Palestine Journal of Mathematics

Vol 13(3)(2024) , 526–540

SIGN-CHANGING RADIAL SOLUTIONS FOR A SUPERLINEAR DIRICHLET PROBLEM ON EXTERIOR DOMAINS

Boubker Azeroual and Abderrahim Zertiti

Communicated by Amjad Tufaha

MSC 2010 Classifications: Primary 35J66; Secondary 35B05, 35A24.

Keywords and phrases: Superlinear Dirichlet problem, Radial sign-changing solution, Exterior domain.

The authors express their debt of gratitude to the editors and the anonymous referee for accurate reading of the manuscript and beneficial comments.

Corresponding Author: B. Azeroual

Abstract In this paper we are interested in the existence and multiplicity of radial solutions to the elliptic equation $\Delta u(x) + K(|x|)f(u) = 0$ on the exterior of the unit ball centered at the origin in $\mathbb{R}^N$ such that $u(x) \to 0$ as $|x| \to \infty$, with any given number of zeros using fairly straightforward tools of the theory of ordinary differential equations where the nonlinearity $f(u)$ is increasing and superlinear for u large enough. We assume $K(|x|) \sim |x|^{-\alpha}$ for large $|x|$ with $\alpha > 2(N-1)$.

1 Introduction

This paper is concerned with the existence of sign-changing radial solutions for the nonlinear boundary-value problem

$$\Delta u(x) + K(|x|)f(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{in } \partial\Omega\,, \tag{1.2}$$

$$\text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1.3}$$

Where $u : \mathbb{R} \to \mathbb{R}$ and Ω is the complement of the ball of the radius $R > 0$ centered at the origin with $|x|^2 = x_1^2 + \cdots + x_N^2$ is the standard norm of $\mathbb{R}^N$.
We furthermore impose the following assumptions:
(H1) $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian such that

$$f(0) = 0, \quad \text{and } \lim_{s\to 0} \sup \frac{f(s)}{s} < 0\,.$$

(H2) $u \to f(u)$ is increasing for $|u|$ large enough and f is superlinear at infinity, i.e

$$\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty\,. \tag{1.4}$$

From **(H1)** and (1.4) we see that f has at least one positive and one negative zero.
(H3) Let β_0^+ (resp. β_1^-) be the least positive (resp. negative) zero of f and β_1^+ (resp. β_0^-) be the greatest positive (resp. negative) zero of f where

$$\beta_1^- \le \beta_0^- < 0 < \beta_0^+ \le \beta_1^+\,.$$

(H4) $u \to F(u) = \int_0^u f(t)dt$ has exactly two zeros γ^-, γ^+ not both 0 such that $\gamma^- < 0 < \gamma^+$ and

$$F < 0 \quad \text{on all } (\gamma^-, \gamma^+) - \{0\}\,.$$

(H5) Furthermore we assume that $r \to K(r)$ is C^1 on $[R, \infty)$ and there are three positive constants α, k_0 and k_1 such that,

$$k_0\, r^{-\alpha} \le K(r) \le k_1\, r^{-\alpha} \quad \text{for any } r \ge R\,, \tag{1.5}$$

$$2(N-1) + \frac{r\,K'}{K} < 0 \quad \text{for any } r \ge R\,, \tag{1.6}$$

where $\alpha > 2(N-1)$ and $N > 2$.

Remark 1.1.

(i) From **(H1)–(H3)** we see that $f < 0$ on $(0, \beta_0^+) \cup (-\infty, \beta_1^-)$ and $f > 0$ on $(\beta_0^-, 0) \cup (\beta_1^+, \infty)$.

(ii) From **(H1)–(H4)** $F > 0$ on $(-\infty, \gamma^-) \cup (\gamma^+, \infty)$ also, $\gamma^+ > \beta_1^+$ and $\gamma^- < \beta_1^-$.

(iii) By (1.4) and Hospital's rule we assert that

$$\lim_{|u| \to \infty} \frac{F(u)}{u^2} = \infty\,. \tag{1.7}$$

(iv) Consequently, it follows that there is $F_0 > 0$ such that

$$F(u) \ge -F_0 \quad \text{for all } u \in \mathbb{R}\,. \tag{1.8}$$

(v) At first, we can assume that f is odd and $\beta_i^- = -\beta_i^+$ (for $i = 0, 1$), although we provide proofs for this in the general case.

Theorem 1.2. *If* (H1)–(H5) *are satisfied, then* (1.1)-(1.3) *has infinitely many radially symmetric solutions. In addition, for each integer n there exists a radially symmetric solution of problem* (1.1)-(1.3) *which has exactly n zeros.*

The existence of radial solutions to the superlinear Dirichlet problem (1.1) when $K(r) = 1$ on different domains (bounded domain or $\Omega = \mathbb{R}^N$) has been extensively studied. Most of these results are obtained through variational, sub-solutions and super-solutions, dynamical methods and the computation of the angular velocity in the phase plane, we see for example [2],[3],[14],[16],[17],[6],[7],[13] et [9]. Recently there has been an interest in studying these problems on exterior domains we see [4],[5],[17] et [15]. In particular, when the nonlinearity is odd and has one positive zero and $f(u) \sim_\infty u|u|^{p-1}$,$p > 1$ the author Iaia in [11] and [12] proves the existence of infinitely many radial solutions of (1.1)- (1.3) by using a scaling argument.
Here we deal with the Dirichlet problem (1.1)- (1.3) in a more general case of superlinearity of $f(u)$ using fairly simple tools from the theory of ordinary differential equations and an approach more recent to prove the existence and multiplicity of radial solutions with change of sign inspired by our paper [1].

Our paper is organized as follows: In Section 2 we begin by establishing some preliminary results concerning the existence of radial solutions of (1.1)- (1.2) by making the change of variable $u(r) = v(t)$ with $t = r^{2-N}$ and transforming our problem to the compact set $[0, T]$ where $T = R^{2-N}$ and then we study the new initial-value problem by using the shooting argument with $v'(t) = -p < 0$. The rest of section two is devoted to showing that $v_p = v(t)$ stays positive if $p > 0$ stays sufficiently small and to assert that the energy function associated to (2.4) is nonincreasing and positive on all $[0, T]$. In Section 3 we obtain the nature of zeros of solution v_p also, we show that v_p has a large number of zeros for p sufficiently large. In section 4 we prove the main theorem by choosing appropriate values of the parameter $p > 0$ such that v_p is a solution of (2.4)- (2.5) with exactly n zeros on $(0, T)$ for each nonnegative integer n and $v_p(0) = 0$. Hence by converting to the famous change of variable, we obtain a solution of our original problem (1.1)- (1.3) with exactly n zeros on (R, ∞). Lastly in section 5 we do some simulations by using Mathlab for an example of the generalized Matukuma equation, see [8].

2 Preliminaries

Since we are interested in radial solutions of (1.1)- (1.3) we denote $r = |x|$ and $u(x) = u(r)$ satisfies

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)\,f(u) = 0, \tag{2.1}$$

$$u(R) = 0 \quad \text{and} \quad \lim_{r\to\infty} u(r) = 0. \tag{2.2}$$

Now, we employ the following transformation

$$t = r^{2-N} \quad \text{and } u(r) = v(t)\,. \tag{2.3}$$

It then follows that the initial value problem (2.1)-(2.2) is converted to

$$v''(t) + h(t)\,f(v) = 0 \quad \text{if } 0 < t < T, \tag{2.4}$$

$$v(T) = 0 \quad \text{and} \quad \lim_{t\to 0} v(t) = 0 \tag{2.5}$$

where $T = R^{2-N}$ and

$$h(t) = \Big(\frac{1}{N-2}\Big)^2\, t^{-\frac{2(N-1)}{N-2}}\, K(t^{-\frac{1}{N-2}})\,.$$

Furthermore from (1.5) we get

$$h_0\, t^{\mu} \le h(t) \le h_1\, t^{\mu} \quad \text{on } (0,T]\,, \tag{2.6}$$

where $\mu = \frac{2(N-1)-\alpha}{N-2}$, $h_0 = \frac{k_0}{(N-2)^2} > 0$ and $h_1 = \frac{k_1}{(N-2)^2} > 0$.
Notice that, since $\alpha > 2(N-1)$ then $\mu > 0$ which implies that $\lim_{t\to 0^+} h(t) = 0$ and consequently h is continuous on $[0,T]$. In addition, from **(H5)** we have that h is C^1 on $(0,T]$ and also

$$h'(t) = -\frac{t^{-\frac{3N-4}{N-2}}\, K(t^{-\frac{1}{N-2}})}{(N-2)^3}\Big[2(N-1) + t^{-\frac{1}{N-2}}\,\frac{K'(t^{-\frac{1}{N-2}})}{\varphi(t^{-\frac{1}{N-2}})}\Big] > 0\,,$$

which means that h is strictly increasing.

To solve (2.4)-(2.5), we apply the shooting method, by considering the following initial value problem

$$v''(t) + h(t)\,f(v) = 0 \quad \text{if} \quad 0 < t < T, \tag{2.7}$$

$$v(T) = 0 \quad \text{and} \quad v'(T) = -p\,. \tag{2.8}$$

We will occasionally write $v = v_p(t)$ to emphasize the dependence of the solution on parameter $p > 0$. As this initial value problem is not singular so, the existence, uniqueness and continuous dependence with respect to p of the solution of (2.7)-(2.8) on $[T-\epsilon, T]$ for some $0 < \epsilon < T$ follows by the standard existence-uniqueness and dependence theorem for ordinary differential equations [10].

For v_p solution of (2.7)-(2.8) we define the energy function as follows

$$\mathcal{E}_p(t) = \frac{v_p'^2}{2\,h(t)} + F(v_p)\,. \tag{2.9}$$

A simple calculation by using (2.4) shows that

$$\mathcal{E}_p'(t) = -\frac{h'(t)\,v_p'^2(t)}{2h^2(t)}. \tag{2.10}$$

As h is strictly increasing it then follows the energy $\mathcal{E}_p$ is strictly decreasing.
From (2.10) we get

$$\big(h(t)\,\mathcal{E}_p(t)\big)' = h'(t)F(v_p).$$

Integrating this from t to T and using (2.9)-(2.8) gives

$$\frac{v_p'^2(t)}{2} + h(t)\,F(v_p) = \frac{p^2}{2} - \int_t^T h'(x)F(V_p)\,dx.$$

From (1.8), since h' and h are positive we assert that

$$v_p'^2(t) \le p^2 + 2\,F_0 h(T)\,.$$

Thus, by using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for all positive reals a and b we see that

$$|v_p'(t)| \le c_{1,p} = p + \sqrt{2\,F_0 h(T)}\,. \tag{2.11}$$

Also we apply the mean value theorem with the initial conditions we get

$$|v_p(t)| \le T\,c_{1,p} = c_{2,p}\,. \tag{2.12}$$

Thus v_p and v_p' are bounded on wherever they are defined. For $p > 0$ fixed it then follows that there is a unique solution v_p of (2.7)-(2.8) defined on all $[0, T]$. Finally, by (2.7) it thus follows that v_p'' is bounded on all $[0, T]$ and consequently, the solution v_p is class C^1 on $[0, T]$.

Lemma 2.1. *Let v_p be a solution of (2.7)-(2.8). Then $v_p(t) > 0$ on $(0, T]$ if p is sufficiently small.*

Proof. As $v_p'(T) = -p < 0$ so either,

$$\begin{cases} \text{Case } (A) & v_p'(t) < 0 \quad \text{on all} \quad t \in (0, T]\,, \\ \text{Case } (B) & v_p \text{ has a local maximum at some } M_p \in (0, T). \end{cases}$$

If (A) holds. Since v_p is strictly decreasing we get $v_p(t) > v_p(T) = 0$ on $(0, T]$ and so we are done in this case.
We then consider the case (B) and we will firstly claim that, if p is sufficiently small we have

$$0 < v_p < \beta_0^+ \quad \text{on } (0, T].$$

Indeed, if not we suppose that for any $p > 0$ sufficiently small there is $\tau_p \in (M_p, T)$ such that $v_p(\tau_p) = \beta_0^+$ and $v_p' < 0$ on (τ_p, T).
Let $t \in [\tau_p, T]$. By integrating (2.7) from t to T with the initial conditions (2.8) yields

$$v_p'(t) + p = \int_t^T h(x)f(v_p)\,dx\,. \tag{2.13}$$

By integrating again this from t to T we get

$$v_p(t) = p(T-t) - \int_t^T \Big(\int_s^T h(x)f(v_p)\,dx\Big)ds\,. \tag{2.14}$$

Notice that by condition **(H1)** there is $c_3 > 0$ such that

$$f(u) \ge -c_3\,u \quad \text{for all } u \in \mathbb{R}^+\,.$$

Since $v_p > 0$ is strictly decreasing on $[\tau_p, T]$ and using (2.14) it then follows that

$$v_p(t) \le pT + c_3 \int_t^T H(s)\,v_p(s)\,ds\,,$$

where $H(t) = \int_t^T h(x)\,dx$ is a continuous and positive function on $[0, T]$. We can apply the Gronwall inequality [10] it thus follows that

$$v_p(t) \le pT\,exp\big(c_3 \int_t^T H(x)\,dx\big). \tag{2.15}$$

We observe that the function $t \to exp\big(c_3 \int_t^T H(x)\,dx\big) > 0$ is bounded above by some $c_4 > 0$ on $[0,T]$. Thus by taking $t = \tau_p$ in (2.15) we get

$$0 < v_p(\tau_p) = \beta_0^+ \le c_4\, p.$$

By making $p \to 0^+$ in this we get $\beta_0^+ = 0$. This is a contradiction and consequently, we have $0 < v_p < \beta_0^+$ on $(0,T]$ for p sufficiently small. Which mean that $v_p(t) > 0$ on $(0,T]$ if p is sufficiently small. □

Lemma 2.2. *Let v_p be a solution of (2.7)-(2.8). Then v_p has a local maximum M_p on $(0,T)$ if p is sufficiently large. In addition,*

$$M_p \to T \quad \text{as } p \to \infty\,, \tag{2.16}$$

$$\text{and } v_p(M_p) \to \infty \quad \text{as } p \to \infty\,. \tag{2.17}$$

Proof. From the above discussion at the beginning to the proof of lemma 2.1, we will assert that the case (A) does not occurs, if $p > 0$ is large enough.
To the contrary we suppose that $v_p' < 0$ on $(0,T]$ for any $p > 0$ large enough, which implying that

$$v_p > 0 \quad \text{on } (0,T) \text{ for all } p > 0 \text{ sufficiently large.} \tag{2.18}$$

Let us $0 < T_1 < T$ is fixed. Firstly, we claim the following result

$$v_p(T_1) \to \infty \quad \text{as } p \to \infty. \tag{2.19}$$

Indeed, by contradiction we suppose that there exists $m > 0$ such that $0 < v_p(T_1) \le m$ for all $p > 0$ large enough. From continuity of f so, there is $M > 0$ such that $0 < f\big(v_p(t)\big) \le M$ on $[T_1, T]$ for p large enough.
From (2.13)-(2.6) it follows that

$$v_p'(t) + p = \int_t^T h(x) f\big(v_p(x)\big)\,dx \le \frac{M\,h_1 T^{\mu+1}}{\mu+1}.$$

Integrating this on (T_1, T) with initial conditions (2.8) gives

$$p\,\big(T - T_1\big) - \frac{M\,h_1 T^{\mu+2}}{\mu+1} \le v_p(T_1). \tag{2.20}$$

Notice that the left-hand side of (2.20) goes to infinity as $p \to \infty$ but the one on the right-hand side is bounded above. This is a contradiction and (2.19) is proven .

Secondly, by (2.19) for p sufficiently large we have

$$v_p(T_1) > \beta_1^+. \tag{2.21}$$

Let us fixed $p > 0$ and $0 < T_0 < T_1$ we denote

$$\Omega_p = \inf_{T_0 \le t \le T_1} \{h(t)\,\frac{f(v_p)}{v_p}\}\,.$$

From (2.21) by using (i) of the remark 1.1 and the fact that v_p is strictly decreasing it follows then that $v_p > \beta_1^+$ and $f(v_p) > 0$ on $[T_0, T_1]$. As h is positive and is increasing we see that

$$\Omega_p \ge h(T_0) \inf_{v_p(T_1) \le x \le v_p(T_0)} \left\{\frac{f(x)}{x}\right\} \quad \text{for } p \text{ sufficiently large.} \tag{2.22}$$

Combining (2.19)-(2.22) with the superlinearity of f we obtain

$$\Omega_p \to \infty \quad \text{as } p \to \infty\,. \tag{2.23}$$

It is well known the eigenvectors of the operator $-\frac{d^2}{dt^2}$ in (T_0, T_1) with Dirichlet boundary conditions can be chosen as

$$\psi_k(t) = \sqrt{\frac{2}{T_1 - T_0}}\,\sin\Big(\frac{k\pi(t - T_0)}{T_1 - T_0}\Big)\,,$$

of eigenvalues $\lambda_k = \big(\frac{k\,\pi}{T_1 - T_0}\big)^2$ where k is nonnegative integer. Also, $z = \frac{T_0 + T_1}{2}$ is a zero of the second eigenfunction ψ_2 on (T_0, T_1). In addition, from (2.23) therefore for suitable large $p > 0$ it follows that $\Omega_p > \lambda_2$. This allows us to apply the Sturm comparison theorem [10] and consequently, v_p has at least one zero in (T_0, T_1) which contradicts to (2.18). Hence, v_p has a local maximum at some $M_p \in (0, T)$ for p sufficiently large.

Next, we will to claim (2.16). We argue by contradiction. We suppose that there is an $\epsilon > 0$ for all p sufficiently large we have that

$$M_p \le T - \epsilon = T_\epsilon. \tag{2.24}$$

Let us denote $T^* = \frac{T + T_\epsilon}{2}$. By a similar way in the previous proof with assuming that $T_1 = T^*$ and $T_0 = T_\epsilon$ and using the fact that $V_p > 0$ is nonincreasing on (T_ϵ, T^*) we can show that v_p has at least one zero on $[T_\epsilon, T^*]$. This is a contradiction. Hence, $M_p \to T$ as $p \to \infty$.
Now, since $\mathcal{E}_p$ is nonincreasing we see that

$$\mathcal{E}_p(t) \ge \frac{p^2}{2\,h(T)} > 0 \quad \text{for all } t \in (0, T] \quad \text{and } \mathcal{L}_p = \lim_{t \to 0^+} \mathcal{E}_p(t) > 0. \tag{2.25}$$

and consequently,

$$\mathcal{L}_p = \inf_{t \in [0,T]} \mathcal{E}_p(t) \to \infty \quad \text{as } p \to \infty\,. \tag{2.26}$$

In particular we have that $\mathcal{E}_p(M_p) = F\big(v_p(M_p)\big) \to \infty$ as $p \to \infty$. Using (1.7) we see that $F(u) \to \infty$ as $u \to \infty$ which implies that, $v_p(M_p) \to \infty$ as $p \to \infty$. This completes the proof of Lemma 2.2. □

3 Solution with a prescribed number of zeros

In this section we show that the solution v_p of initial value problem (2.4)-(2.5) has a large number of zeros for p sufficiently large.

Lemma 3.1. *The solution v_p of* (2.4)-(2.5) *has,*

(i) only simple zero,

(ii) a finite number of zeros.

Proof. (i) Suppose there is some point $t_0 \in [0, T]$ such that $v_p(t_0) = v_p'(t_0) = 0$ which implies that $\mathcal{E}_p(t_0) = 0$. This is a contradiction to (2.25). Thus v_p has only simple zero on $[0, T]$.
(ii) By contradiction, we suppose an infinite number of zeros of v_p denoted $z_n < z_{n+1}$ on $[0, T]$. So, there is a subsequence (again label z_n) of (z_n) such $z_n \to z \in [0, T]$ as $n \to \infty$. Then, by mean value theorem there is a local extrema $z_n < m_n < z_{n+1}$ and also $m_n \to z$ as $n \to \infty$. Therefore, taking $n \to \infty$ gives $v_p(z) = v_p'(z) = 0$ which contradicts (i). □

Now, by (i) and (ii) of remark 1.1 it follows that $F(u) > 0$ for all $|u| > \min(-\gamma^-, \gamma^+)$ and F is increasing on (γ^+, ∞) and is decreasing on $(-\infty, \gamma^-)$. Since $\mathcal{L}_p > 0$ for any $p > 0$ we assert that the equation $F(u) = \frac{\mathcal{L}_p}{2}$ has exactly two solutions $\sigma_p^- < \gamma^-$ and $\sigma_p^+ > \gamma^+$ such that

$$F(\sigma_p^\pm) = \frac{1}{2} \inf_{t \in [0,T]} \mathcal{E}_p(t) > 0\,. \tag{3.1}$$

From (2.26) and since $F(u) \to +\infty$ as $u \to \pm\infty$ we see that

$$\lim_{p \to +\infty} \sigma_p^+ = +\infty \quad \text{and } \sigma_p^+ > \gamma^+, \tag{3.2}$$

$$\lim_{p \to +\infty} \sigma_p^- = -\infty \quad \text{and } \sigma_p^- < \gamma^-. \tag{3.3}$$

By Lemma 2.2, we see that v_p has a local maximum M_p on $(0,T)$ if p is sufficiently large and $M_p \to T$ as $p \to \infty$.

Lemma 3.2. *For p large enough there is $t_p \in (0, M_p)$ such that,*

$$v_p(t_p) = \sigma_p^+ \quad \textit{and } \sigma_p^+ < v_p \leq v_p(M_p) \quad \textit{on } (t_p, M_p],$$

and

$$t_p \to T \quad \textit{as} \;\; p \to \infty. \tag{3.4}$$

Proof. By contradiction, we suppose that, for all p sufficiently large

$$\sigma_p^+ < v_p(t) \quad \text{on } (0, M_p]. \tag{3.5}$$

Integrating (2.4) from t to M_p gives,

$$v_p'(t) = \int_t^{M_p} h(x) f(v_p)\, dx\,. \tag{3.6}$$

Now let $s \in (0, M_p)$ is fixed. By integrating (3.6) over $\left(M_p - s, M_p - \frac{s}{2}\right)$ we obtain

$$v_p(M_p - \frac{s}{2}) = v_p(M_p - s) + \int_{M_p - s}^{M_p - \frac{s}{2}} \Big(\int_t^{M_p} h(x)\, f(v_p)\, \mathrm{d}x\Big)\, \mathrm{d}t. \tag{3.7}$$

By (3.2) and (3.5) we see that $v_p(t) > \beta_1^+$ on $(0, M_p)$ if p is sufficiently large, which implies that $f\big(v_p(t)\big) > 0$ on $(0, M_p)$. From (3.6) we deduce that v_p is increasing on all $(0, M_p)$ for p large enough. Since $u \to f(u)$ is increasing for $|u|$ large enough and using (3.2) it then follows that

$$f\big(v_p(x)\big) \geq f\big(v_p(M_p - \frac{s}{2})\big) \quad \forall x \in \big(M_p - \frac{s}{2}, M_p\big) > 0.$$

Multiplying this by $h > 0$ and integrating the resultant on (t, M_p) and using the fact that h is increasing gives

$$\begin{aligned}\int_t^{M_p} h(x) f(v_p)\, dx &\geq \int_{M_p - \frac{s}{2}}^{M_p} h(x) f(v_p)\, dx \\ &\geq \frac{s}{2}\, h(M_p - \frac{s}{2})\, f\big(v_p(M_p - \frac{s}{2})\big).\end{aligned}$$

By integrating this on $\big(M_p - s, M_p - \dfrac{s}{2}\big)$ and using (3.7) it then follows that

$$v_p(M_p - \frac{s}{2}) \geq \frac{s^2}{4}\, h(M_p - \frac{s}{2})\, f\big(v_p(M_p - \frac{s}{2})\big) \quad \forall\, s \in (0, M_p). \tag{3.8}$$

Taking $s = M_p$ and dividing by $f\big(v_p(\frac{M_p}{2})\big) > 0$ in (3.8) it follows that

$$\frac{v_p(\frac{M_p}{2})}{f\big(v_p(\frac{M_p}{2})\big)} \geq \frac{M_p^2}{4}\, h(\frac{M_p}{2}) \quad \text{if } p \text{ is sufficiently large.} \tag{3.9}$$

Since $v_p(\frac{M_p}{2}) > \sigma_p^+$ and using (3.2)-(1.4) we deduce that the left-side hand of (3.9) goes to 0 as $p \to \infty$. But from (2.16) the right-side hand (3.9) converges to $\frac{T^2}{4}\, h(T) \neq 0$ as $p \to \infty$. This is a contradiction. Hence, for p large enough there is $t_p \in (0, M_p)$ such that,

$$v_p(t_p) = \sigma_p^+ \quad \text{and } \sigma_p < v_p \leq v_p(M_p) \quad \text{on } (t_p, M_p]\,.$$

Next, we will show (3.4). Since $(t_p)_p$ is bounded, so a subsequence (again label $(t_p)_p$) such that $t_p \to t_* \in [0, T]$ as $p \to \infty$. Firstly, we claim that $t_* \neq 0$.
Otherwise, we suppose that $t_* = 0$. Since $M_p \to T$ as $p \to \infty$ so, for any p is sufficiently large

we have $t_p < \frac{T}{4} < \frac{T}{2} < M_p$. Since v_p is increasing and $v_p > \sigma_p^+ > \gamma^+ > \beta_1^+$ on $(\frac{T}{4}, \frac{T}{2})$ for p sufficiently large then by integrating (3.6) on $(\frac{T}{4}, \frac{T}{2})$ we get

$$\begin{aligned} v_p(\frac{T}{2}) &= v_p(\frac{T}{4}) + \int_{\frac{T}{4}}^{\frac{T}{2}} \Big(\int_t^{M_p} h(x)\, f(v_p)\, \mathrm{d}x \Big)\, \mathrm{d}t\,, \\ &\geq \int_{\frac{T}{4}}^{\frac{T}{2}} \Big(\int_t^{M_p} h(x)\, f(v_p)\, \mathrm{d}x \Big)\, \mathrm{d}t \quad (v_p(\frac{T}{2}) > 0)\,, \\ &\geq \int_{\frac{T}{4}}^{\frac{T}{2}} \Big(\int_{\frac{T}{2}}^{M_p} h(x)\, f(v_p)\, \mathrm{d}x \Big)\, \mathrm{d}t \quad (f(v_p) > 0\,, h > 0)\,, \\ &\geq \frac{T}{4}\, (M_p - \frac{T}{2})\, h(\frac{T}{2})\, f\big(v_p(\frac{T}{2})\big) \quad (h \text{ and } f \text{ are increasing})\,. \end{aligned}$$

Thus,

$$\frac{v_p(\frac{T}{2})}{f\big(v_p(\frac{T}{2})\big)} \geq \frac{T}{4}\, (M_p - \frac{T}{2})\, h(\frac{T}{2}) > 0 \quad \text{for } p \text{ large enough.} \tag{3.10}$$

Since $v_p(\frac{T}{2}) \to \infty$ as $p \to \infty$ and by (1.4) (superlinearity of f) we have that the left-side hand of (3.10) goes to 0 which implies that $M_p \to \frac{T}{2}$ as $p \to \infty$. This is a contradiction and consequently, $t_* \in (0, T]$.
Secondly, we will prove that $t_* = T$. Denoting

$$C_p = \frac{1}{2} \min_{t \in [t_p, M_p]} \big\{ h(t)\, \frac{f(v_p)}{v_p} \big\}.$$

As $f\big(v_p(t)\big) > 0$ for p large enough on $[t_p, M_p]$ and h and v_p are increasing on $[t_p, M_p]$ it then follows

$$C_p \geq h(t_p) \min_{u \in [\sigma_p, v_p(M_p)]} \frac{f(u)}{u}\,. \tag{3.11}$$

By continuity of h and $t_p \to T$ as $p \to \infty$ we see that $h(t_p) \to h(T) > 0$ as $p \to \infty$. From (3.2)-(1.4) it then follows that the right-side hand of (3.11) goes to infinity and consequently,

$$\lim_{p \to +\infty} C_p = +\infty. \tag{3.12}$$

We now compare the problem

$$v_p''(t) + h(t) \big\{ \frac{f(v_p)}{v_p} \big\}\, v_p = 0\,, \tag{3.13}$$

with

$$y''(t) + C_p\, y = 0\,, \tag{3.14}$$

with the initial conditions

$$v_p(M_p) = y(M_p) \quad \text{and} \quad v_p'(M_p) = y'(M_p) = 0. \tag{3.15}$$

From (3.13)-(3.15) we have that $v_p''(M_p) = -\big(h(M_p)\, \frac{f\big(v_p(M_p)\big)}{v_p(M_p)}\big) v_p(M_p) \leq -C_p\, y(M_p) = y''(M_p)$. And there is $\eta >$ such that $\big(v_p - y\big)'' < 0$ on $(M_p - \eta, M_p)$. Which implies that $\big(v_p - y\big)' > 0$ on $(M_p - \eta, M_p)$ and consequently, $v_p < y$ on all $(M_p - \eta, M_p)$. Denoting

$$\tau = \inf \big\{ t \in (t_p, M_p) : v_p < y \quad \text{on } (t, M_p) \big\}\,.$$

Next, we will show that $\tau = t_p$ for p sufficiently large. Otherwise, we suppose that

$$v_p < y \quad \text{on } (\tau, M_p] \quad \text{and } v_p(\tau) = y(\tau)\,.$$

From (3.2) and (2.17) if p is sufficiently large we see that

$$C_p > 0 \quad \text{and } v_p(M_p) > 0\,, \tag{3.16}$$

and by using (3.13)-(3.14) we get

$$\big(v_p'\,y - v_p\,y'\big)' = -v_p\,y\Big(h(t)\frac{f(v_p)}{v_p} - C_p\Big). \tag{3.17}$$

Integrating this on (t, M_p) and using the initial conditions, gives

$$v_p'(t)\,y(t) - v_p(t)\,y'(t) = \int_t^{M_p} v_p\,y\Big(h(x)\frac{f(v_p)}{v_p} - C_p\Big)dx. \tag{3.18}$$

From (3.16) for p large enough it then follows that

$$v_p\,y > 0, \quad h(x)\frac{f(v_p)}{v_p} - C_p \geq 2C_p - C_p > 0 \quad \text{on } [\tau, M_p].$$

Consequently, $v_p'(t)\,y(t) - v_p(t)\,y'(t) > 0$ on $[\tau, M_p]$.
In particular, for $t = \tau$ we obtain $v_p'(\tau) < y'(\tau)$ and since $v_p(t) - v_p(\tau) < y(t) - y(\tau)$ for all $t \in (\tau, M_p)$, it follows that

$$v_p'(\tau) = \lim_{t\to\tau^+}\frac{v_p(t) - v_p(\tau)}{t-\tau} \leq \lim_{t\to\tau^+}\frac{y(t)-y(\tau)}{t-\tau} = y'(\tau)\,.$$

Which contradicts to $v_p'(\tau) < y'(\tau)$. Hence, $\tau = t_p$ and $v_p < y$ on (t_p, M_p) for p large enough.
Finally, we know that every interval of length $\frac{\pi}{\sqrt{C_p}}$ contains at least one zero of $y(t)$ and $y > v_p > 0$ on $(t_p, M_p]$ it then follows that,

$$M_p - \frac{\pi}{\sqrt{C_p}} < t_p < M_p \quad \text{if } p \text{ is sufficiently large.} \tag{3.19}$$

Since $M_p \to T$ as $p \to \infty$ consequently, $t_p \to T$ as $p \to \infty$ which completes the proof of Lemma 3.2. □

Lemma 3.3. *For p sufficiently large v_p has a first zero $z_{1,p}$ on $(0, T)$. In addition,*

$$\lim_{p\to\infty} z_{1,p} = T. \tag{3.20}$$

Proof. We argue by contradiction. We suppose that $v_p > 0$ on $(0, T)$. Since $0 < v_p(t) < v_p(t_p) = \sigma_p^+$ on $(0, t_p)$ it follows that $F\big(v_p\big) < F(\sigma_p^+) = \frac{1}{2}\inf_{t\in[0,T]}\mathcal{E}_p(t)$ which implies that $F\big(v_p\big) < \frac{v_p'^2}{2\,h(t)}$. Thus

$$\sqrt{2\,h(t)\,F\big(\sigma_p^+\big)} < |v_p'| = v_p' \quad (0, t_p). \tag{3.21}$$

Integrating this on $(0, t_p)$ and using (2.6) it then follows that

$$\frac{2\sqrt{2h_0}}{2+\mu}\big(t_p^{\frac{2+\mu}{2}} - t^{\frac{2+\mu}{2}}\big)\sqrt{F\big(\sigma_p^+\big)} < \sigma_p^+ - v_p(t).$$

Since $v_p(t) > 0$ we deduce that

$$\frac{2\sqrt{2h_0}}{2+\mu}\big(t_p^{\frac{2+\mu}{2}} - t^{\frac{2+\mu}{2}}\big) < \frac{\sigma_p^+}{\sqrt{F\big(\sigma_p^+\big)}}. \tag{3.22}$$

By making $t = 0$ in (3.22) we get

$$\frac{2\sqrt{2h_0}}{2+\mu}\,t_p^{\frac{2+\mu}{2}} \leq \frac{\sigma_p^+}{\sqrt{F\big(\sigma_p^+\big)}}. \tag{3.23}$$

Combining (3.2)-(1.7) we obtain

$$\lim_{p\to+\infty} \frac{\sigma_p^+}{\sqrt{F(\sigma_p^+)}} = 0\,. \tag{3.24}$$

Consequently, the left-side hand of (3.23) goes to $\frac{2\sqrt{2\,h_0}}{2+\mu}\,T^{\frac{2+\mu}{2}} \neq 0$ as $p \to \infty$. This is a contradiction. Hence, there is $z_{1,p}$ the first zero of v_p on $(0, T)$.
Now, making $t = z_{1,p}$ in (3.22) we get

$$0 < \frac{2\sqrt{2\,h_0}}{2+\mu}\,(t_p^{\frac{2+\mu}{2}} - z_{1,p}^{\frac{2+\mu}{2}}) < \frac{\sigma_p^+}{\sqrt{F(\sigma_p^+)}}\,. \tag{3.25}$$

Since $t_p \to T$ as $p \to \infty$ and by using (3.24)-(3.25) it thus follows that $\lim_{p\to+\infty} z_{1,p} = T$. □

Lemma 3.4. *For p sufficiently large, the solution v_p has a local minimum at $m_p \in (0, t_p)$ and moreover $m_p \to T$ as $p \to \infty$.*

Proof. We begin by establishing the following claim.
Claim: v_p attains the value σ_p^- at some $s_p \in (0, z_{1,p})$ if p is sufficiently large. In addition,

$$\lim_{p\to\infty} s_p = T\,. \tag{3.26}$$

Indeed, if not we suppose that $v_p(t) > \sigma_p^-$ on $(0, z_{1,p})$ for all p large enough. Since v_p has only simple zeros therefore $v_p'(z_{1,p}) > 0$ and $v_p' > 0$ on a maximum interval $(m^*, z_{1,p})$ for p sufficiently large. Consequently, we have $2\,F\big(v_p\big) < \frac{v_p'^2}{2\,h(t)} + F(v_p)$ and $F\big(v_p\big) < \frac{v_p'^2}{2\,h(t)}$ on $(m^*, z_{1,p})$. Thus,

$$\sqrt{2\,h(t)\,F\big(\sigma_p^-\big)} < |v_p'| = v_p' \quad (m^*, z_{1,p}). \tag{3.27}$$

Letting $t = m^*$ in (3.27) we get $\sqrt{2\,h(m^*)\,F\big(\sigma_p^-\big)} \leq 0 = v_p'(m^*)$. Which implies that $m^* = 0$ and v_p is strictly increasing on $(0, z_{1,p})$ for p large enough.
By integrating (3.27) on $(t, z_{1,p})$ and using (2.6) gives

$$0 < \frac{2\sqrt{2\,h_0}}{2+\mu}\,(z_{1,p}^{\frac{2+\mu}{2}} - t^{\frac{2+\mu}{2}}) < \frac{\sigma_p^-}{\sqrt{F\big(\sigma_p^-\big)}}\,. \tag{3.28}$$

Letting $t = 0$ in (3.28) and using (3.20) we assert that the left-side hand of (3.28) goes to $\frac{2\sqrt{2\,h_0}}{2+\mu}\,T^{\frac{2+\mu}{2}}$ as $p \to \infty$. Similarly as in (3.24) by using (3.3) and (1.7) we see that the right-side hand of (3.28) converges to 0 as $p \to \infty$ which is a contradiction. Finally by taking $t = s_p$ in (3.28) and using (3.20)-(3.24) it follows that $s_p \to T$ as $p \to \infty$ and the claim is proven.

Next, we will prove the Lemma 3.4. Again by contradiction, we suppose that v_p is strictly increasing on $(0, z_{1,p})$ for all p large enough. Therefore we get $v_p < \sigma_p^- < 0$ on $(0, s_p)$.
Since $s_p \to T$ as $p \to \infty$ then for p large enough we have that $\frac{T}{2} < s_p$. Denoting

$$C_p = \frac{1}{2} \min_{t\in[\frac{T}{2}, s_p]} \{h(t)\,\frac{f(v_p)}{v_p}\}.$$

Since h and v_p are increasing on $[\frac{T}{2}, s_p]$ it then follows

$$C_p \geq \frac{1}{2} h(\frac{T}{2}) \min_{u\leq\sigma_p^-} \{\frac{f(u)}{u}\}\,, \tag{3.29}$$

for p large enough. Consequently by using (1.4) and (3.3) the right-side hand of (3.29) goes to infinity which implies that

$$\lim_{p\to+\infty} C_p = +\infty. \tag{3.30}$$

We compare the problem

$$v_p''(t) + h(t)\{\frac{f(v_p)}{v_p}\}\, v_p = 0\,, \tag{3.31}$$

with

$$y''(t) + C_p\, y = 0\,, \tag{3.32}$$

where $t \in (\frac{T}{2}, s_p)$ and the initial conditions

$$v_p(s_p) = y(s_p) \quad \text{and} \quad v_p'(s_p) = y'(s_p). \tag{3.33}$$

From (3.31)-(3.33) and since $v_p(s_p) < 0$ we have

$$v_p''(s_p) = -h(s_p)\, f\big(v_p(s_p)\big) \geq -C_p\, y(s_p) = y''(s_p).$$

And by continuity there is $\epsilon >$ such that $\big(v_p - y\big)'' > 0$ on $(\frac{T}{2} - \epsilon, s_p)$. Which implies that $\big(v_p - y\big)' < 0$ on $(\frac{T}{2} - \epsilon, s_p)$ and consequently, $v_p > y$ on all $(\frac{T}{2} - \epsilon, s_p)$. Denoting

$$\zeta = \inf\big\{t \in (\frac{T}{2} - \epsilon, s_p) : v_p > y \quad \text{on } (t, s_p)\big\}\,.$$

We will show that $\zeta = \frac{T}{2}$. If not, suppose that

$$v_p(\zeta) = y(\zeta) \quad \text{and } v_p(t) > y(t) \quad \text{for any } t \in (\zeta, s_p).$$

For $\zeta < t < s_p$ we have $\frac{v_p(t) - v_p(\zeta)}{t - \zeta} > \frac{y(t) - y(\zeta)}{t - \zeta}$ and making $t \to \zeta^+$ it follows that $v_p'(\zeta) \geq y'(\zeta)$. On other hand, Since $v_p < 0$ on $(\frac{T}{2}, s_p)$ we see that $y < v_p < 0$ on $[\zeta, s_p]$ and also

$$C_p - h(t)\frac{f(v_p)}{v_p} \leq -C_p < 0\,. \tag{3.34}$$

Integrating (3.17) on (ζ, s_p) and using the initial conditions (3.33), gives

$$v_p'(\zeta)\, y(\zeta) - v_p(\zeta)\, y'(\zeta) = \int_\zeta^{s_p} v_p(x)\, y(x)\Big(h(x)\frac{f(v_p)}{v_p} - C_p\Big)dx < 0\,.$$

Combining this with (3.33), for p large enough it then follows that $v_p'(\zeta) < y'(\zeta)$ which contradicts to $v_p'(\zeta) \geq y'(\zeta)$. Consequently, $y < v_p$ on all $(\frac{T}{2}, s_p)$ for p large enough.
From (3.30) and since $s_p \to T$ as $p \to \infty$ we have that

$$s_p - \frac{\pi}{\sqrt{C_p}} > \frac{T}{2} \quad \text{if } p \text{ is sufficiently large.}$$

We know that every interval of length $\frac{\pi}{\sqrt{C_p}}$ contains at least one zero of $y(t)$ and $y < v_p < 0$ on $(\frac{T}{2}, s_p)$ we assert that v_p would has one least zero on $(\frac{T}{2}, s_p)$. This is a contradiction and consequently, v_p has a local minimum m_p on $(0, s_p)$ and also, if p is sufficiently large

$$s_p - \frac{\pi}{\sqrt{C_p}} < m_p < s_p\,. \tag{3.35}$$

Hence, $m_p \to T$ as $p \to \infty$ which completes the proof of Lemma 3.4. □

Now, since $F\big(v_p(m_p)\big) = \mathcal{E}_p(m_p) \to \infty$ as $p \to \infty$ it follows that $v_p(m_p) \to -\infty$ as $p \to \infty$. Proceeding in the same way as the proof of Lemma 3.3, we can show that for p sufficiently large, v_p has a second zero at $z_{2,p} \in (0, s_p)$ and $z_{2,p} \to T$ as $p \to \infty$.

Continuing in the same way we can obtain as many zeros of v_p as desired on $(0, T)$ and we deduce the following result

Lemma 3.5. *If p is sufficiently large, v_p has an arbitrary large number of zeros on $(0, T)$.*

Lastly, we end this section with a technical lemma 3.6 in the proof of our main result.

Lemma 3.6. *Let us suppose that v_{p_*} has exactly k zeros on $(0,T)$ and $v_{p_*}(0) = 0$. If p is sufficiently close to p_* then v_p has at most $k+1$ zeros on $(0,T)$.*

Proof. Since v_p and v_{p_*} have a finite number of zeros by Lemma 3.1 then this result will follow if we can prove that $v_p \to v_{p_*}$ and $v_p' \to v_{p_*}'$ uniformly on $[0,T]$ if p is sufficiently close to p_*. Indeed, if $p_j \to p_*$ as $j \to \infty$ we denote $v_{p_j}(t) = v_j$. By using (2.11)-(2.12) it follows that

$$\forall j \geq 0 \quad |v_j'| \leq p_j + \sqrt{2F_0\, h(T)} = c_{1,p_j}$$
$$\text{and } |v_j| \leq T\, c_{1,p_j} = c_{2,p_j}\,,$$

and the sequences $(c_{1,p_j})_j$ and $(c_{2,p_j})_j$ are bounded. Thus (v_j) are uniformly bounded and equicontinuous. Thus, by Arzela-Ascoli's theorem, we have a subsequence (still denoted by v_j) of (v_j) such that $v_j \to v_{p_*}$ uniformly on $[0,T]$ as $j \to \infty$. Consequently, by using (2.13) we get

$$p_* + \lim_{j \to +\infty} v_j'(t) = \lim_{j \to +\infty} \int_t^T h(x) f(v_j)\, dx = \int_t^T h(x) f(v_{p_*})\, dx\,.$$

Therefore $v_j' \to w$ converges uniformly on $[0,T]$ as $j \to \infty$.
We now show that $w' = v_{p_*}'$. By (2.8) we have

$$-v_j(t) = \int_t^T v_j'\, dx\,,$$

and making $j \to \infty$ in this gives

$$-v_{p_*}(t) = \int_t^T w\, dx\,.$$

By differentiating this we obtain $v_{p_*}' = w$ and $v_j' \to v_{p_*}'$ uniformly on $[0,T]$ as $j \to \infty$.

□

4 Proof of the main result

In what follows, let v_p is the solution of (2.7)-(2.8) and for any integer $k \geq 1$ we construct the following sets

$$S_k = \{p > 0 : \quad v_p \text{ has at least } k \text{ zeros on } (0,T)\}\,.$$

By Lemma 3.5 the set S_1 is not empty also from Lemma 2.1 we see that S_1 is bounded from below by some positive constant. Thus, let

$$p_0 = \inf S_1 > 0\,.$$

Lemma 4.1.

$$v_{p_0} > 0 \quad \text{on } (0,T).$$

Proof. To the contrary, we suppose that $v_{p_0}(z) = 0$ for some point $z \in (0,T)$.
Since $v_p > 0$ for any $p > p_0$ and by continuous dependence of solutions on initial conditions it follows that $v_{p_0} \geq 0$ on $(0,T)$. Thus $v_{p_0}(z) = v_{p_0}'(z) = 0$. Which contradicts that z is a simple zero and consequently, $v_{p_0} > 0$ on $(0,T)$. □

Lemma 4.2. $v_{p_0}(0) = 0$.

Proof. By the definition of p_0 it follows that v_p must have a zero z_p on $(0,T)$ for $p > p_0$. In the first we will claim the following result

$$z_p \to 0 \quad \text{as } p \to p_0^+\,, \tag{4.1}$$

Otherwise, so a subsequence of (z_p) would converge to a $z \in (0,T]$ (still denoted (z_p)). By continuous dependence of solutions on initial conditions we get $v_{p_0}(z) = 0$.
Since $v_{p_0}'(T) = -p_0^* < 0$ then it follows that $z \in (0,T)$ which contradicts the fact that $v_{p_0} > 0$ on $(0,T)$. Thus (4.1) is proven and also $v_{p_0}(0) = 0$. □

By referring to the change variables (2.3) and using the Lemma 4.1 there is a positive solution u_{p_0} of (2.1)-(2.2) such that $u_{p_0}(r) \to 0$ as $r \to \infty$.

Next, from Lemmas 3.5 and 2.1 the set S_2 is non empty and is bounded from below by some positive constant. Therefore we let

$$p_1 = \inf S_2 .$$

From lemma 3.6 it follows that v_p has at most one zero on $(0, T)$ as $p \to p_0$. Thus $p_1 > p_0$ and by the same argument in the proof of Lemma 4.2 we assert that $v_{p_1}(0) = 0$. Hence there is a solution u_{p_1} of (2.1)-(2.2) which has exactly one zero on (R, ∞) and $u_{p_1}(r) \to 0$ as $r \to \infty$.

Proceeding inductively, we can show that for every nonnegative integer n there is a solution of (2.1)-(2.2) which has exactly n zeros on (R, ∞). Finally, the proof of Theorem 1.2 is complete as well.

5 Simulations

In this section we are interested in doing some simulations to the problem (2.1)-(2.2) satisfying (H1)–(H5) using MATLAB, with the aim of validating our results.

We consider the generalized Matukuma equation in exterior ball:

$$\Delta u(x) + \frac{1}{1+|x|^\alpha} f(u) = 0 \quad \text{if } |x| > R,$$
$$N \geq 2 \quad \text{and} \quad 2(N-1) < \alpha .$$

Taking $N = 3$, $\alpha = 5$, $R = 1$ and $K(r) = \dfrac{1}{1+r^5}$ satisfies (H5) and let v_p the solution of the problem,

$$v_p''(t) + \frac{t}{1+t^5} f(v_p) = 0 \quad \text{if } 0 < t < 1, \tag{5.1}$$
$$v_p(1) = 0 \quad \text{and} \quad v_p'(1) = -p , \tag{5.2}$$

for the two following cases of nonlinearity f :

(i) $f(u) = u(u^2-1)(u^2-2)(u^2-3)$, we see the Figure 1: the nonlinearity is odd, satisfies (H1)–(H4) superlinear, increasing for $|u| > 2$ and f has three positive zeros with $\beta_0^+ = 1$ and $\beta_1^+ = \sqrt{3}$. Also, F is even and has exactly one positive zero $\gamma^+ = 2$. For different value of parameter $p > 0$ we give $\mathcal{N}_p$ the number of zeros of solution v_p of (5.1)-(5.2) on interval $(0, 1)$ and satisfies $\lim_{t\to 0} v_p(t) = 0$ (we see the Figures 2, 3 and 4) are graphs generated numerically using Mathlab.

In particular, the solution remains positive when $p = 5.6$ with $v_p(0) = 0$ and v_p has exactly five zeros when $p = 56$ with $v_p(0) = 0$.

(ii) $f(u) = u^3 - 2u + e^u - 1$, we see the Figure 5: the nonlinearity satisfies (H1)–(H4), $f(0) = 0, f'(0) = -1$ and f is superlinear, increasing for $|u| > 2$ and f has one positive zero $\beta_0^+ \approx 0.73$ and one negative zero $\beta_0^- \approx -1.19$. Also, F has exactly two zeros not both 0, $\gamma^+ \approx 1.04$ and $\gamma^- \approx -1.66$. For different value of parameter $p > 0$ we give $\mathcal{N}_p$ the number of zeros of solution v_p of (5.1)-(5.2) on interval $(0, 1)$ and satisfies $\lim_{t\to 0} v_p(t) = 0$, (we see the Figures 6, 7 and 8) are graphs generated numerically using Mathlab.

In particular, the solution has exactly one zero when $p = 47.7$ with $v_p(0) = 0$ and v_p has exactly seven zeros when $p = 484.5$ with $v_p(0) = 0$.

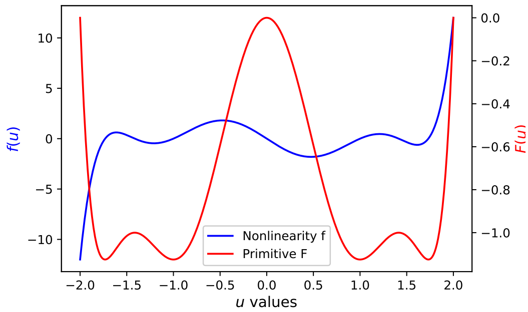

Figure 1. $f(u) = u(u^2 - 1)(u^2 - 2)(u^2 - 3)$

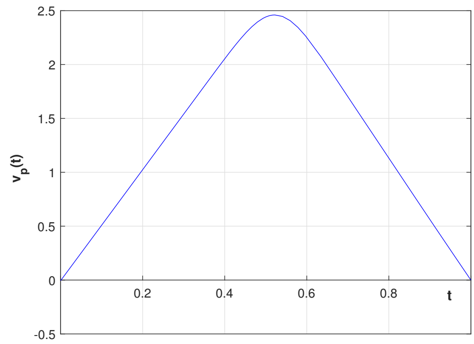

Figure 2. $v_p > 0$ on $(0, 1)$ for $p = 5.6$

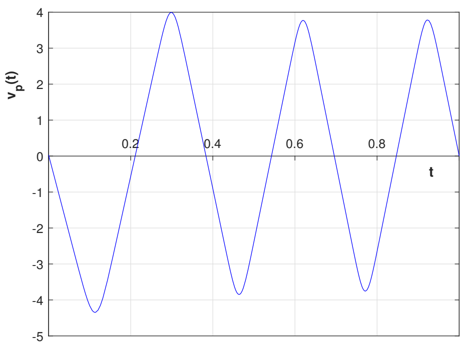

Figure 3. $p = 56$, $\mathcal{N}_p = 5$

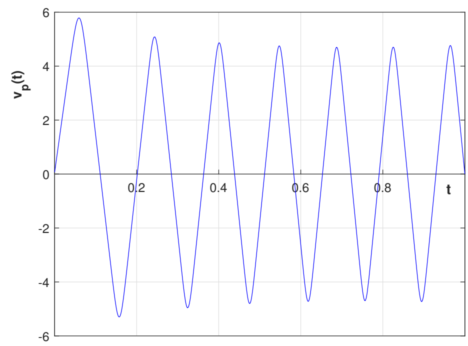

Figure 4. $p = 154.8$, $\mathcal{N}_p = 12$

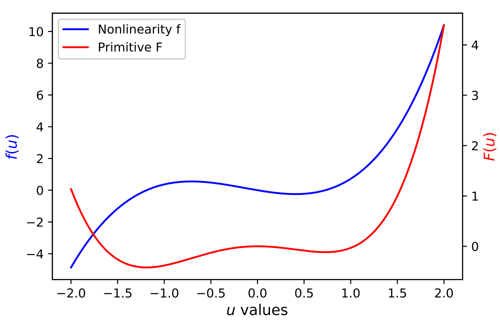

Figure 5. $f(u) = u^3 - 2u + e^u - 1$

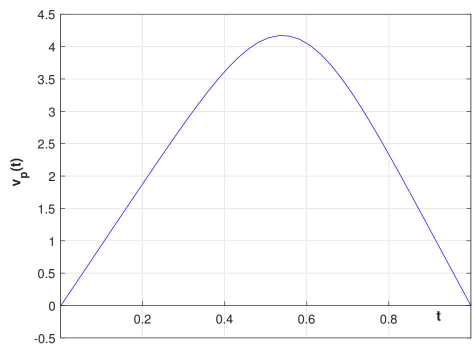

Figure 6. $v_p > 0$ on $(0, 1)$ for $p = 11.75$

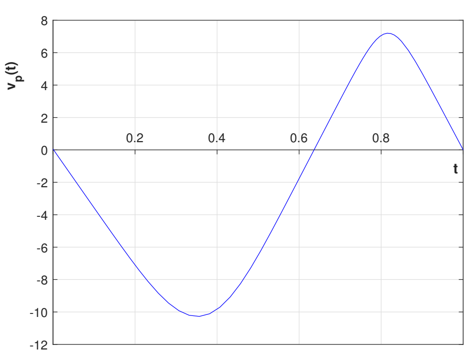

Figure 7. $p = 47.7$, $\mathcal{N}_p = 1$

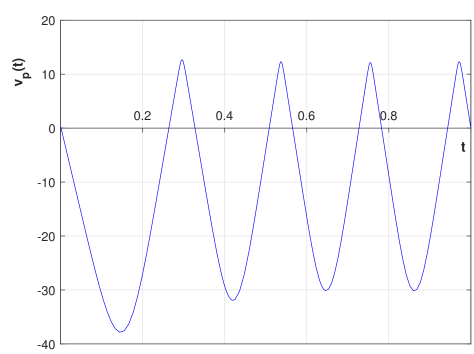

Figure 8. $p = 484.5$, $\mathcal{N}_p = 7$

References

[1] B. Azeroual, A. Zertiti; *Localized radial solutions to a superlinear Dirichlet problem in annular domain*, Commun.Fac.Sci.Univ.Ank.Series A1, Volume 67, Number 1(2018), pp. 129-140.

[2] H. Berestycki, P. L. Lions, and L.A. Peletier; *An ODE Approach to the Existence of Positive Solutions for Semilinear Problems in* R^N, Ind. Univ. Math. J., 30(1) (1981), pp 141-157.

[3] H. Berestycki, P. L. Lions; *Non-linear scalar field equations II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. 82(1983), pp 347-375.

[4] A. Castro, L. Sankar and R. Shivaji; *Uniqueness of nonnegative solutions for semipositone problems on exterior domains*, J. Math. Anal. Appl, 394(2012), pp. 432-437.

[5] M. Chhetri, L. Sankar and R. Shivaji; *Positive solutions for a class of superlinear semipositone systems on exterior domains*, Bound. Value Probl, (2014), pp. 198-207.

[6] C. Cortázar, M. Garcia-Huidobro, and C. S. Yarur; *On the uniqueness of sign changing bound state solutions of a semilinear equation*, Ann. Inst. H. Poincar´e Analyse non lin., 28 (2011), pp. 599-621.

[7] C. Cortázar, J. Dolbeault, M. Garcia-Huidobro, and R. Manàsevich; *On the existence of sign changing bound state solutions of a quasilinear equation*, J. Differential Equations, 254 (2013), pp. 2603-2625.

[8] C. Cortázar, J. Dolbeault, M. Garcia-Huidobro, and R. Manàsevich; *Existence of sign changing solutions for an equation with a weighted -Laplace operator*, J. Nonlinear analysis, vol 110 (2014), pp. 1-22.

[9] C. Dharuman, N. Prabaharan, E. Thandapani and E. Tunç; *Oscillatory behavior of even-order functional differential equations with a superlinear neutral term*, Palestine Journal of Mathematics, Vol. 12(1)(2023), 722-731.

[10] P. Hartman; *Ordinary Differential Equation, second edition* , Society for Industrial and Applied Mathematics, Philadelphia (2002).

[11] J. Iaia; *Existence of infinitely many solutions for semilinear problems on exterior domains*, Communications on Pure and Applied Analysis, Vol. 19, N. 9(2020), pp. 4269-4284.

[12] J. Iaia; *Existence of solutions for semilinear problems on exterior domains*, Electronic journal of differential equations, 34(2020), 1-10.

[13] F. Jaafri, A. Ayoujil and M. Berrajaa; *Multiple solutions for a bi-nonlocal elliptic problem involving p(x)-biharmonic operator*, Palestine Journal of Mathematics, Vol. 12(1)(2023), 197-203.

[14] C.K.R.T. Jones and T. Kupper; *On the infinitely many solutions of a semilinear elliptic equation*, SIAM J. Math. Anal. 17 (1986), pp 803-835.

[15] E. K. Lee, R. Shivaji and B. Son; *Positive solutions for infinite semipositone problems on exterior domains*, Differ. Integral Equ., 24(2011), 861-875.

[16] K. Mcleod, W. C. Troy, F. B. Weissler; *Radial solution of* $\Delta u + f(u) = 0$ *with prescribed numbers of zeros*, Journal of Differential Equation, Volume 83,(1990), pp. 368-378.

[17] L. Sankar, S. Sasi and R. Shivaji; *Semipositone problems with falling zeros on exterior domains*, J. Math. Anal. Appl., 401(2013), 146-153.

Author information

Boubker Azeroual, Mathematics computer science and decision making(Lab), National school of applied sciences, Abdelmalek Essaadi University, BP 2222, Tetouan, Morocco.
E-mail: bazeroual@uae.ac.ma

Abderrahim Zertiti, Mathematics Department, Faculty of Science, Abdelmalek Essaadi University, BP 2121, Tetouan, Morocco.
E-mail: abdzertiti@hotmail.fr

Received: 2023-10-24
Accepted: 2024-02-22