{L^{2*}_x u}y, v>\\&+<\mathcal{L}^{1*}_{R^{2*}_x u}y, v>-<\mathcal{L}^{1*}_{L^{2*}_y u}x, v>-<\mathcal{L}^{1*}_{R^{2*}_y u}x, v>-<\mathcal{R}^{1*}_{L^{2*}_y u}x, v>-<\mathcal{R}^{1*}_{R^{2*}_y u}x, v>\\=&<-\triangle_1([x,y]_{g2})+(1\otimes R^2_y)\triangle_1 x+(1\otimes L^2_x)\triangle_1 y+(L^2_x\otimes 1)\triangle_1 y, u\otimes v>\\&+<(R^2_x\otimes 1)\triangle_1 y, -(L^2_y\otimes 1)\triangle_1 x-(R^2_y\otimes 1)\triangle_1 x-\tau_{12}((1\otimes L^2_y)\triangle_1 x), u\otimes v>\\&-<\tau_{12}((1\otimes R^2_y)\triangle_1 x), u\otimes v>.\end{aligned}$$

We deduct that, equation (ii) $c)$ 5) is then equivalent to

$$\begin{aligned}&\triangle_1[x,y]_{g2}+\triangle_2[x,y]_{g1}\\&\quad=(1\otimes R^2_y-L^2_y\otimes 1-R^2_y\otimes 1-\tau_{12}\circ(1\otimes L^2_y)-\tau_{12}\circ(1\otimes R^2_y))\triangle_1 x\\&\qquad+(1\otimes L^2_x+L^2_x\otimes 1+R^2_x\otimes 1)\triangle_1 y\\&\qquad+(1\otimes R^1_y-L^1_y\otimes 1-R^1_y\otimes 1-\tau_{12}\circ(1\otimes L^1_y)-\tau_{12}\circ(1\otimes R^1_y))\triangle_2 x\\&\qquad+(1\otimes L^1_x+L^1_x\otimes 1+R^1_x\otimes 1)\triangle_2 y.\end{aligned} \tag{5.5}$$

From this we deduce that (5.4) and (5.5) $\Longleftrightarrow (ii)$ $c)$ 5) and (ii) $c)$ 6). Let us show that (5.4) and (5.5) $\Longrightarrow (ii)$ $c)$ 4). The left hand side of equation (ii) $c)$ (4) act on an arbitrary element v $\in g^*$, we have

$$\begin{aligned}&-<\mathcal{L}^{1*}_u([x,y]_{g2}), v>-<\mathcal{R}^{1*}_u([x,y]_{g2}), v>+<[x,\mathcal{L}^{1*}_u y]_{g2}, v>+<[x,\mathcal{R}^{1*}_u y]_{g2}, v>\\&-<[y,\mathcal{L}^{1*}_u x]_{g2}, v>-<[y,\mathcal{R}^{1*}_u x]_{g2}, v>+<\mathcal{L}^{1*}_{L^{2*}_y u}x, v>+<\mathcal{R}^{1*}_{L^{2*}_y u}x, v>\\&-<\mathcal{L}^{1*}_{L^{2*}_x u}y, v>-<\mathcal{R}^{1*}_{L^{2*}_x u}y, v>-<\mathcal{L}^{2*}_u([x,y]_{g1}), v>-<\mathcal{R}^{2*}_u([x,y]_{g1}), v>\\&+<[x,\mathcal{L}^{2*}_u y]_{g1}, v>+<[x,\mathcal{R}^{2*}_u y]_{g1}, v>-<[y,\mathcal{L}^{2*}_u x]_{g1}, v>-<[y,\mathcal{R}^{2*}_u x]_{g1}, v>\\&+<\mathcal{L}^{2*}_{L^{1*}_y u}x, v>+<\mathcal{R}^{2*}_{L^{1*}_y u}x, v>-<\mathcal{L}^{2*}_{L^{1*}_x u}y, v>-<\mathcal{R}^{2*}_{L^{1*}_x u}y, v>.\end{aligned}$$

A simple calculation yields

$$
\begin{aligned}
&< \mathcal{L}_u^{1*}([x,y]_{g2}), v >= - < \triangle_1([x,y]_{g2}), u \otimes v > \\
&< \mathcal{R}_u^{1*}([x,y]_{g2}), v >= - < \tau_{12} \triangle_1 ([x,y]_{g2}), u \otimes v > \\
&< [x, \mathcal{L}_u^{1*} y]_{g2}, v > =< y, [u, L_x^{2*} v]_{g*1} >= - < (1 \otimes L_x^2) \triangle_1 y, u \otimes v > \\
&< [x, \mathcal{R}_u^{1*} y]_{g2}, v >=< y, [L_x^{2*} v, u]_{g*1} >=< \tau_{12}(L_x^2 \otimes 1) \triangle_1 y, u \otimes v > \\
&< \mathcal{L}^{1*}_{L_y^{2*} u} x, v >= - < x, [L_y^{2*} u, v]_{g*1} >=< (L_y^2 \otimes 1) \triangle_1 x, u \otimes v > \\
&< \mathcal{R}^{1*}_{L_y^{2*} u} x, v >= - < x, [v, L_y^{2*} u]_{g*1} >=< \tau_{12}((1 \otimes L_y^2) \triangle_1 x, u \otimes v > .
\end{aligned}
$$

By swapping the subscripts 1 and 2 or x and y or both, as appropriate, in the above identities, we calculate the remaining terms. Equation (ii) $c)$ (4) is written as follows

$$
\begin{aligned}
&\triangle_1([x,y]_{g2}) + \tau_{12}(\triangle_1([x,y]_{g2})) + \triangle_2([x,y]_{g1}) + \tau_{12}(\triangle_2([x,y]_{g1})) \\
&\quad = - (1 \otimes L_y^2 + \tau_{12} \circ (L_y^2 \otimes 1) + L_y^2 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^2)) \triangle_1 x \\
&\qquad + (1 \otimes L_x^2 + \tau_{12} \circ (L_x^2 \otimes 1) + \tau_{12} \circ (1 \otimes L_x^2) + (L_x^2 \otimes 1)) \triangle_1 y \\
&\qquad - (1 \otimes L_y^1 + \tau_{12} \circ (L_y^1 \otimes 1) + L_y^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^1)) \triangle_2 x \\
&\qquad + ((1 \otimes L_x^1) + \tau_{12} \circ (L_x^1 \otimes 1) + \tau_{12} \circ (1 \otimes L_x^1) + (L_x^1 \otimes 1)) \triangle_2 y.
\end{aligned}
$$

From another point of view, by (5.5), we have

$$
\begin{aligned}
&\triangle_1([x,y]_{g2}) + \tau_{12}(\triangle_1([x,y]_{g2})) + \triangle_2([x,y]_{g1}) + \tau_{12}(\triangle_2([x,y]_{g1})) \\
&\quad = - (1 \otimes L_y^2 + \tau_{12} \circ (L_y^2 \otimes 1) + L_y^2 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^2)) \triangle_1 x \\
&\qquad + (1 \otimes L_x^2 + \tau_{12} \circ (L_x^2 \otimes 1) + \tau_{12} \circ (1 \otimes L_x^2) + (L_x^2 \otimes 1)) \triangle_1 y \\
&\qquad - (1 \otimes L_y^1 + \tau_{12} \circ (L_y^1 \otimes 1) + L_y^1 \otimes 1 + \tau_{12} \circ (1 \otimes L_y^1)) \triangle_2 x \\
&\qquad + ((1 \otimes L_x^1) + \tau_{12} \circ (L_x^1 \otimes 1) + \tau_{12} \circ (1 \otimes L_x^1) + (L_x^1 \otimes 1)) \triangle_2 y \\
&\qquad - (R_y^2 \otimes 1) \triangle_1 x - (R_y^1 \otimes 1) \triangle_2 x + \tau_{12} \circ (R_x^2 \otimes 1) \triangle_1 y + \tau_{12} \circ (R_x^1 \otimes 1) \triangle_2 y \\
&\qquad + (R_x^2 \otimes 1) \triangle_1 y + (R_x^1 \otimes 1) \triangle_2 y - \tau_{12} \circ (R_y^2 \otimes 1) \triangle_1 x - \tau_{12} \circ (R_y^1 \otimes 1) \triangle_2 x.
\end{aligned}
$$

By (5.4), Equation (ii) $c)$ (4) is true. Similarly, we show the other implications.
$(ii) \iff (iii)$ The representations $(g, (L^1, R^1), (L^2, R^2))$ and $(g^*, (\mathcal{L}^1, \mathcal{R}^1), (\mathcal{L}^2, \mathcal{R}^2))$ are admissible, $(g^*, (L^{1*}, -L^{1*} - R^{1*}), (L^{2*}, -L^{2*} - R^{2*}))$ and $(g, (\mathcal{L}^{1*}, -\mathcal{L}^{1*} - \mathcal{R}^{1*}), (\mathcal{L}^{2*}, -\mathcal{L}^{2*} - \mathcal{R}^{2*}))$ are therefore representations of algebras $(g, [,]_{g1}, [,]_{g2})$ and $(g^*, [,]_{g^*1}, [,]_{g^*2})$, respectively. By Proposition 2.14., $g \oplus g^*$ is equipped with a compatible Leibniz structure. Let ω be the natural bilinear form on $g \oplus g^*$ (see (1.3)). We have, for $x, y, z \in g$ and $u, v \in g^*$

$$
\begin{aligned}
&\omega(x+u, [y+v, z+w]_1) =< u, [y,z]_{g1} > + < u, \mathcal{L}_v^{1*} z > - < u, \mathcal{L}_w^{1*} y > - < u, \mathcal{R}_w^{1*} y > \\
&- < L_y^{1*}(w), x > + < L_z^{1*}(v), x > + < R_z^{1*}(v), x > - < [v,w]_{g*1}, x > \\
&=< u, [y,z]_{g1} > - < [v,u]_{g*1}, z > + < [w,u]_{g*1}, y > + < [u,w]_{g*1}, y > + < w, [y,x]_{g1} > \\
&- < v, [z,x]_{g1} > - < v, [x,z]_{g1} > - < [v,w]_{g*1}, x > .
\end{aligned}
$$

On the other, we have $\omega([x+u, z+w]_1 + [z+w, x+u]_1, y+v)$

$$
\begin{aligned}
&\omega([x+u, z+w]_1 + [z+w, x+u]_1, y+v) \\
&=- < R_z^{1*} u, y > + < [u,w]_{g*1}, y > - < v, [x,z]_{g1} > + < v, \mathcal{R}_w^{1*} x > - < R_x^{1*} w, y > \\
&\quad + < [w,u]_{g*1}, y > - < v, [z,x]_{g1} > + < v, \mathcal{R}_u^{1*} z > \\
&= < u, [y,z]_{g1} > + < [u,w]_{g*1}, y > - < v, [x,z]_{g1} > - < [v,w]_{g*1}, x > + < w, [y,x]_{g1} > \\
&\quad + < [w,u]_{g*1}, y > - < v, [z,x]_{g1} > - < [v,u]_{g*1}, z > .
\end{aligned}
$$

Similarly, we show invariance for $[,]_2$. Hence the result. □

6 Examples for Compatible non-Lie Leibniz algebras.

In this article, we have defined Leibniz algebras by (1.1)*. It is about left Leibniz algebras, but we can always consider the right algebras. In what follows, we study the compatibility of the classes of isomorphisms given in [11]. These are right algebras, left algebras, and symmetric algebras (left and right algebras). Moreover, they are all non-lie. There are nine classes* $(L_i)_{1\leq i\leq 9}$ *where* L_2 *is a family which depends on a complex parameter* α*. The issue used in this study is the property* (1.3) *for left algebras and the similar property for right algebras. We checked these properties and the Leibniz identity (to determine the nature of the structures) for these algebras, two by two, for the elements of the basis. The tables below show the results obtained. The notation used in the following tables is as follows: "*c_{-2}*" means compatibility only for* $\alpha = -2$*, "*c*" compatible, "*nc*" not compatible and "* $-$ *" in the diagonal, means that both structures are trivial, outside the diagonal means that the two structures are of different natures (so compatibility is not defined). Note that the arrays are symmetrical concerning the diagonal.*

Table 1: Left Compatibility, $\mathbb{K} = \mathbb{C}$

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
L_1	-	-	-	-	-	-	-	-	-
L_2		-	-	-	-	-	-	-	-
L_3			-	nc	nc	-	-	-	-
L_4				-	c	-	-	-	-
L_5					-	-	-	-	-
L_6						-	-	-	-
L_7							-	-	-
L_8								-	-
L_9									-

Table 2: Right Compatibility, $\mathbb{K} = \mathbb{C}$

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9
L_1	-	c_{-2}	nc	nc	c	nc	nc	nc	nc
L_2		-	c	nc	c_{-2}	nc	nc	nc	nc
L_3			-	nc	nc	nc	nc	nc	nc
L_4				-	c	nc	nc	nc	nc
L_5					-	nc	nc	nc	nc
L_6						-	c	c	c
L_7							-	c	c
L_8								-	c
L_9									-

Remark 6.1. Note that among the algebras $(L_i)_{1\leq i\leq 9}$, there is only (g, L_4, L_5) which is a compatible left Leibniz algebra.

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