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# On $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-fuzzy primary submodules

Abhishek Kumar Rath and A. S. Ranadive

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**Corresponding Author: Abhishek Kumar Rath**

**Abstract** This paper introduces $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-fuzzy primary submodules, providing characterizations and exploring their behavior under $R$-module homomorphisms. It also investigates quotient fuzzy submodules, the direct product of fuzzy submodules, and the intersection of fuzzy submodules within this context. Finally, it establishes that radical of $(\in, \in \vee q)$-residual quotient of $(\in, \in \vee q)$-fuzzy primary submodule with respect to characteristic function forms an $(\in, \in \vee q)$-fuzzy prime ideal.

## 1 Introduction

The concept of fuzzy prime and fuzzy primary submodules is explored in several papers (see, [1, 14, 18, 13]). Different authors (see, [5, 10, 2, 3, 4, 12]) further developed these topics in detail. In [6], the concept of a fuzzy point and its quasi-coincidence with a fuzzy set is utilized to define a new type of fuzzy subgroup, called an $(\in, \in \vee q)$-fuzzy subgroup. This new subgroup effectively generalizes the fuzzy subgroup concept introduced by Rosenfeld in [17]. Further investigation of this type of generalization of different fuzzy subsystems has also been done in the literature. In this sequence, some important algebraic structures such as subrings, ideals (like prime, semiprime, primary, semiprimary, and maximal) have been thoroughly examined (see, [7]). Our paper introduces the concepts of $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-fuzzy primary submodules, followed by an in-depth discussion of specific results.

A fundamental question arises: why do we need this generalization when we have already developed effective fuzzification for both prime and primary submodules? The answer lies in the historical focus of research in this area, which has predominantly centered on two-valued fuzzy sets. Therefore, we find it necessary to adjust the definitions of fuzzy prime and fuzzy primary submodules to accommodate a wider range of fuzzy set characteristics. This is precisely the reason that motivates us to modify the definitions of fuzzy prime and fuzzy primary submodules in such a way that their range sets can encompass a greater variety. This modification opens the door for us to explore the possibility of defining infinite valued fuzzy prime submodules and fuzzy primary submodules. In this research, we have succeeded in defining infinite-valued $(\in, \in \vee q)$-fuzzy prime submodules and $(\in, \in \vee q)$-fuzzy primary submodules.

A characterization of $(\in, \in \vee q)$-fuzzy prime (primary) submodules has been provided (See Theorem 3.6 and 4.4). We have demonstrated that for two modules, $M_1$ and $M_2$, over commutative rings $R_1$ and $R_2$, respectively, any $(\in, \in \vee q)$-fuzzy prime submodule $\mu$ of $M = M_1 \times M_2$ is of the form $\mu_1 \times \mu_2$ such that either $\mu_1$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_1$ and $(\mu_2)_{0.5} = M_2$ or $\mu_2$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_2$ and $(\mu_1)_{0.5} = M_1$ (Theorem 3.12). Furthermore, we have investigated how $(\in, \in \vee q)$-fuzzy prime submodules respond under $R$-module homomorphisms. The same investigation has been conducted for $(\in, \in \vee q)$-fuzzy primary submodules. Additionally, we have proven that certain alpha-cuts of $(\in, \in \vee q)$-fuzzy prime(primary) submodules are prime(primary) submodules of the given $R$-module. Finally, we have established that the radical of $(\in, \in \vee q)$-residual quotient of $(\in, \in \vee q)$-fuzzy primary submodule and characteristic function of $M$ is $(\in, \in \vee q)$-fuzzy prime ideal.

## 2 Preliminaries

Throughout the paper, $R$ represents a commutative ring having unity, while $M$ denotes a unital $R$-module.

**Definition 2.1.** [15] Consider $\mu$ as a fuzzy subset of an $R$-module $M$. We refer to $\mu$ as a fuzzy submodule of $M$ if, for all $r \in R$ and $m, m_1, m_2 \in M$, it satisfies the following conditions ;
(i) $\mu(0) = 1$,
(ii) $\mu(m_1 + m_2) \geq \min(\mu(m_1), \mu(m_2))$,
(iii) $\mu(rm) \geq \mu(m)$.
We denote the collection of all fuzzy submodules of $M$ as $F(M)$.

**Definition 2.2.** [9] Consider $\nu$ as a fuzzy subset of a ring $R$. We define $\nu$ as a fuzzy ideal of $R$ if it satisfies the following conditions;
(i) $\nu(r_1 + r_2) \geq \min(\nu(r_1), \nu(r_2))$ for all $r_1, r_2 \in R$.
(ii) $\nu(r_1 r_2) \geq \max(\nu(r_1), \nu(r_2))$ for all $r_1, r_2 \in R$.
We represent the set of all fuzzy ideals of $R$ by $FI(R)$.

**Definition 2.3.** [16] Assume $\nu \in F(M)$ and consider $N$ as a submodule of $M$. Define a fuzzy submodule $\xi$ of $M/N$ as follows: $\xi(x+N) = \vee\{\, \nu(z) \mid z \in x+N\} \,\forall\, x \in M$. The fuzzy module $\xi$ defined is called the quotient fuzzy submodule of $M$ with respect to $\nu$.

**Definition 2.4.** [11] Let $\nu \in FI(R)$ and $\mu \in F(M)$. For $m \in M$, define $\nu \circ \mu(m)$ and $\nu\mu(m)$ as follows;

$$\nu \circ \mu(m) = \begin{cases} \sup\{\inf(\nu(r), \mu(x)) \mid m = rx \text{ for some } r \in R \text{ and } x \in M\}, \\ 0, \text{ if } m \neq rx \text{ for all } r \in R \text{ and } x \in M \end{cases}$$

$$\nu\mu(m) = \begin{cases} \sup\{\inf(\nu(r_1), \nu(r_2), \ldots, \nu(r_m), \mu(x_1), \mu(x_2), \ldots, \mu(x_m)) \\ \qquad \mid m = \sum_{i=1}^{m} r_i x_i \text{ for some } r_i \in R \text{ and } x_i \in M\}, \\ 0, \text{ if for all } r_i \in R, x_i \in M, \ m \neq \sum_{i=1}^{m} r_i x_i. \end{cases}$$

**Definition 2.5.** [11] Consider $M$ and $N$ as $R$-modules, and suppose $f : M \to N$ is an $R$-module homomorphism. A fuzzy subset $\mu$ of $M$ is regarded as $f$-invariant if, for any $x, y \in M$, the equality $f(x) = f(y)$ implies $\mu(x) = \mu(y)$.

**Definition 2.6.** [11] Define $m_t$ as a fuzzy subset of $M$ in the following manner;

$$m_t(x) = \begin{cases} t, & \text{if } x = m, \\ 0, & \text{if } x \neq m. \end{cases}$$

This fuzzy subset is referred to as a fuzzy point with support $m$ and value $t$, denoted by $m_t$.

**Proposition 2.7.** *[11] For any $r_t \in R$, $m \in M$ and $\mu \in F(M)$;*

$$r_t \circ \mu(m) = r_t\mu(m) = \begin{cases} \sup\{\inf(t, \mu(x)) \mid m = rx \textit{ for some } x \in M\}, \\ 0, \textit{ if } m \neq rx \textit{ for all } x \in M. \end{cases}$$

*Considering the above result, it is easy to verify that $r_t m_s = rm_{min\{t,s\}}$ for any $r_t \in R$ and $m_s \in F(M)$.*

**Corollary 2.8.** *For any $r_t \in R$, $m \in M$ and $\mu \in F(M)$;*

$$r_t \circ \chi_M(m) = r_t\chi_M(m) = \begin{cases} t, & \textit{if } m = rx \textit{ for some } x \in M, \\ 0, & \textit{else}. \end{cases}$$

**Definition 2.9.** [6] A fuzzy point $m_t$ is considered to belong to a fuzzy subset $\mu$, denoted as $m_t \in \mu$ if and only if the value of $\mu$ at its support, is greater than or equal to $t$, i.e., $\mu(m) \geq t$. Otherwise we say $m_t$ does not belong to $\mu$ and we denote it by $m_t \overline{\in} \mu$.

Furthermore, it is identified as quasi-coincident with $\mu$, denoted by $m_t \, q \, \mu$, when the sum of $\mu(m)$ and $t$ exceeds 1, i.e. $\mu(m) + t > 1$. Otherwise, we write $m_t \, \overline{q} \, \mu$.

**Definition 2.10.** [7] Consider a non-constant fuzzy ideal $\nu$ of $R$. It is termed a fuzzy prime ideal of $R$ if, for fuzzy points $x_t, y_s \in FI(R)$, $x_t y_s \in \nu$ implies $x_t \in \nu$ or $y_s \in \nu$.

$\nu$ is said to be $(\in, \in \vee q)$-fuzzy prime ideal of R if, for fuzzy points $x_t, y_s \in FI(R)$, $x_t y_s \in \nu$ implies $x_t \in \vee \, q \, \nu$ or $y_s \in \vee \, q \, \nu$.

**Definition 2.11.** [7] A non-constant fuzzy ideal $\nu$ of the commutative ring $R$ is refered to as a fuzzy primary ideal of $R$ if, for any $x_t, y_s \in R$, whenever the product $x_t y_s \in \nu$, it must be the case that either $x_t \in \nu$ or there exists some positive integer $n \in \mathbb{N}$ such that $y_s^n \in \nu$.

Consider a non-constant fuzzy ideal $\nu$ in the commutative ring $R$. It is refered to as an $(\in, \in \vee q)$-fuzzy primary ideal of $R$ if, for any $x_t, y_s \in R$, whenever the product $x_t y_s \in \nu$, it must be the case that either $x_t \in \vee \, q \, \nu$ or there exists some positive integer $n \in \mathbb{N}$ such that $y_s^n \in \vee \, q \, \nu$.

**Definition 2.12.** [1] Consider $\nu$ as a fuzzy submodule of $\mu$ (i.e., $\mu, \nu \in F(M)$ such that $\nu \subseteq \mu$). We define $\nu$ as a fuzzy prime submodule of $\mu$ if, for any $r_t \in FI(R)$ and $m_s \in F(M)$, whenever the product $r_t m_s \in \nu$ then it follows that either $m_s \in \nu$ or $r_t \mu \subseteq \nu$.

Additionally, if we consider $\mu = \chi_M$, then we refer to $\nu$ as a fuzzy prime submodule of $M$.

**Definition 2.13.** [14] Consider $\nu$ as a fuzzy submodule of $\mu$. We define $\nu$ as a fuzzy primary submodule of $\mu$ if, for any $r_t \in R$ and $m_s \in M$, whenever the product $r_t m_s \in \nu$, it must be the case that either $m_s \in \nu$ or $r_t^n \mu \subseteq \nu$ for some $n \in \mathbb{N}$.

**Definition 2.14.** [15] Consider $\mu$ as a fuzzy subset of the set $M$, and $\nu$ as a fuzzy subset of set $N$. The direct product of these fuzzy subsets, denoted as $\mu \times \nu$, is a fuzzy subset defined on the direct product of $M$ and $N$, denoted as $M \times N$, with the following membership function;

$$(\mu \times \nu)(m, n) = \min\{\mu(m), \nu(n)\},$$

where for any $(m, n) \in M \times N$, the membership value of $\mu \times \nu$ at $(m, n)$ is the minimum of the membership values of $\mu$ at $m$ and $\nu$ at $n$.

## 3 $(\in, \in \vee q)$-fuzzy prime submodule

**Definition 3.1.** If $\nu$ is a fuzzy submodule of $\mu$, then it is termed as an $(\in, \in \vee q)$-fuzzy prime submodule of $\mu$ whenever, for any $r_t \in FI(R)$ and $x_s \in F(M)$, where $r \in R$, $x \in M$, and $t, s \in (0, 1]$, we have $r_t x_s \in \nu$, then at least one of the following conditions are satisfied;

(i) $x_s \in \vee \, q \, \nu$,

(ii) For all $s \in (0, 1]$ and $m \in M$, if $m_s \in r_t \mu$, then $m_s \in \vee \, q \, \nu$.

If we set $\mu = \chi_M$, we then say that $\nu$ is an $(\in, \in \vee q)$-fuzzy prime submodule of $M$.

**Definition 3.2.** Suppose $\mu$ and $\nu$ are fuzzy subsets of $M$. $\nu$ is considered as a quasi-subset of $\mu$ if, for every $m \in M$, the following condition is satisfied;

$$\nu(m) \leq \mu(m) \quad or \quad \nu(m) + \mu(m) > 1.$$

We will denote it by $\nu \subseteq_1 \mu$.

**Proposition 3.3.** *Given $\mu, \nu \in F(M)$. $\nu \subseteq_1 \mu$ if and only if for all $m \in M$ and $s \in (0, 1]$, whenever $m_s \in \nu$, we have $m_s \in \vee \, q \, \mu$.*

*Proof.* Assume that for all $m \in M$ and $s \in (0, 1]$, $m_s \in \nu$, implies $m_s \in \vee \, q \, \mu$ and $\nu \not\subseteq_1 \mu$. In this scenario, there will be an element $x \in M$ such that $\nu(x) > \mu(x)$ and $\nu(x) + \mu(x) \leq 1$. Let $\mu(x) = s$. Consequently, we find that $x_s \in \nu$, but $x_s \overline{\in \vee q} \, \mu$, resulting in a contradiction.

The converse of the proposition can be readily demonstrated using similar arguments. □

In light of the above proposition, for the rest of the paper, we will use $r_t\mu \subseteq_1 \nu$ in place of condition (ii) of Definition 3.1.

**Proposition 3.4.** *If we let $M = R$, then $\nu \in F(M)$ is defined as an $(\in, \in \vee q)$-fuzzy prime submodule of $M$ if and only if $\nu \in FI(R)$ is an $(\in, \in \vee q)$-fuzzy prime ideal of $R$, and $\nu(0) = 1$.*

*Proof.* Assume $\nu$ to be an $(\in, \in \vee q)-$fuzzy prime submodule of $M$. Since $R$ is a commutative ring thus we have $\nu \in FI(R)$. Now if $x_s, y_t \in FI(M)$ such that $x_s y_t \in \nu$ then we have either $x_s \in \vee\, q\, \nu$ or we have $y_t\chi_M \subseteq_1 \nu$. If $x_s \in \vee\, q\, \nu$, then by definition, $\nu$ is an $(\in, \in \vee q)$-fuzzy prime ideal.

If $y_t\chi_M \subseteq_1 \nu$ then since $y_t\chi_M(y.1) = t$ therefore $y_t \in y_t\chi_M$ this implies $y_t \in \vee\, q\, \nu$ making $\nu$ again an $(\in, \in \vee q)$-fuzzy prime ideal.
Conversely, consider $\nu$ as an $(\in, \in \vee q)$-fuzzy prime ideal of $R$ with $\nu(0) = 1$. It is clear that $\nu$ is a fuzzy submodule of $M$. Suppose $r_t \in R$, $x_s \in M, r_t x_s \in \nu$. If $x_s \in \vee\, q\, \nu$ then $\nu$ becomes an $(\in, \in \vee q)$-fuzzy prime submodule of $M$.

If $x_s \overline{\in \vee\, q}\, \nu$ then $r_t \in \vee\, q\, \nu$. So $\nu(r) \geq t$ or $\nu(r) + t > 1$. Since $\forall\, m \in M$, $r_t\chi_M(rm) = t$ therefore $\nu(r) \geq r_t\chi_M(rm)$ or $\nu(r) + r_t\chi_M(rm) > 1$. Now $\nu(rm) \geq \nu(r)$($\nu$ is fuzzy ideal of $R$) so we have $\nu(rm) \geq r_t\chi_M(rm)$ or $\nu(rm) + r_t\chi_M(rm) > 1$. This is eqivalent of saying that $r_t\chi_M \subseteq_1 \nu$. Hence again we have $\nu$ as an $(\in, \in \vee q)$-fuzzy prime submodule of $M$. □

**Proposition 3.5.** *Suppose $\nu$ is an $(\in, \in \vee q)$-fuzzy prime submodule of $\mu$, then $\nu_t$ is a prime submodule of $\mu_t$, provided $\nu_t \neq \mu_t$ and $0 < t \leq 0.5$.*

*Proof.* Let $\nu_t \neq \mu_t$ and $0 < t \leq 0.5$. Suppose $rm \in \nu_t$ for some $r \in R$ , $m \in M$. If $rm \in \nu_t$ then $\nu(rm) \geq t$ or $r_t m_t \in \nu$. Since $\nu$ is $(\in, \in \vee q)$-fuzzy prime submodule of $\mu$ therefore either $m_t \in \vee q\, \nu$ or $r_t\mu \subseteq_1 \nu$.
**Case (i):** If $m_t \in \vee\, q\, \nu$ then either $\nu(m) \geq t$ or $\nu(m) + t > 1$. Since $t \leq 0.5$ therefore in both situations $\nu(m) \geq t$ or $m \in \nu_t$.
**Case (ii):** Assume $r_t\mu \subseteq_1 \nu$. For any $w \in r\mu_t$, we can express it as $w = rz$ for some $z \in \mu_t$. This implies $\mu(z) \geq t$ which implies $t \leq r_t\mu(w)$ (as per the definition of $r_t\mu$). Consequently, we have $w_t \in r_t\mu$. So, we obtain $w_t \in \vee\, q\, \nu$. Thus, either $\nu(w) \geq t$ or $\nu(w) + t > 1$. Since $t \leq 0.5$, this leads us to the conclusion that $\nu(w) \geq t$ or $w_t \in \nu$, ultimately leading to $r\mu_t \subseteq \nu_t$. □

The following result paves the path for obtaining an infinite-valued $(\in, \in \vee q)$-fuzzy prime submodule.

**Theorem 3.6.** *Consider $M_1$ as a prime submodule of $M$. If $\mu$ is defined on $M$ such that, for every $m \in M$;*

$$\mu(m) = \begin{cases} 1, & \text{if } m = 0 \\ p_i\ (\geq 0.5), & \text{if } m \in M_i/M_{i+1} \text{ for } i = 1, 2, 3, \ldots \\ q\ (< 0.5), & \text{otherwise} \end{cases}$$

*Where, for $i \in \mathbb{N}$, $M_{i+1} \subseteq M_i$ and $p_{i+1} \geq p_i$, then $\mu$ is indeed an $(\in, \in \vee q)$-fuzzy prime submodule of $M$. Moreover, converse is also true i.e. any $(\in, \in \vee q)$-fuzzy prime submodule of $M$ can be expressed in the provided manner.*

*Proof.* Let's begin with the assumption that there exist $r_t \in F(R)$ and $m_s \in F(M)$ such that $r_t m_s \in \mu$ and $m_s \overline{\in \vee\, q}\, \mu$.

So, we find that $\mu(m) < s$ and $\mu(m) + s \leq 1$. This implies $\mu(m) < 0.5$ because if $\mu(m) \geq 0.5$, then adding $s$ to it would result in $\mu(m) + s > 1$, which yields a contradiction. Therefore, we can conclude that $m \notin M_1$. Let us consider these two cases;
**Case (i):** If $\mu(rm) = 1$ or $p_i$ then $rm \in M_1$. Because $m \notin M_1$ and $M_1$ being a prime submodule of $M$ implies $rM \subseteq M_1$. This futher implies either $r_t\chi_M(rm) = t \leq \mu(rm)$ or $\mu(rm) + t > 1$.
**Case (ii):** If $\mu(rm) = q$, then $\mu(m) = q$ as well. This leads to the conclusion that $s \nleq q$. Given that $r_t m_s \in \mu$ implies $min\{t, s\} \leq q < 0.5$. Consequently, we arrive at $t \leq q$ (since $s \nleq q$). This implies that $r_t\chi_M(w) \leq q \leq \mu(w)\ \forall\ w \in M$.
In both of these cases, we have $r_t\chi_M \subseteq_1 \mu$.

Conversely, if $\mu$ is an $(\in, \in \vee q)$-fuzzy prime submodule of $M$, then it follows, according to proposition 3.5, that $\mu_{0.5}$ is a prime submodule of $M$. Therefore, $\mu_{0.5} \neq M$, so $m \in M$ exists such that $m \notin \mu_{0.5}$. Now, we claim that $\mu(y) = \mu(m)$ for all $y \in M$ such that $y \notin \mu_{0.5}$.

Because $m \notin \mu_{0.5}$ so $m_{0.5} \notin \mu$. Now $m_{\mu(m)} = 1_{\mu(m)} m_{0.5} \in \mu$. By $(\in, \in \vee q)$-fuzzy primeness of $\mu$, we now obtain $1_{\mu(m)}\chi_M \subseteq_1 \mu$. This implies that $\mu(m) = 1_{\mu(m)}\chi_M(w) \leq \mu(w)$ or $1_{\mu(m)}\chi_M(w) + \mu(w) > 1$ for all $w \in M$. The latter condition is only possible if $\mu(w) > 0.5$, therefore, $\mu(m) = 1_{\mu(m)}\chi_M(w) \leq \mu(w)$ for all $w \in M$. because m is arbitrary, we obtain $\mu(m) = \mu(y)$ for all $m, y \notin \mu_{0.5}$. This concludes the argument. □

To support the preceding result, we provide an example.

**Example 3.7.** Consider $\mathbb{Z}$-module $\mathbb{Z}$ and define a fuzzy submodule $\mu$ on it as follows;

$$\mu(z) = \begin{cases} 1, & \text{if } z = 0 \\ (m+1)/m & \text{if } z \in m\mathbb{Z}/2m\mathbb{Z}, \text{ m=2, 4,8,...} \\ 0, & \text{otherwise} \end{cases}$$

for all $z \in \mathbb{Z}$.

It is a straightforward to confirm that $\mu$ meets the conditions to be considered as a fuzzy submodule of $\mathbb{Z}$. Furthermore, $\mu_{0.5} = 2\mathbb{Z}$, which is a prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$. Hence, according to the previous theorem, $\mu$ becomes an $(\in, \in \vee q)$-fuzzy prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$.

In the upcoming result, we establish a connection between $(\in, \in \vee q)$-fuzzy prime submodule $\nu$ in the context of a module $M$ and its quotient fuzzy module $\xi$ of $M/N$. The theorem demonstrates that when $N$ is contained within $\nu_{0.5}$, then the quotient fuzzy module $\xi$ inherits the crucial characteristics of an $(\in, \in \vee q)$-fuzzy prime submodule.

**Proposition 3.8.** *Consider $\nu$ as an $(\in, \in \vee q)$-fuzzy prime submodule of $M$, and $\xi$ as a quotient fuzzy module of $M/N$ with respect to $\nu$, where $N$ is a submodule of $M$ such that $N \subseteq \nu_{0.5}$. Then, $\xi$ becomes an $(\in, \in \vee q)$-fuzzy prime submodule of $M/N$.*

*Proof.* Suppose $r_t(x+N)_s \in \xi$ for $r \in R$, $x + N \in M/N$ and $t, s \in (0, 1]$. Assume $(x + N)_s \overline{\in \vee q}\, \xi$. We claim that $r_t\chi_{M/N} \subseteq_1 \xi$. Since $(x+N)_s \overline{\in \vee q}\, \xi$ therefore we have;

$$\begin{aligned} & \xi(x+N) < s \text{ and } \xi(x+N) + s \leq 1 \\ \implies & \sup\{\nu(z) \mid z \in x+N\} < s \text{ and } \sup\{\nu(z) \mid z \in x+N\} + s \leq 1 \\ \implies & \nu(x) < s \quad and \quad \nu(x) + s \leq 1 \\ \implies & x_s \overline{\in \vee q}\, \nu. \end{aligned}$$

**Case I:** If $r_t x_s \in \nu$. Since $\nu$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M$ and $x_s \overline{\in \vee q}\, \nu$ thus we get,

$$r_t\chi_M \subseteq_1 \nu$$

$$\begin{aligned} \implies & \nu(rm) \geq t \quad or \quad \nu(rm) + t > 1 \\ \implies & \sup\{\nu(z) \mid z \in rm+N\} \geq t \quad or \quad \sup\{\nu(z) \mid z \in rm+N\} + t > 1 \\ \implies & \xi(rm+N) \geq t \quad or \quad \xi(rm+N) + t > 1 \quad \forall m \in M \\ \implies & r_t\chi_{M/N} \subseteq_1 \xi. \end{aligned}$$

**Case II:** If $r_t x_s \notin \nu$
This implies $min\{t, s\} > \nu(rx)$. Since $\xi(rx + N) \geq min\{t, s\}$
so we get, $\xi(rx + N) \geq min\{t, s\} > \nu(rx)$.
**Subcase I:** If $rx \in \nu_{0.5}$ then since $\nu_{0.5}$ is prime submodule of $M$ therefore either $x \in \nu_{0.5}$ or

$rM \subseteq \nu_{0.5}$. If $x \in \nu_{0.5}$ then $0.5 \leq \nu(x) \leq \nu(rx) < min\{t, s\}$ which gives $\xi(x + N) + s > 1$ a contradiction. Therefore $x \notin \nu_{0.5}$. Thus we get $rM \subseteq \nu_{0.5}$.

$$\implies \nu(rm) \geq 0.5 \quad \forall\, m \in M$$
$$\implies \xi(rm + N) \geq \nu(rm) \geq 0.5$$
$$\implies \xi(rm + N) \geq t \quad or \quad \xi(rm + N) + t > 1$$
$$\implies r_t\chi_{M/N} \subseteq_1 \xi.$$

**Subcase II:** If $rx \notin \nu_{0.5}$ then $\nu(rx) < 0.5$. Thus $\nu(rx) = q$ (say)(where $q$ is some fixed number). Now since we have $N \subseteq \nu_{0.5}$ therefore;

$$\xi(rx + N) = \sup\{\nu(z) \mid z \in rx + N\} = q$$
$$\implies q = \xi(rx + N) \geq \min\{t, s\} > \nu(rx) = q$$
$$\implies \xi(rx + N) = \min\{t, s\} = q = \nu(rx)$$

Now, we will claim that $q \neq s$. Suppose $q = s$ then,

$$\xi(rx + N) = s = \xi(x + N)$$
$$\implies (x + N)_s \in \xi$$

which is a contradiction. Thus we get $q = t$ or $\xi(rx + N) = t$ which implies $r_t\chi_{M/N} \subseteq_1 \xi$. This concludes the proof. □

Next, we investigate the concept of $(\in, \in \vee\, q)$-fuzzy primeness within the intersection of two fuzzy submodules of a module.

**Proposition 3.9.** *Suppose $\mu$ is a fuzzy submodule of $M$ and $\nu$ is an $(\in, \in \vee\, q)$-fuzzy prime submodule of $M$. Then, the intersection of $\mu$ and $\nu$, denoted as $\mu \cap \nu$, forms an $(\in, \in \vee\, q)$-fuzzy prime submodule of $\mu_{0.5}$.*

*Proof.* Suppose $r \in R$, $x \in \mu_{0.5}$, and $r_tx_s \in \mu \cap \nu$ then it follows that $r_tx_s$ belongs to both $\mu$ and $\nu$. Since $\mu(x) \geq 0.5$, we can conclude that $x_s \in \vee\, q\, \mu$. Moreover, as $\nu$ is $(\in, \in \vee\, q)$-fuzzy prime submodule of $M$, we have $x_s \in \vee\, q\, \nu$ or $r_t\chi_M \subseteq_1 \nu$. Thus, we can either have $x_s \in \vee\, q\, \mu \cap \nu$ or $r_t\chi_{\mu_{0.5}} \subseteq_1 \mu \cap \nu$. □

Since homomorphism is the fundamental topic in algebra, it becomes essential to study the characteristics of $(\in, \in \vee\, q)$-fuzzy prime submodules under homomorphisms. The forthcoming results illuminate this aspect.

**Proposition 3.10.** *Let $M$ and $N$ be $R$-modules. Assume $\mu$ is $f$-invariant $(\in, \in \vee\, q)$-fuzzy prime submodule of $M$ having supremum property. Then $f(\mu)$ becomes an $(\in, \in \vee\, q)$-fuzzy prime submodule of $N$.*

*Proof.* Given the supremum property of $\mu$, we can assert that $f(\mu_{0.5}) = (f(\mu))_{0.5}$.

First, we claim that $f(\mu_{0.5})$ is prime submodule of $N$. Let $r \in R$ and $y \in N$ such that $ry \in f(\mu_{0.5})$. Then $\exists\, x \in M$ such that $f(x) = y$, and consequently, $rf(x) \in f(\mu_{0.5})$. By the $f$-invariance property of $\mu$, we conclude that $rx \in \mu_{0.5}$. Now $\mu_{0.5}$ is prime submodule of $M$, so either $x \in \mu_{0.5}$ or $rN \subseteq f(\mu_{0.5})$. Hence, $f(\mu_{0.5})$ becomes a prime submodule of $N$.
Next, we claim, for any $n \notin f(\mu_{0.5})$, $f(\mu)(n) = q$, where $0 \leq q < 0.5$ is some fixed number. Since $n \notin f(\mu_{0.5})$, there doesn't exist $m$ in $\mu_{0.5}$ for which $f(m) = n$. However, $f$ is an epimorphism, so there does exist an element $x \in M$ for which $f(x) = n$. By the $(\in, \in \vee\, q)$-fuzzy prime submodule property of $\mu$, we get $\mu(x) = q$ where $0 \leq q < 0.5$.Which implies;

$$f(\mu)(n) = \vee\,\{\mu(x) | f(x) = n\} = q.$$

Hence the result. □

**Proposition 3.11.** *Consider an $R$-module homomorphism $f$ from $M$ to $N$. If $\nu$ is an $(\in, \in \vee\, q)$-fuzzy prime submodule of $N$ then $f^{-1}(\nu)$ also becomes an $(\in, \in \vee\, q)$-fuzzy prime submodule of $M$.*

*Proof.* We claim, $f^{-1}(\nu_{0.5})$ is a prime submodule of $M$. Suppose $0 \neq rx \in f^{-1}(\nu_{0.5})$. Then, it follows that $f(rx) \in \nu_{0.5}$ or $rf(x) \in \nu_{0.5}$. Since $\nu_{0.5}$ is prime submodule of $N$, we have either $f(x) \in \nu_{0.5}$ or $rN \subseteq \nu_{0.5}$. This implies either $x \in f^{-1}(\nu_{0.5})$ or $rM \subseteq f^{-1}(\nu_{0.5})$.

Furthermore, it is straightforward to demonstrate that for every $x \notin f^{-1}(\nu_{0.5})$, $f^{-1}(\nu)(x) = \nu(f(x)) = q$, where $0 \leq q < 0.5$. □

Next, we investigate $(\in, \in \vee q)$-fuzzy primeness in the context of direct product of modules.

**Proposition 3.12.** *Let $M_1$ and $M_2$ be $R_1$ and $R_2$ modules respectively. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $\mu \in F(M)$. Then $\mu$ becomes an $(\in, \in \vee q)$-fuzzy prime module of $M$ iff $\mu = \mu_1 \times \mu_2$ where either $\mu_1$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_1$ and $(\mu_2)_{0.5} = M_2$ or $(\mu_1)_{0.5} = M_1$ and $\mu_2$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_2$.*

*Proof.* First consider the sufficient part. Assume $\mu = \mu_1 \times \mu_2$ such that, $\mu_1$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_1$ and $(\mu_2)_{0.5} = M_2$.
**Claim I:** $(\mu_1 \times \mu_2)_{0.5}$ is a prime submodule of $M$. Since,

$$\begin{aligned}(\mu_1 \times \mu_2)_{0.5} =& \{(m_1, m_2) \in M_1 \times M_2 \mid min\{\mu_1(m_1), \mu_2(m_2)\} \geq 0.5\}\\ =& \{(m_1, m_2) \in M_1 \times M_2 \mid \mu_1(m_1) \geq 0.5,\ \mu_2(m_2) \geq 0.5\}\\ =& (\mu_1)_{0.5} \times (\mu_2)_{0.5}\\ =& (\mu_1)_{0.5} \times M_2\end{aligned}$$

Because $\mu_1$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M_1$ so $(\mu_1)_{0.5} \times M_2$ will become prime submodule of $M_1 \times M_2$.
**Claim II:** If a pair $(m_1, m_2) \in M_1 \times M_2$ such that $(m_1, m_2) \notin (\mu_1 \times \mu_2)_{0.5}$, then there exists a fixed $q \in [0, 0.5)$ such that $(\mu_1 \times \mu_2)(m_1, m_2) = q$.
Given that $(m_1, m_2) \notin (\mu_1 \times \mu_2)_{0.5}$, it follows $(\mu_1 \times \mu_2)(m_1, m_2) < 0.5$, which can be expressed as $min\{\mu_1(m_1), \mu_2(m_2)\} < 0.5$. Since $(\mu_2)_{0.5} = M_2$, this implies $\mu_1(m_1) < 0.5$, and therefore, we conclude that $m_1 \notin (\mu_1)_{0.5}$. We also have $\mu_1$ as a $(\in, \in \vee q)$-fuzzy prime submodule of $M_1$ therefore for all such elements not in $(\mu_1)_{0.5}$, we have $\mu_1(m_1) = q$.

Hence, for all $(m_1, m_2) \notin (\mu_1 \times \mu_2)$, we have $(\mu_1 \times \mu_2)(m_1, m_2) = q$. This implies that $\mu_1 \times \mu_2$ is $(\in, \in \vee q)$-fuzzy prime submodule of $M$.
Conversely, let's define $\mu_1$ and $\mu_2$ as fuzzy submodules of $M_1$ and $M_2$, respectively, such that;

$$\mu_1(m_1) = sup\{\mu(m_1, m_2) | m_2 \in M_2,\ \mu(m_1, m_2) > 0\}$$

and

$$\mu_2(m_2) = sup\{\mu(m_1, m_2) | m_1 \in M_1,\ \mu(m_1, m_2) > 0\}$$

Now, it's easy to verify that $\mu = \mu_1 \times \mu_2$. Now $\mu$ is a $(\in, \in \vee q)$-fuzzy prime submodule of $M$, so $(\mu)_{0.5} = (\mu_1 \times \mu_2)_{0.5} = (\mu_1)_{0.5} \times (\mu_2)_{0.5}$, is prime submodule of $M = M_1 \times M_2$.

This property implies that either $(\mu_1)_{0.5} = M_1$ or $(\mu_2)_{0.5} = M_2$. Suppose $(\mu_2)_{0.5} = M_2$, then $(\mu_1)_{0.5} \times M_2$ is a prime submodule of $M_1 \times M_2$, or equivalently, $(\mu_1)_{0.5}$ is a prime submodule of $M_1$.

Furthermore, if $m_1 \notin (\mu_1)_{0.5}$, then $(m_1, m_2) \notin (\mu_1)_{0.5} \times M_2$, which implies $(\mu_1 \times \mu_2)(m_1, m_2) = q(< 0.5)$. Since $\mu_1(m_1) = \sup\{\mu(m_1, m_2) | m_2 \in M_2\}$, we conclude that $\mu_1(m_1) = q$. □

## 4 $(\in, \in \vee q)$-fuzzy primary submodule

**Definition 4.1.** A fuzzy submodule $\nu$ of $\mu$ is termed as $(\in, \in \vee q)$-fuzzy primary submodule of $\mu$ if, for any $r_t \in FI(R)$ and $x_s \in F(M)$ ($r \in R$, $x \in M$, and $t, s \in (0, 1]$), whenever $r_t x_s \in \nu$, we have either $x_s \in \vee q\ \nu$ or $r_t^n \mu \subseteq_1 \nu$ for some $n \in \mathbb{N}$.

If we set $\mu = \chi_M$, we then say that $\nu$ is $(\in, \in \vee q)$-fuzzy primary submodule of $M$.

**Theorem 4.2.** *In the case of $M = R$, $\nu \in F(M)$ becomes $(\in, \in \vee q)$-fuzzy primary submodule of $M$ if and only if $\nu$ is $(\in, \in \vee q)$-fuzzy primary ideal of $R$ and $\nu(0) = 1$.*

*Proof.* Consider $\nu$ as an $(\in, \in \vee q)$-fuzzy submodule of $M$. Then it is easy to check, $\nu \in FI(R)$. Let $x_t y_s \in \nu$. If $y_s \in \vee\, q\, \nu$ then nothing to show. Assume $y_s \overline{\in \vee\, q}\, \nu$ then $x_t^n \chi_M \subseteq_1 \nu$, for some $n \in \mathbb{N}$. Since we have,

$$x_t^n \chi_M(m) = \begin{cases} t, & if\ m = x^n c\ for\ some\ c \in M \\ 0, & otherwise \end{cases}$$

and $x^n = x^n.1$ so by $x_t^n \chi_M \subseteq_1 \nu$ we get,

$$(t =) x_t^n \chi_M(x^n) + \nu(x^n) > 1 \quad or \quad \nu(x^n) \geq t.$$

This implies $x_t^n \in \vee\, q\, \nu$.

In converse, let $\nu$ be an $(\in, \in \vee q)$-fuzzy primary ideal of $R$ with the condition $\nu(0) = 1$. Then it is easy to check that $\nu \in F(M)$. Now consider $x_t y_s \in \nu$. If $y_s \in \vee\, q\, \nu$ then nothing to prove. Assume $y_s \overline{\in \vee\, q}\, \nu$ then, $x_t^n \in \vee\, q\, \nu$, for some $n \in \mathbb{N}$ ($\nu$ is $(\in, \in \vee q)$-fuzzy primary ideal of $R$). Now;

$$x_t^n \chi_R(m) = \begin{cases} t, & if\ m = x^n c\ for\ some\ c \in R \\ 0, & else \end{cases}$$

and $x_t^n \in \vee\, q\, \nu$ so either $t \leq \nu(x^n) \leq \nu(x^n c)$ or $t + \nu(x^n c) > 1$. Hence $x_t^n \chi_R \subseteq_1 \nu$. □

**Theorem 4.3.** *If $\nu$ is an $(\in, \in \vee q)$-fuzzy submodule of $\mu$, then $\nu_t$ becomes a primary submodule of $\mu_t$ under the condition that $\nu_t \neq \mu_t$ and $t \in (0, 0.5]$.*

*Proof.* Assume $rx \in \nu_t$ for $r \in R$, $x \in M$, and $t \in (0, 0.5]$. We want to show that either $x \in \nu_t$ or $r^n \mu_t \subseteq \nu_t$, for some $n \in \mathbb{N}$.

Because $rx \in \nu_t$, so $\nu(rx) \geq t$, which means $(rx)_t \in \nu$. This implies that either $x_t \in \vee\, q\, \nu$ or $r_t^n \mu \subseteq_1 \nu$ for some $n \in \mathbb{N}$.
If $x_t \in \vee\, q\, \nu$, then, since $t \leq 0.5$, we get $\nu(x) \geq t$, which means $x \in \nu_t$, and we are done.

Otherwise, if $r_t^n \mu \subseteq_1 \nu$. For any $w \in r^n \mu_t$, we can express $w$ as $w = r^n z$ for some $z \in \mu_t$. This gives $\mu(z) \geq t$. Now we have two possibilities;

(i) $r_t^n \mu(w) \leq \nu(w)$. Which implies $t \leq \nu(w)$ or $w \in \nu_t$.

(ii) $\nu(w) + r_t^n \mu(w) > 1$ or $\nu(w) + t > 1$. Since $t \leq 0.5$ therefore we get $\nu(w) \geq t$ or $w \in \nu_t$.

Hence, $r^n \mu_t \subseteq \nu_t$. □

**Theorem 4.4.** *Consider $\nu$ as a fuzzy submodule of $M$ defined such that, for every $m \in M$;*

$$\nu(m) = \begin{cases} 1, & if\ m = 0, \\ t_i(\geq 0.5), & if\ m \in M_i/M_{i+1}, i = 1, 2, 3... \\ q(< 0.5), & otherwise \end{cases}$$

*where $M_1$ is a primary submodule of $M$ and $t_i \leq t_{i+1}\ \forall\, i \in \mathbb{N}$, then $\nu$ is $(\in, \in \vee q)$-fuzzy primary submodule of $M$. Moreover, converse is also true i.e. any $(\in, \in \vee q)$-fuzzy primary submodule of $M$ can be expressed in the given form.*

*Proof.* Assuming $r_t \in R$ and $x_s \in M$ such that $r_t x_s \in \nu$ but $x_s \overline{\in \vee\, q}\, \nu$, we claim that $r_t^n \chi_M \subseteq_1 \nu$, for some $n \in \mathbb{N}$.

Since $x_s \overline{\in \vee\, q}\, \nu$, we have $\nu(x) < s$, and $\nu(x) + s \leq 1$. This implies that $\nu(x) < 0.5$, otherwise, we cannot satisfy both of these inequalities simultaneously. Thus, we conclude that $x \notin M_1$.

**Case I:** Assume $\nu(rx) \geq 0.5$, then $rx \in M_1$, and since $x \notin M_1$, we have $r^n M \subseteq M_1$, for some $n \in \mathbb{N}$. Therefore, $r^n z \in M_1\ \forall\, z \in M$. Consequently, for any $w \in M$, we either have $r_t^n \chi_M(w) \leq \nu(w)$ or $r_t^n \chi_M(w) + \nu(w) > 1$. Thus, we conclude that $r_t^n \chi_M \subseteq_1 \nu$.

**Case II:** Assume $\nu(rx) = q$ where $q < 0.5$. Since $r_t x_s \in \nu$ therefore $\nu(rx) \geq min\{t, s\}$ or $q \geq t$ (since $\nu(rx) = \nu(x) = q$). Thus we have,

$$r_t^n \chi_M(w) \leq t \leq q \leq \nu(w) \qquad \forall\, w \in M$$

or

$$r_t^n \chi_M \subseteq_1 \nu \qquad \forall\, n \in \mathbb{N}.$$

Conversely, suppose $\nu$ is any $(\in, \in \vee q)$-fuzzy primary submodule of $M$. Since $\nu_{0.5}$ is primary submodule of $M$ therefore $\nu_{0.5} \neq M$. This means $\exists\, z \in M$ such that $z \notin \nu_{0.5}$. Now, we claim that $\forall\, y \in M$ such that $y \notin \nu_{0.5}$, we have $\nu(z) = \nu(y)$. Since $z \notin \nu_{0.5}$ we have $z_{0.5} \notin \nu$. Again $z_{\nu(z)} = 1_{\nu(z)} z_{0.5} \in \nu$ so by $(\in, \in \vee q)$-fuzzy primaryness of $\nu$ we get $1_{\nu(z)}\chi_M = 1^n_{\nu(z)}\chi_M \subseteq_1 \nu$. This implies that $\nu(z) = 1_{\nu(z)}\chi_M(w) \leq \nu(w)$ or $1_{\nu(z)}\chi_M(w) + \nu(w) > 1$. The latter condition is only possible when $\nu(w) > 0.5$. Therefore, we ultimately obtain;

$$1_{\nu(z)}\chi_M(w) \leq \nu(w) \quad or \quad \nu(z) \leq \nu(w)\ \forall\, w \in M$$

Since $z$ is arbitrary therefore we get $\nu(z) = \nu(y)\ \forall\, y, z \notin \nu_{0.5}$. This completes the proof. □

**Example 4.5.** Consider $\mathbb{Z}$-module $\mathbb{Z}$ and define a fuzzy subset $\nu$ on it as follows;

$$\nu(z) = \begin{cases} 1, & \text{if } z = 0 \\ (n+1)/n & \text{if } z \in \, < p^n > / < p^{n+1} >, \text{ n= 2,3,...} \\ 0, & \text{otherwise} \end{cases}$$

for all $z \in \mathbb{Z}$.

It is straightforward to verify that $\nu$ qualifies as a fuzzy submodule of $\mathbb{Z}$. Furthermore, $\nu_{0.5} =< p^2 >$, which is nothing but the primary submodule of $\mathbb{Z}$-module $\mathbb{Z}$. Hence, according to the previous theorem, $\nu$ becomes an $(\in, \in \vee q)$-fuzzy primary submodule of $\mathbb{Z}$-module $\mathbb{Z}$.

Next, we will only state the upcoming two results, which can be easily proven with minor modifications to the proofs of Propositions 3.10 and 3.11.

**Proposition 4.6.** *Consider an R-module homomorphism $f$ from $M$ to $N$ and suppose $\mu$ is an $f$-invariant $(\in, \in \vee q)$-fuzzy primary submodule of $M$ with the supremum property. Then $f(\mu)$ becomes an $(\in, \in \vee q)$-fuzzy primary submodule of $N$.*

**Proposition 4.7.** *Consider an R-module homomorphism $f$ from $M$ to $N$. If $\nu$ is an $(\in, \in \vee q)$-fuzzy primary submodule of $N$, then the preimage $f^{-1}(\nu)$ forms an $(\in, \in \vee q)$-fuzzy primary submodule of $M$.*

If $K$ and $L$ are two submodules of the $R$-module $M$ then, $(K : L) := \{r \in R | rL \subseteq K\}$ is called residual quotient of $N$ by $L$. Next, we give definition of $(\in, \in \vee q)$-residual quotient of $\nu$ by $\mu$.

**Definition 4.8.** Given $\mu$ and $\nu$ in $F(M)$, we define the $(\in, \in \vee q)$-residual quotient of $\nu$ with respect to $\mu$ as $(\nu : \mu)_1(r)$, where, $\forall\, r \in R$ and $t \in [0, 1]$, it is given by,

$$(\nu : \mu)_1(r) := sup\{min\{t, 0.5\} |\, r_t\mu \subseteq \nu\}.$$

**Theorem 4.9.** *If $\mu,\ \nu \in F(M)$ then $(\nu : \mu)_1 \in FI(R)$.*

*Proof.* First we claim, $(\nu : \mu)_1(r + s) \geq min((\nu : \mu)_1)(r), (\nu : \mu)_1(s))\ \forall\, r, s \in R$.

Let $p, q \in [0, 1]$ be such that $r_p\mu \subseteq \nu$ and $s_q\mu \subseteq \nu$. We show that in this case we also have;

$$(r + s)_{min\{p,q,0.5\}}\mu \subseteq \nu.$$

For $m \in M$ if $(r + s)_{min\{p,q,0.5\}}\mu(m) = 0$ then $(r + s)_{min\{p,q,0.5\}}\mu(m) \leq \nu(m)$ so nothing to prove. Otherwise,

$$(r + s)_{min\{p,q,0.5\}}\mu(m) = sup\,\{min(min(p, q, 0.5), \mu(x))|m = (r + s)x \ for\ some\ x \in M\}$$

Also,

$$r_p\mu(rx) = sup\{min(p, \mu(w))|rx = rw,\ for\ some\ w \in M\} \leq \nu(rx)$$

$$s_q\mu(sx) = sup\,\{min(q, \mu(z))|sx = sz,\ for\ some\ z \in M\} \leq \nu(sx)$$

Now,

$$\begin{aligned} min(min(p,q,0.5),\mu(x)) &= min(min(p,\mu(x)), min(q,\mu(x)), min(0.5,\mu(x))) \\ &\leq min(r_p\mu(rx), s_q\mu(sx), min(0.5,\mu(x))) \\ &\leq min(\nu(rx),\nu(sx), min(0.5,\mu(x))) \\ &\leq min(\nu(rx),\nu(sx)) \\ &\leq \nu(rx+sx) \qquad (since,\ \nu\in F(M)) \\ &= \nu(m)\ \forall\ m=(r+s)x. \end{aligned}$$

Thus, $(r+s)_{min(p,q,0.5)}\mu \subseteq \nu\ \forall\ p,q\in[0,1]$ such that $r_p\mu\subseteq\nu$ and $s_q\mu\subseteq\nu$.
Now,

$$\begin{aligned} min((\nu:\mu)_1(r),(\nu:\mu)_1(s)) &= min(sup\{min(p,0.5)|r_p\mu\subseteq\nu\}), sup\{min(q,0.5)|s_q\mu\subseteq\nu\} \\ &= sup(min(p,q,0.5)|r_p\mu\subseteq\nu,\ s_q\mu\subseteq\nu) \\ &\leq sup\,\{min(h,0.5)|(r+s)_h\mu\subseteq\nu\} \\ &= (\nu:\mu)_1(r+s). \end{aligned}$$

Next we claim that, $(\nu:\mu)_1(rs)\geq(\nu:\mu)_1(r)$.
Let $p\in[0,0.5]$ be such that $r_p\mu\subseteq\nu$. we claim that $(rs)_p\mu\subseteq\nu$. Now if for $m\in M$, $(rs)_p\mu(m)=0\leq\nu(m)$ so nothing to prove. Otherwise, $(rs)_p\mu(m)=sup\,\{min(p,\mu(x))|m=rsx, for\ some$
Therefore;

$$\begin{aligned} (rs)_p\mu(m) &\leq sup\{min(p,\mu(sx))|m=r(sx)\} \\ &\leq sup\{min(p,\mu(sy))|m=ry\} \\ &= r_p\mu(m)\leq\nu(m) \end{aligned}$$

Thus, we get $(rs)_p\mu(m)\leq\nu(m)\ \forall\ m\in M$ and hence $(rs)_p\mu\subseteq\nu$ whenever $r_p\mu\subseteq\nu$. Therfore, $sup\{min\{p,0.5\}|r_p\mu\subseteq\nu\}\leq sup\{min\{h,0.5\}|(rs)_h\mu\subseteq\nu\}$ which implies $(\nu:\mu)_1(r)\leq(\nu:\mu)_1(rs)$. □

**Definition 4.10.** Let $\mu,\nu\in F(M)$. We define;

$$R_\mu(\nu) := Rad(\nu:\mu)_1.$$

**Theorem 4.11.** *If $\nu$ is an $(\in,\in\vee q)$-fuzzy primary submodule of R-module $M$ then $R_{\chi_M}(\nu)$ forms an $(\in,\in\vee q)$-fuzzy prime ideal of R.*

*Proof.* Given that $\nu$ is $(\in,\in\vee q)$-fuzzy primary submodule of $M$, it follows, $\nu$ can be expressed in the following form;

$$\nu(m)=\begin{cases} 1 & if\ m=0; \\ t_i\ (\geq 0.5) & if\ m\in M_i/M_{i+1}, i=1,2,... \\ q\ (<0.5) & otherwise \end{cases}$$

where $M_1=\nu_{0.5}$ is primary submodule of $M$. Consider $R_M(\nu_{0.5})=Rad(\nu_{0.5}:M)$. Evidently, $R_M(\nu_{0.5})$ is a prime ideal of $R$. Now define $\xi\in FI(R)$ as follows;

$$\xi(r)=\begin{cases} 1 & if\ r=0, \\ 0.5 & if\ r\in Rad(M_1:M)/\{0\}, \\ q\ (<0.5) & otherwise \end{cases}$$

We claim that $R_{\chi_M}(\nu)=\xi$.
First we show that $(\nu:\chi_M)_1\subseteq\xi$. If $r\in R_M(\nu_{0.5})$ then $\xi(r)\geq 0.5$ and we know that $(\nu:\chi_M)_1(r)\leq 0.5$ this gives $(\nu:\chi_M)_1\subseteq\xi$. Assume $r\notin R_M(\nu_{0.5})$. Since $R_M(\nu_{0.5})=\{r\in R: r^nM\subseteq\nu_{0.5},\ for\ some\ n\in\mathbb{N}\}$ therefore $r^nM\nsubseteq\nu_{0.5}$ for all $n\in\mathbb{N}$. Hence there does exist an

element $z \in M$ such that $rz \notin \nu_{0.5}$ and so, $\nu(rz) = q$.
Suppose $p \in [0, 0.5]$ be such that $r_p\chi_M \subseteq \nu$. Since

$$r_p\chi_M(w) = \begin{cases} p & if\ w = rx\ for\ some\ x \in M \\ 0 & otherwise \end{cases}$$

So we have $p = r_p\chi_M(rz) \leq \nu(rz) = q$. Hence $p \leq q\ \forall\ p \in [0, 0.5]$ such that $r_p\chi_M \subseteq \nu$. Since $(\nu : \chi_M)_1(r) = sup\{p : r_p\chi_M \subseteq \nu,\ p \in [0, 0.5]\}$ Thus, we get

$$(\nu : \chi_M)_1(r) \leq q = \xi(r)\ \forall r \notin R_M(\nu_{0.5}).$$

Now we will claim that $\xi$ becomes the smallest $(\in, \in \vee q)$-fuzzy prime ideal containing $(\nu : \chi_M)_1$.
Let $\xi^{'} \supseteq (\nu : \chi_M)_1$ be an $(\in, \in \vee q)$-fuzzy prime ideal. It is evident that $r_q\chi_M \subseteq \nu$. Therefore;

$$q \in \{p \in [0, 0.5] | r_p\chi_M \subseteq \nu\ \forall\ r \in R\}$$

Since $(\in, \in \vee q)$-fuzzy prime ideal of $R$ is of the form;

$$\xi^{'}(x) = \begin{cases} 1 & if\ x = 0 \\ t_i^{'}(\geq 0.5) & if\ x \in R_i/R_{i+1},\ i = 1, 2, ... \\ q^{'}(< 0.5) & otherwise \end{cases}$$

where $R_1 = \xi^{'}_{0.5}$. Suppose $r \notin \xi^{'}_{0.5}$. Then, we get $q^{'} = \xi^{'}(r) \geq (\nu : \chi_M)_1(r) \geq q$ this implies $q^{'} > q$. Now for $s \in (\nu_{0.5} : M)$, $sM \subseteq \nu_{0.5}$ and therefore $\nu(sx) \geq 0.5\ \forall\ x \in M$. This implies $s_{0.5}\chi_M(m) \leq \nu(m)\ \forall\ m \in M$. Hence $s_{0.5}\chi_M \subseteq \nu$ and so $0.5 \in \{p \in [0, 0.5] | s_p\chi_M \subseteq \nu\}$.
Now $\xi^{'} \supseteq (\nu : \chi_M)_1$ so we have $\xi^{'}(s) \geq (\nu : \chi_M)_1(s) = 0.5$.
Therefore $\xi^{'}(s) \geq 0.5$ or $s \in \xi^{'}_{0.5}$. Thus $\xi^{'}_{0.5} \supseteq (\nu_{0.5} : M)$ implies $\xi^{'}_{0.5} \supseteq Rad(\nu_{0.5} : M)$. Thus we get $\xi^{'} \supseteq \xi \supseteq (\nu : \chi_M)_1$.
Ultimately we get $R_{\chi_M}(\nu) = Rad(\nu : \chi_M)_1 = \xi$. Which forms an $(\in, \in \vee q)$-fuzzy prime ideal of $R$. □

## 5 Conclusion

In this research, we have broadened the scope of fuzzy prime and primary submodule theory through the introduction of the concepts of $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-fuzzy primary submodules. Our research has redefined traditional definitions of fuzzy prime and fuzzy primary submodules, opening up new possibilities and enriching the theory. Characterization theorems have provided deeper insights into the fundamental nature of these submodules, while our exploration of module homomorphisms and their alpha-cuts has unveiled dynamic aspects of their behavior. We have also bridged $(\in, \in \vee q)$-fuzzy primary submodules with $(\in, \in \vee q)$-fuzzy prime ideals, offering a novel theoretical perspective. In essence, our work not only expands the scope of fuzzy prime(primary) submodule theory but also invites further exploration into these evolving concepts.

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## Author information


Abhishek Kumar Rath, Department of Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India.
E-mail: rathresearch495@gmail.com

A. S. Ranadive, Department of Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India.
E-mail: asranadive04@yahoo.co.in



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