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# Geometry of bilinear forms on the plane with hexagonal norms

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**Abstract** Let $0 < w_1, w_2 < 1$. We denote by $\mathbb{R}^2_{h(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x,y)\|_{h(w_1,w_2)} = \max\Big\{|y|, w_1|x| + w_2|y|\Big\}.$$

We denote by $\mathbb{R}^2_{h'(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x,y)\|_{h'(w_1,w_2)} = \max\Big\{|x|, w_1|x| + w_2|y|\Big\}.$$

In this paper, we classify the extreme bilinear forms of the unit balls of $\mathcal{L}(^2X)$ and $\mathcal{L}_s(^2X)$, where $X = \mathbb{R}^2_{h(w_1,w_2)}$ or $\mathbb{R}^2_{h'(w_1,w_2)}$.

From this, we induce that

$$\text{ext}\, B_{\mathcal{L}_s(^2X)} = \text{ext}\, B_{\mathcal{L}(^2X)} \cap \mathcal{L}_s(^2X).$$

We show that every extreme bilinear forms on that spaces is exposed.

## 1 Introduction

We write $B_E$ for the closed unit ball of a real Banach space $E$. A point $x \in B_E$ is called an *extreme point* of $B_E$ if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $\text{ext}\, B_E$ the set of extreme points of $B_E$. A point $x \in B_E$ is called an *exposed point* of $B_E$ if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. We denote by $\exp B_E$ the set of exposed points of $B_E$. It is easy to see that $\exp B_E \subseteq \text{ext}\, B_E$.

A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form $L$ on the product $E \times E$ such that $P(x) = L(x,x)$ for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on $E$ endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x,y)|$. $\mathcal{L}_s(^2E)$ denotes the closed subspace of $\mathcal{L}(^2E)$ consisting of all continuous symmetric bilinear forms on $E$. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from $E$ into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [6].

Choi *et al.* [2, 3] characterized the extreme points of the unit ball of $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_2^2)$. Kim [11] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^2\ell_p^2)$ $(1 \le p \le \infty)$, where $\ell_p^2 = \mathbb{R}^2$ with the $\ell_p$-norm. Kim [12, 14] classify the extreme, exposed points of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_{d_*} = \max\{|x|, |y|, \frac{|x|+|y|}{1+w}\}$. Kim [11] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_s(^2\ell_\infty^2)$.

We refer to [1–5, 7–18] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces.

Let $0 < w_1, w_2 < 1$. We denote by $\mathbb{R}^2_{h(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x,y)\|_{h(w_1,w_2)} = \max\Big\{|y|, w_1|x| + w_2|y|\Big\}.$$

We denote by $\mathbb{R}^2_{h'(w_1,w_2)}$ the plane with the hexagonal norm

$$\|(x,y)\|_{h'(w_1,w_2)} = \max\Big\{|x|, w_1|x| + w_2|y|\Big\}.$$

In this paper, we classify the extreme bilinear forms of the unit balls of $\mathcal{L}(^2X)$ and $\mathcal{L}_s(^2X)$, where $X = \mathbb{R}^2_{h(w_1,w_2)}$ or $\mathbb{R}^2_{h'(w_1,w_2)}$.

From this, we induce that

$$\text{ext}\, B_{\mathcal{L}_s(^2X)} = \text{ext}\, B_{\mathcal{L}(^2X)} \cap \mathcal{L}_s(^2X).$$

We show that every extreme bilinear forms on that spaces is exposed.

## 2 Extreme bilinear forms on $\mathbb{R}^2_{h(w_1,w_2)}$

Throughout the paper, we let $0 < w_1, w_2 < 1, k_1 = \frac{w_2}{w_1}$ and $k_2 = \frac{1-w_2}{w_1}$.

Note that $1 < k_1 + k_2$. Let $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. Then there are $a, b, c, d \in \mathbb{R}$ such that

$$T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1.$$

For simplicity, we write $T$ by $(a, b, c, d)$.

**Theorem 2.1.** *Let* $0 < w_1, w_2 < 1$ *and* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. *Then*

$$\begin{aligned}\|T\| &= \max\Big\{(k_1+k_2)^2|a|, (|a|k_2+|c|)(k_1+k_2), (|a|k_2+|d|)(k_1+k_2),\\ &\quad |ak_2^2+b|+|c+d|k_2, |ak_2^2-b|+|c-d|k_2\Big\}.\end{aligned}$$

*Proof.* Note that $extB_{\mathbb{R}^2_{h(w_1,w_2)}} = \{(\pm(k_1+k_2), 0), (k_2, \pm1), (-k_2, \pm1)\}$. By the Krein-Milman theorem, $B_{\mathbb{R}^2_{h(w_1,w_2)}}$ is the closed convex hull of the set $extB_{\mathbb{R}^2_{h(w_1,w_2)}}$.

Let $X_1, X_2 \in B_{\mathbb{R}^2_{h(w_1,w_2)}}$. By the Krein-Milman Theorem, there exist $A_j, B_j \in extB_{\mathbb{R}^2_{h(w_1,w_2)}}$ and $t_1^{(j)}, t_2^{(j)} \in \mathbb{R}$ for $j = 1, 2$ such that

$$|t_1^{(j)}| + |t_2^{(j)}| \le 1 \text{ and } X_j = t_1^{(j)}A_j + t_2^{(j)}B_j \quad (j = 1, 2).$$

By the bilinearity of $T$, it follows that

$$\begin{aligned}
|T(X_1, X_2)| &= \Big|T\Big(t_1^{(1)}A_1 + t_2^{(1)}B_1,\ t_1^{(2)}A_2 + t_2^{(2)}B_2\Big)\Big|\\
&\le \sum_{1\le j_k\le 2, 1\le k\le 2} |t_{j_1}^{(1)}||t_{j_n}^{(2)}|\ \max\{\ |T(A_1, A_2)|,\ |T(A_1, B_2)|,\ |T(B_1, A_2)|,\ |T(B_1, B_2)|\ \}\\
&\le \max\{\ |T(A_1, A_2)|,\ |T(A_1, B_2)|,\ |T(B_1, A_2)|,\ |T(B_1, B_2)|\ \}\\
&\le \max\Big\{|T((x_1, y_1), (x_2, y_2))| : (x_j, y_j) \in \text{ext}\,\mathbb{R}^2_{h(w_1,w_2)} \text{ for } j = 1, 2\Big\}\\
&= \max\{|T((\pm(k_1+k_2), 0), (\pm(k_1+k_2), 0))|, |T((\pm(k_1+k_2), 0), (k_2, \pm1))|,\\
&\quad |T((k_2, \pm1), (\pm(k_1+k_2), 0))|, |T((k_2, \pm1), (k_2, \pm1))|\}\\
&= \max\{|T((k_1+k_2, 0), ((k_1+k_2, 0))|, |T((k_1+k_2, 0), (k_2, 1))|,\\
&\quad |T((k_2, 1), (k_1+k_2, 0))|, |T((k_1+k_2, 0), (k_2, -1))|,\\
&\quad |T(k_2, -1), (k_1+k_2, 0))|, |T((k_2, 1),\ (k_2, 1))|,\\
&\quad |T((k_2, -1), (k_2, -1))|, |T((k_2, 1), (k_2, -1))|, |T((k_2, -1), (k_2, 1))|\}\\
&= \max\Big\{(k_1+k_2)^2|a|, (|a|k_2+|c|)(k_1+k_2), (|a|k_2+|d|)(k_1+k_2),\\
&\quad |ak_2^2+b|+|c+d|k_2, |ak_2^2-b|+|c-d|k_2\Big\}\\
&\le \|T\|.
\end{aligned}$$

This completes the proof. □

Note that if $\|T\| = 1$, then $|a| \leq w_1^2, |b| \leq 1, |c| \leq w_1$ and $|d| \leq w_1$.

Let
$$\begin{aligned}
T_1((x_1,y_1),\ (x_2,y_2)) &:= T((x_2,y_2),\ (x_1,y_1)) = (a,b,d,c),\\
T_2((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,-y_1),\ (x_2,y_2)) = (a,-b,c,-d)\\
T_3((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,y_1),\ (x_2,-y_2)) = (a,-b,-c,d)\\
T_4((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,y_1),\ (-x_2,-y_2)) = (-a,-b,-c,-d)\\
T_5((x_1,y_1),\ (x_2,y_2)) &:= T((x_1,-y_1),\ (x_2,-y_2)) = (a,b,-c,-d).
\end{aligned}$$

Then $\|T_i\| = \|T\|$ $(i = 1, \ldots, 5)$. Hence, without loss of generality, we may assume that $a \geq 0$ and $c \geq d \geq 0$.

**Theorem 2.2.** *Let* $0 < w_1, w_2 < 1$ *and* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ *with* $a \geq 0$ *and* $c \geq d \geq 0$. *Then the followings are equivalent:*
(*1*) $T$ *is extreme;*
(*2*) $T_1 = (a,b,d,c)$ *is extreme;*
(*3*) $T_2 = (a,-b,c,-d)$ *is extreme;*
(*4*) $T_3 = (a,-b,-c,d)$ *is extreme;*
(*5*) $T_4 = (-a,-b,-c,-d)$ *is extreme;*
(*6*) $T_5 = (a,b,-c,-d)$ *is extreme.*

*Proof.* It follows from Theorem 2.1 and the above remark. □

**Theorem 2.3.** *Let* $0 < w_1, w_2 < 1$ *and* $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ *with* $\|T\| = 1, a \geq 0$ *and* $c \geq d \geq 0$.
(*a*) *Let* $0 < w_2 < \frac{1}{2}$. *Then,* $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ *if and only if*

$$T \in \Big\{(0,\ \pm 1,\ 0,\ 0),\ (0,\ \pm w_2,\ w_1,\ 0),\ (w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2),$$
$$(w_1^2,\ -w_2(2-w_2),\ w_1w_2,\ w_1w_2)\Big\}.$$

(*b*) *Let* $\frac{1}{2} \leq w_2 < 1$. *Then,* $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ *if and only if*

$$\begin{aligned}
T \in \Big\{&(0,\ \pm 1,\ 0,\ 0),\ (0,\ \pm w_2,\ w_1,\ 0),\ (w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2),\\
&(0,\ \pm(2w_2-1),\ w_1,\ w_1),\ (w_1^2,\ -(w_2^2-w_2+1),\ w_1w_2,\ w_1(1-w_2)),\\
&(w_1^2,\ -w_2(2-w_2),\ w_1(1-w_2),\ w_1(1-w_2)),\\
&(w_1^2,\ -(3w_2^2-4w_2+2),\ w_1w_2,\ w_1w_2)\Big\}.
\end{aligned}$$

*Proof.* Let $k_1 = \frac{w_2}{w_1}, k_2 = \frac{1-w_2}{w_1}$ and $T((x_1,y_1),\ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$. Note that $T \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $\|T \pm (\epsilon,\ \delta,\ \gamma,\ \rho)\| \leq 1$ for some $\epsilon \geq 0, \delta, \gamma, \rho \in \mathbb{R}$ implies that $\epsilon = \delta = \gamma = \rho = 0$. Notice that

$$\begin{aligned}
&T((0,1),(0,1)) = b,\\
&T((k_1+k_2,0),(k_1+k_2,0)) = (k_1+k_2)^2 a,\\
&T((k_1+k_2,0),(k_2,1)) = (ak_2+c)(k_1+k_2),\\
&T((k_2,1),(k_1+k_2,0)) = (ak_2+d)(k_1+k_2),\\
&T((k_2,1),(k_2,1)) = ak_2^2 + b + (c+d)k_2,\\
&T((k_2,-1),(k_2,1)) = ak_2^2 - b + (c-d)k_2,\\
&T((k_2,-1),(k_2,-1)) = ak_2^2 + b - (c+d)k_2.
\end{aligned}$$

**Case 1**. Let $|b| = 1$

Since $\|T\| = 1$, by Theorem 2.1, $T = (0,\ \pm 1,\ 0,\ 0)$.

**Claim**. $(0,\ 1,\ 0,\ 0) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1 \le w_2 < 1$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \le 1$ for some $\epsilon \ge 0, \delta, \gamma, \rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$. It follows that

$$\begin{aligned}
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((0,1),(0,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((0,1),(0,1))| = 1 + |\delta|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,-1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,-1))|\\
&= 1 + |\epsilon k_2^2 - \delta + (-\gamma+\rho)k_2| = 1 + |\epsilon k_2^2 + (-\gamma+\rho)k_2|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))|\\
&= 1 + |\epsilon k_2^2 + \delta - (\gamma+\rho)k_2| = 1 + |\epsilon k_2^2 - (\gamma+\rho)k_2|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))|\\
&= 1 + |\epsilon k_2^2 + \delta + (\gamma+\rho)k_2| = 1 + |\epsilon k_2^2 + \gamma k_2|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$. Therefore, $T$ is extreme.

By Theorem 2.2, $(0,\ -1,\ 0,\ 0) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$

**Case 2**. Let $|b| < 1$

**Subcase 1**. Let $0 \le b < 1$

Let $a = 0$.

Since $T$ is extreme,

$$(ak_2 + c)(k_1 + k_2) = (ak_2 + d)(k_1 + k_2) = |ak_2^2 + b| + (c + d)k_2 = 1.$$

By calculation, $T = \frac{1}{k_1+k_2}(0,\ k_1 - k_2,\ 1,\ 1) = (0,\ 2w_2 - 1,\ w_1,\ w_1)$ for $\frac{1}{2} \le w_2 < 1$.

**Claim.** $T = (0,\ 2w_2 - 1,\ w_1,\ w_1) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $\frac{1}{2} \le w_2 < 1$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \le 1$ for some $\epsilon \ge 0, \delta, \gamma, \rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$. It follows that

$$\begin{aligned}
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))|\\
&= 1 + |\epsilon k_2(k_1+k_2) + (k_1+k_2)\gamma|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))|\\
&= 1 + |\epsilon k_2(k_1+k_2) + (k_1+k_2)\rho|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_1+k_2,0))|\\
&= 1 + |\epsilon k_2(k_1+k_2) - (k_1+k_2)\rho|,\\
1 &\ge |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))|\\
&= 1 + |\epsilon k_2^2 + \delta + (\gamma+\rho)k_2|,
\end{aligned}$$

which shows that $\epsilon = \delta = \gamma = \rho = 0$ and $T$ is extreme.

Let $0 < a$.

**Claim**. $|ak_2^2 + b| + (c + d)k_2 > |ak_2^2 - b| + (c - d)k_2$

It suffices to show that if $b = 0$, then $d > 0$. Let $b = 0 = d$.

Assume that $|ak_2^2 + b| + (c + d)k_2 < 1$.

Since $\|T\| = 1$, by Theorem 2.1, $(k_1 + k_2)^2 a = 1$ or $(ak_2 + c)(k_1 + k_2) = 1$.

Let $(k_1 + k_2)^2 a = 1$.

By calculation, $T = \left(\frac{1}{(k_1+k_2)^2},\ 0,\ c,\ 0\right)$ for $0 \le c < \frac{k_1}{(k_1+k_2)^2}$ and $0 < w_2 < \frac{3-\sqrt{5}}{2}$ or $0 \le c \le w_1$ and $\frac{3-\sqrt{5}}{2} \le w_2 < 1$. Since $T$ is extreme, $T = (w_1^2,\ 0,\ 0,\ 0)$ or $(w_1^2,\ 0,\ w_1,\ 0)$. Note that $(w_1^2,\ 0,\ 0,\ 0)$ is not extreme and $\left\|(w_1^2,\ 0,\ w_1,\ 0)\right\| > 1$. This is a contradiction.

Let $(ak_2 + c)(k_1 + k_2) = 1$.

By calculation, $T = \Big(a,\ 0,\ \frac{1}{k_1+k_2} - ak_2,\ 0\Big)$ for $0 < a \le \frac{1}{(k_1+k_2)^2}$. Since $T$ is extreme, $T = \frac{1}{(k_1+k_2)^2}(1,\ 0,\ k_1,\ 0)$. This is not extreme because

$$\Big\|\Big(w_1^2,\ 0,\ w_1w_2,\ \pm\frac{1}{n}\Big)\Big\| \le 1 \text{ for a sufficiently large } n \in \mathbb{N}.$$

This is a contradiction.

Assume that $|ak_2^2 + b| + (c+d)k_2 = 1$. It follows that

$$1 = |ak_2^2 + b| + (c+d)k_2 = ak_2^2 + ck_2 < (ak_2 + c)(k_1 + k_2) \le 1,$$

which is impossible. Thus, $d > 0$ and the claim holds.

Since $T$ is extreme, we have

$$1 \quad = \quad (k_1 + k_2)^2 a = (ak_2 + c)(k_1 + k_2) = ak_2^2 + b + (c+d)k_2.$$

By calculation,

$$T = \Big(\frac{1}{(k_1+k_2)^2},\ \frac{k_1(k_1+k_2)}{(k_1+k_2)^2} - dk_2,\ \frac{k_1}{(k_1+k_2)^2},\ d\Big)$$

for $0 \le d \le \frac{k_1}{(k_1+k_2)^2}$. Since $T$ is extreme, $T = \frac{1}{(k_1+k_2)^2}(1,\ k_1(k_1{+}k_2),\ k_1,\ 0)$ or $\frac{1}{(k_1+k_2)^2}(1,\ k_1^2,\ k_1,\ k_1) = (w_1^2, w_2^2, w_1w_2, w_1w_2)$. Note that $\frac{1}{(k_1+k_2)^2}(1,\ k_1(k_1+k_2),\ k_1,\ 0) = (w_1^2,\ w_2,\ w_1w_2,\ 0)$ is not extreme because

$$\Big\|\Big(w_1^2,\ w_2 \pm \frac{1}{n},\ w_1w_2,\ \mp\frac{1}{n}\Big)\Big\| \le 1 \text{ for a sufficiently large } n \in \mathbb{N}.$$

**Claim**. $(w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \le 1$ for some $\epsilon \ge 0, \delta, \gamma, \rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$. It follows that

$$\begin{aligned}
1 \quad &\ge \quad |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))| \\
&= \quad 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))| = 1 + (k_1+k_2)^2|\epsilon|, \\
1 \quad &\ge \quad |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| \\
&= \quad 1 + |\epsilon(k_1+k_2)k_2 + \gamma(k_1+k_2)| = 1 + |\gamma|(k_1+k_2), \\
1 \quad &\ge \quad |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| \\
&= \quad 1 + |\epsilon(k_1+k_2)k_2 + \rho(k_1+k_2)| = 1 + |\rho|(k_1+k_2), \\
1 \quad &\ge \quad |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_2,1))| \\
&= \quad 1 + |\epsilon(k_2)^2 + \delta + (\gamma+\rho)k_2| = 1 + |\delta|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$. Therefore, $T$ is extreme.

**Subcase 2**. Let $-1 < b < 0$

Suppose that $|ak_2^2 + b| + (c+d)k_2 > |ak_2^2 - b| + (c-d)k_2$.

Let $a = 0$.

Then $d > 0, c = \frac{1}{k_1+k_2}$. By calculation, $T = \Big(0,\ -\frac{k_1}{k_1+k_2} + dk_2,\ \frac{1}{k_1+k_2},\ d\Big)$ for $0 < d \le \frac{1}{k_1+k_2}$. Since $T$ is extreme, $d = \frac{1}{k_1+k_2}$ and $T = \frac{1}{k_1+k_2}(0,\ -k_1+k_2,\ 1,\ 1) = (0,\ -(2w_2-1),\ w_1,\ w_1)$ for $\frac{1}{2} < w_2 < 1$.

**Claim.** $T = (0,\ -(2w_2-1),\ w_1,\ w_1) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $\frac{1}{2} < w_2 < 1$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \le 1$ for some $\epsilon \ge 0, \delta, \gamma, \rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$.

It follows that

$$\begin{aligned}
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| \\
&= 1 + |\epsilon k_2(k_1+k_2) + (k_1+k_2)\gamma|, \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| \\
&= 1 + |\epsilon k_2(k_1+k_2) + (k_1+k_2)\rho|, \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_1+k_2,0))| \\
&= 1 + |\epsilon k_2(k_1+k_2) - (k_1+k_2)\rho|, \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| \\
&= 1 + |\epsilon k_2^2 + \delta - (\gamma+\rho)k_2|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$ and $T$ is extreme.

Let $0 < a$.

Since $T$ is extreme, we have

$$1 = (k_1+k_2)^2 a = (ak_2+c)(k_1+k_2) = |ak_2^2+b| + (c+d)k_2.$$

**Claim.** $ak_2^2 + b < 0$

Suppose not. $ak_2^2+b = 0$ or $ak_2^2+b > 0$.

Let $ak_2^2+b = 0$.

By calculation, $w_1w_2 \geq d = \frac{w_1(w_2^2-w_2+1)}{1-w_2} > w_1w_2$ , which is impossible.

Let $ak_2^2+b > 0$.

By calculation, $T = (w_1^2,\ w_2 - dk_2,\ w_1w_2,\ d)$ for $0 \leq d \leq w_1w_2$.

Since $-1 < b = w_2 - dk_2 < 0$, $w_1w_2 < \frac{w_2}{k_2} < d < \frac{1+w_2}{k_2}$, which is a contradiction because $w_1w_2 < d \leq w_1w_2$. Therefore, the claim holds.

By calculation, $T = \left(w_1^2,\ -2+3w_2-2w_2^2 + (\frac{1-w_2}{w_1})d,\ w_1w_2,\ d\right)$ for $0 < d \leq w_1w_2$. Since $T$ is extreme, $T = (w_1^2,\ -(3w_2^2-4w_2+2),\ w_1w_2,\ w_1w_2)$ for $\frac{1}{2} < w_2 < 1$.

**Claim**. $(w_1^2,\ -(3w_2^2-4w_2+2),\ w_1w_2,\ w_1w_2) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $\frac{1}{2} < w_2 < 1$

Let $\|T \pm (\epsilon,\delta,\gamma,\rho)\| \leq 1$ for some $\epsilon \geq 0, \delta,\gamma,\rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$.

It follows that

$$\begin{aligned}
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))| \\
&= 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))| = 1 + (k_1+k_2)^2|\epsilon|, \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| \\
&= 1 + |\epsilon(k_1+k_2)k_2 + \gamma(k_1+k_2)| = 1 + |\gamma|(k_1+k_2), \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,1),(k_1+k_2,0))| \\
&= 1 + |\epsilon(k_1+k_2)k_2 + \rho(k_1+k_2)| = 1 + |\rho|(k_1+k_2), \\
1 &\geq |T \pm (\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| = 1 + |(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| \\
&= 1 + |\epsilon k_2^2 + \delta - (\gamma+\rho)k_2| = 1 + |\delta|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$. Therefore, $T$ is extreme.

Suppose that $|ak_2^2+b| + (c+d)k_2 < |ak_2^2-b| + (c-d)k_2$.

Note that $a > 0$. Since $T$ is extreme, we have

$$1 \;=\; (k_1+k_2)^2 a = (ak_2+c)(k_1+k_2) = ak_2^2 - b + (c-d)k_2.$$

By calculation, $T = (w_1^2,\ -w_2 - dk_2,\ w_1w_2,\ d)$ for $0 \leq d \leq w_1w_2$. Since $T$ is extreme, $T = (w_1^2,\ -w_2,\ w_1w_2,\ 0)$ or $T = (w_1^2,\ -w_2(2-w_2),\ w_1w_2,\ w_1w_2)$ for $0 < w_2 < \frac{1}{2}$. Note that $T = (w_1^2,\ -w_2,\ w_1w_2,\ 0)$ is not extreme because

$$\left\|\left(w_1^2,\ -w_2 \pm \frac{1}{n},\ w_1w_2,\ \mp\frac{1}{nk_2}\right)\right\| \leq 1 \text{ for a sufficiently large } n \in \mathbb{N}.$$

**Claim**. $(w_1^2,\ -w_2(2-w_2),\ w_1w_2,\ w_1w_2) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_2 < \frac{1}{2}$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \leq 1$ for some $\epsilon \geq 0, \delta, \gamma, \rho \in \mathbb{R}$.
We will show that $\epsilon = \delta = \gamma = \rho = 0$. It follows that

$$\begin{aligned}
1 &\geq |T \pm (\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_1+k_2, 0))| \\
&= 1 + |(\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_1+k_2, 0))| = 1 + (k_1+k_2)^2|\epsilon|, \\
1 &\geq |T \pm (\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_2, 1))| = 1 + |(\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_2, 1))| \\
&= 1 + |\epsilon(k_1+k_2)k_2 + \gamma(k_1+k_2)| = 1 + |\gamma|(k_1+k_2), \\
1 &\geq |T \pm (\epsilon, \delta, \gamma, \rho)((k_2, 1), (k_1+k_2, 0))| = 1 + |(\epsilon, \delta, \gamma, \rho)((k_2, 1), (k_1+k_2, 0))| \\
&= 1 + |\epsilon(k_1+k_2)k_2 + \rho(k_1+k_2)| = 1 + |\rho|(k_1+k_2), \\
1 &\geq |T \pm (\epsilon, \delta, \gamma, \rho)((k_2, -1), (k_2, 1))| = 1 + |(\epsilon, \delta, \gamma, \rho)((k_2, -1), (k_2, 1))| \\
&= 1 + |\epsilon k_2^2 - \delta + (\gamma - \rho)k_2| = 1 + |\delta|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$. Therefore, $T$ is extreme.
Suppose that $|ak_2^2 + b| + (c+d)k_2 = |ak_2^2 - b| + (c-d)k_2$.

**Claim.** $|ak_2^2 + b| + (c+d)k_2 = |ak_2^2 - b| + (c-d)k_2 = 1$

Suppose not. Without loss of generality we may assume that

$$1 = (k_1+k_2)^2 a = (ak_2 + c)(k_1+k_2) = (ak_2 + d)(k_1+k_2)$$

because $T$ is extreme. Notice that

$$\left\| \Big( w_1^2\ , b \pm \frac{1}{n},\ w_1(1-w_2),\ w_1(1-w_2) \Big) \right\| \leq 1$$

for a sufficiently large $n \in \mathbb{N}$. Thus, $T$ is not extreme, This is a contradiction.

**Claim**. $ak_2^2 + b < 0$

Suppose not. Since

$$1 = ak_2^2 + b + (c+d)k_2 = ak_2^2 - b + (c-d)k_2,$$

$$1 = ak_2^2 + ck_2 < (ak_2 + c)(k_1+k_2) \leq 1.$$

This is a contradiction. Therefore, the claim holds.
Thus,

$$1 = -(ak_2^2 + b) + (c+d)k_2 = |ak_2^2 - b| + (c-d)k_2.$$

By calculation,

$$T = \Big(a, ck_2 - 1, c, ak_2\Big) \text{ for } 0 \leq a \leq w_1^2.$$

Let $a = 0$. Then,

$$T = \Big(0, ck_2 - 1, c, 0\Big) \text{ for } 0 < c \leq w_1.$$

Since $T$ is extreme, $T = (0,\ -w_2,\ w_1,\ 0)$ for $0 < w_1, w_2 < 1$.

**Claim.** $(0,\ -w_2,\ w_1,\ 0) \in \operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let $\|T \pm (\epsilon, \delta, \gamma, \rho)\| \leq 1$ for some $\epsilon \geq 0, \delta, \gamma, \rho \in \mathbb{R}$. We will show that $\epsilon = \delta = \gamma = \rho = 0$. It follows that

$$1 \geq |T \pm (\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_2, 1))| = 1 + |(\epsilon, \delta, \gamma, \rho)((k_1+k_2, 0), (k_2, 1))|$$

$$\begin{aligned}
&= 1+|\epsilon k_2(k_1+k_2)+(k_1+k_2)\gamma|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,-1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,-1))|\\
&= 1+|\epsilon k_2(k_1+k_2)-(k_1+k_2)\gamma|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))|\\
&= 1+|\epsilon k_2^2+\delta-(\gamma+\rho)k_2|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,1))|\\
&= 1+|\epsilon k_2^2-\delta+(\gamma-\rho)k_2|.
\end{aligned}$$

Thus, $\epsilon=\delta=\gamma=\rho=0$ and $T$ is extreme.

By Theorem 2.2, $(0,\ w_2,\ w_1,\ 0)\in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$.

Let $a>0$.

Since $T$ is extreme and $0<a\leq w_1^2$,

$$T=\Big(w_1^2,\ ck_2-1,\ c,\ w_1(1-w_2)\Big) \text{ for } w_1(1-w_2)\leq c\leq w_1w_2$$

because $(ak_2+c)(k_1+k_2)\leq 1$. Since $T$ is extreme,

$$T=(w_1^2,\ -w_2(2-w_2),\ w_1(1-w_2),\ w_1(1-w_2)) \text{ for } \frac{1}{2}\leq w_2<1$$

or

$$(w_1^2,\ -(w_2^2-w_2+1),\ w_1w_2,\ w_1(1-w_2)) \text{ for } \frac{1}{2}\leq w_2<1.$$

**Claim**. $T=(w_1^2,\ -w_2(2-w_2),\ w_1(1-w_2),\ w_1(1-w_2))\in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $\frac{1}{2}\leq w_2<1$

Let $\|T\pm(\epsilon,\delta,\gamma,\rho)\|\leq 1$ for some $\epsilon\geq 0,\delta,\gamma,\rho\in\mathbb{R}$. We will show that $\epsilon=\delta=\gamma=\rho=0$. It follows that

$$\begin{aligned}
1 &\geq |T\pm(\epsilon,\delta,\gamma,\gamma)((k_1+k_2,0),(k_1+k_2,0))|\\
&= 1+|(\epsilon,\delta,\gamma,\gamma)((k_1+k_2,0),(k_1+k_2,0))| = 1+(k_1+k_2)^2|\epsilon|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\gamma)((k_2,-1),(k_2,1))| = 1+|(\epsilon,\delta,\gamma,\gamma)((k_2,-1),(k_2,1))|\\
&= 1+|\epsilon k_2^2-\delta+(\gamma-\rho)k_2|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\gamma)((k_2,1),(k_2,-1))| = 1+|(\epsilon,\delta,\gamma,\gamma)((k_2,1),(k_2,-1))|\\
&= 1+|\epsilon k_2^2-\delta-(\gamma-\rho)k_2|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\gamma)((k_2,-1),(k_2,-1))| = 1+|(\epsilon,\delta,\gamma,\gamma)((k_2,-1),(k_2,-1))|\\
&= 1+|\epsilon k_2^2+\delta-(\gamma+\rho)k_2|.
\end{aligned}$$

Thus, $\epsilon=\delta=\gamma=\rho=0$. Therefore, $T$ is extreme.

**Claim**. $(w_1^2,\ -(w_2^2-w_2+1),\ w_1w_2,\ w_1(1-w_2))\in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $\frac{1}{2}\leq w_2<1$
Let $\|T\pm(\epsilon,\delta,\gamma,\rho)\|\leq 1$ for some $\epsilon\geq 0,\delta,\gamma,\rho\in\mathbb{R}$. We will show that $\epsilon=\delta=\gamma=\rho=0$. It follows that

$$\begin{aligned}
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))|\\
&= 1+|(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_1+k_2,0))| = 1+(k_1+k_2)^2|\epsilon|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_1+k_2,0),(k_2,1))|\\
&= 1+|\epsilon(k_1+k_2)k_2+\gamma(k_1+k_2)| = 1+|\gamma|(k_1+k_2),\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,1))|\\
&= 1+|\epsilon k_2^2-\delta+(\gamma-\rho)k_2| = 1+|\delta|,\\
1 &\geq |T\pm(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))| = 1+|(\epsilon,\delta,\gamma,\rho)((k_2,-1),(k_2,-1))|\\
&= 1+|\epsilon k_2^2+\delta-(\gamma+\rho)k_2| = 1+|\delta|.
\end{aligned}$$

Thus, $\epsilon = \delta = \gamma = \rho = 0$. So, $T$ is extreme.

This completes the proof. □

We are in a position to classify the extreme points of the unit balls of $\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$ and $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$ for $0 < w_1, w_2 < 1$.

**Theorem 2.4.** *Let* $0 < w_1, w_2 < 1$.
$(a)$ *Let* $w_2 < \frac{1}{2}$. *Then,*

$$\begin{aligned}\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{&(0,\ \pm1,\ 0,\ 0), \pm(0,\ \pm w_2,\ w_1,\ 0), \pm(0,\ \pm w_2,\ 0,\ w_1),\\
&\pm(w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(w_1^2,\ -w_2^2,\ -w_1w_2,\ w_1w_2),\\
&\pm(w_1^2,\ -w_2^2,\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ w_2^2,\ -w_1w_2,\ -w_1w_2),\\
&\pm(w_1^2,\ -w_2(2-w_2),\ w_1w_2,\ w_1w_2), \pm(w_1^2,\ w_2(2-w_2),\ -w_1w_2,\ w_1w_2),\\
&\pm(w_1^2,\ w_2(2-w_2),\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ -w_2(2-w_2),\ -w_1w_2,\ -w_1w_2)\Big\}.\end{aligned}$$

$(b)$ *Let* $\frac{1}{2} \le w_2$.
*Then,*

$$\begin{aligned}\operatorname{ext} B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{&(0,\ \pm1,\ 0,\ 0), \pm(0,\ \pm w_2,\ w_1,\ 0), \pm(0,\ \pm w_2,\ 0,\ w_1),\\
&\pm(w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(w_1^2,\ -w_2^2,\ -w_1w_2,\ w_1w_2),\\
&\pm(w_1^2,\ -w_2^2,\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ w_2^2,\ -w_1w_2,\ -w_1w_2),\\
&\pm(0,\ 2w_2-1,\ w_1,\ w_1), \pm(0,\ -(2w_2-1),\ -w_1,\ w_1), \pm(0,\ 2w_2-1,\ w_1,\ -w_1),\\
&\pm(0,\ 2w_2-1,\ -w_1,\ -w_1), \pm(w_1^2,\ -(w_2^2-w_2+1),\ w_1w_2,\ w_1(1-w_2)),\\
&\pm(w_1^2,\ w_2^2-w_2+1,\ -w_1w_2,\ w_1(1-w_2)), \pm(w_1^2,\ w_2^2-w_2+1,\ w_1w_2,\ -w_1(1-w_2)),\\
&\pm(w_1^2,\ -(w_2^2-w_2+1),\ -w_1w_2,\ -w_1(1-w_2)),\\
&\pm(w_1^2,\ -(w_2^2-w_2+1),\ w_1(1-w_2), w_1w_2,\ ),\\
&\pm(w_1^2,\ w_2^2-w_2+1,\ w_1(1-w_2),\ -w_1w_2),\\
&\pm(w_1^2,\ w_2^2-w_2+1,\ -w_1(1-w_2),\ w_1w_2),\\
&\pm(w_1^2,\ -(w_2^2-w_2+1),\ -w_1(1-w_2),\ -w_1w_2),\\
&\pm(w_1^2,\ -w_2(2-w_2),\ w_1(1-w_2),\ w_1(1-w_2)),\\
&\pm(w_1^2,\ w_2(2-w_2),\ -w_1(1-w_2),\ w_1(1-w_2)),\\
&\pm(w_1^2,\ w_2(2-w_2),\ w_1(1-w_2),\ -w_1(1-w_2)),\\
&\pm(w_1^2,\ -w_2(2-w_2),\ -w_1(1-w_2),\ -w_1(1-w_2)),\\
&\pm(w_1^2,\ -(3w_2^2-4w_2+2),\ w_1w_2,\ w_1w_2), \pm(w_1^2,\ 3w_2^2-4w_2+2,\ -w_1w_2,\ w_1w_2),\\
&\pm(w_1^2,\ 3w_2^2-4w_2+2,\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ -(3w_2^2-4w_2+2),\ -w_1w_2,\ -w_1w_2)\Big\}.\end{aligned}$$

*Proof.* It follows from Theorems 2.2 and 2.3. □

Note that $\{x_1x_2,\ y_1y_2,\ x_1y_2 + x_2y_1\}$ is a basis for $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$. Thus, if $T = (a, b, c, c) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$, we will write $T = (a, b, c)$.

**Theorem 2.5.** *Let* $0 < w_1, w_2 < 1$.
$(a)$ *Let* $w_2 < \frac{1}{2}$. *Then,*

$$\operatorname{ext} B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{(0,\ \pm1,\ 0), \pm(w_1^2,\ w_2^2,\ \pm w_1w_2), \pm(w_1^2,\ -w_2(2-w_2),\ \pm w_1w_2)\Big\}.$$

$(b)$ *Let* $w_2 = \frac{1}{2}$.
*Then,*

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{(0,\ \pm 1,\ 0), \pm(w_1^2,\ 1/4,\ \pm w_1/2), \pm(w_1^2,\ -3/4,\ \pm w_1/2)\Big\}.$$

$(c)$ *Let* $\frac{1}{2} < w_2$.
*Then,*

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \Big\{(0,\ \pm 1,\ 0), \pm(w_1^2,\ w_2^2,\ \pm w_1 w_2), \pm(0,\ \pm(2w_2 - 1),\ \pm w_1),$$
$$\pm(w_1^2,\ -w_2(2 - w_2),\ \pm w_1(1 - w_2)), \pm(w_1^2,\ -(3w_2^2 - 4w_2 + 2),\ \pm w_1 w_2)\Big\}.$$

*Proof.* It follows from Theorem 2.2 and slight modifications in the proof of Theorem 2.3. □

We may ask the following questions: Is it true that

$$\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})$$

for $0 < w_1, w_2 < 1$?
In general, it is true that

$$\text{ext}\, B_{\mathcal{L}_s({}^2E)} \supseteq \text{ext}\, B_{\mathcal{L}({}^2E)} \cap \mathcal{L}_s({}^2E)$$

for a Banach space.
Theorems 2.4 and 2.5 show the following:

**Remark 2.6.** It is true that $\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})$ for $0 < w_1, w_2 < 1$.

## 3 Exposed bilinear forms on $\mathbb{R}^2_{h(w_1,w_2)}$

**Theorem 3.1.** *([13]) Let $E$ be a real Banach space such that $extB_E$ is finite. Suppose that $x \in \text{ext}\, B_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \text{ext}\, B_E \backslash \{\pm x\}$. Then $x \in \exp B_E$.*

Note that $\{x_1x_2,\ y_1y_2,\ x_1y_2,\ x_2y_1\}$ is a basis for $\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})$.

**Theorem 3.2.** *Let* $0 < w_1, w_2 < 1$ *and* $f \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$. *Let* $\alpha = f(x_1x_2), \beta = f(y_1y_2), u = f(x_1y_2), v = f(x_2y_1)$.
$(a)$ *Let* $0 < w_2 < \frac{1}{2}$. *Then,*

$$\|f\| = \max\Big\{|\beta|,\ |w_1^2\alpha \pm w_2^2\beta| + w_1w_2|u \pm v|,\ |w_1^2\alpha \mp w_2(2 - w_2)\beta| + w_1w_2|u \pm v|,$$
$$w_2|\beta| + w_1|u|,\ w_2|\beta| + w_1|v|\Big\}.$$

$(b)$ *Let* $\frac{1}{2} \leq w_2 < 1$. *Then,*

$$\|f\| = \max\Big\{|\beta|,\ |w_1^2\alpha \pm w_2^2\beta| + w_1w_2|u \pm v|,\ w_2|\beta| + w_1|u|,\ w_2|\beta| + w_1|v|,$$
$$(2w_2 - 1)|\beta| + w_1(|u| + |v|),\ |w_1^2\alpha \mp (w_2^2 - w_2 + 1)\beta| + w_1|w_2u \pm (1 - w_2)v|,$$
$$|w_1^2\alpha \mp (w_2^2 - w_2 + 1)\beta| + w_1|(1 - w_2)u \pm w_2v|,$$
$$|w_1^2\alpha \mp w_2(2 - w_2)\beta| + w_1(1 - w_2)|u \pm v|,$$
$$|w_1^2\alpha \mp (3w_2^2 - 4w_2 + 2)\beta| + w_1w_2|u \pm v|\Big\}.$$

*Proof.* If follows from Theorem 2.4 and the fact that

$$\|f\| = \sup_{T \in \text{ext } B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}} |f(T)|.$$

□

By Theorem 2.1, if $f \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ with $\|f\| = 1$, then

$$|\alpha| \leq \frac{1}{w_1^2}, |\beta| \leq 1, |u| \leq \frac{1}{w_1}, |v| \leq \frac{1}{w_1}.$$

If $f \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$, we will write $f = \Big(f(x_1x_2), f(y_1y_2), f(x_1y_2), f(x_2y_1)\Big)$.

**Theorem 3.3.** *Let* $0 < w_1, w_2 < 1$ *and* $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})$. *Then the followings are equivalent:*
(*1*) $T$ *is exposed;*
(*2*) $T_1 = (a, b, d, c)$ *is exposed;*
(*3*) $T_2 = (a, -b, c, -d)$ *is exposed;*
(*4*) $T_3 = (a, -b, -c, d)$ *is exposed;*
(*5*) $T_4 = (-a, -b, -c, -d)$ *is exposed;*
(*6*) $T_5 = (a, b, -c, -d)$ *is exposed.*

*Proof.* We only show that (1)⇔(3) since the proofs of the other cases are similar. (1)⇒(3): Let $f = (\alpha, \beta, u, v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$ be such that $f(T) = \|f\| = 1$ and $f(S) < 1$ for all $S \in B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{T\}$. Let $g = (\alpha, -\beta, u, -v) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. Then $g(T_2) = f(T) = 1$. By Theorem 3.2, $\|g\| = 1$.

**Claim.** $g$ exposes $T_2$.

Let $S = (a', b', c', d') \in B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{T_2\}$. Then

$$\|(a', b', c', d')\| \leq 1,\ g(S) = a'\alpha - b'\beta + c'u - d'v = f(a', -b', c', -d') < 1$$

because $(a', -b', c', -d') \neq T$. Thus, the claim holds.

The proof of (3)⇒(1) is similar. □

Note that $\{x_1x_2,\ y_1y_2,\ x_1y_2{+}x_2y_1\}$ is a basis for $\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})$. Thus, if $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$, we will write $f = \Big(f(x_1x_2), f(y_1y_2), f(x_1y_2 + x_2y_1)\Big)$.

**Theorem 3.4.** *Let* $0 < w_1, w_2 < 1$ *and* $f \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. *Let* $\alpha = f(x_1x_2), \beta = f(y_1y_2), \theta = f(x_1y_2 + x_2y_1)$.
(*a*) *Let* $w_2 < \frac{1}{2}$. *Then,*

$$\|f\| = \max\Big\{|\beta|,\ |w_1^2\alpha + w_2^2\beta| + w_1w_2|\theta|,\ |w_1^2\alpha - w_2(2 - w_2)\beta| + w_1w_2|\theta|\Big\}.$$

(*b*) *Let* $\frac{1}{2} \leq w_2$. *Then,*

$$\begin{aligned} \|f\| = \max\Big\{&|\beta|,\ |w_1^2 + w_2^2\beta| + w_1w_2|\theta|,\ (2w_2 - 1)|\beta| + w_1|\theta|, \\ &|w_1^2\alpha - w_2(2 - w_2)\beta| + w_1(1 - w_2)|\theta|,\ |w_1^2\alpha - (3w_2^2 - 4w_2 + 2)\beta| + w_1w_2|\theta|\Big\}. \end{aligned}$$

*Proof.* If follows from Theorem 2.5 and the fact that

$$\|f\| = \sup_{T \in \text{ext } B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}} |f(T)|.$$

□

Note that if $f \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$ with $\|f\| = 1$, then

$$|\alpha| \leq \frac{1}{w_1^2}, |\beta| \leq 1, |\theta| \leq \frac{1}{w_1}.$$

The following shows that every extreme point of the unit balls of $\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})$ and $\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})$ is exposed.

**Theorem 3.5.** *Let* $0 < w_1, w_2 < 1$. *Then,*
$(a)$ $\exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$;
$(b)$ $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \mathrm{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

*Proof.* (a) It is enough to show that $\mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \subseteq \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$. Let $T = (a, b, c, d) \in \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$. By Theorem 3.3, we may assume that $a \geq 0$ and $c \geq d \geq 0$.

**Claim.** $T = (0,\ \pm 1,\ 0,\ 0) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let $f = (0,\ 1,\ 0,\ 0) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^*$. By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (0,\ -w_2,\ w_1,\ 0) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let

$$f = \Big(\frac{1 - w_2}{3w_1^2},\ -\frac{2}{3},\ \frac{3 - 2w_2}{3w_1},\ 0\Big) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

By Theorem 3.3, $(0,\ w_2,\ w_1,\ 0) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$.

**Claim.** $T = (w_1^2,\ w_2^2,\ w_1 w_2,\ w_1 w_2) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let

$$f = \Big(\frac{(2 - w_2)^2}{4w_1^2},\ \frac{1}{4},\ \frac{2 - w_2}{4w_1},\ \frac{2 - w_2}{4w_1}\Big) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ -w_2(2 - w_2),\ w_1 w_2,\ w_1 w_2) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_2 < \frac{1}{2}$

Let

$$f = \Big(\frac{(2 - w_2)^2}{4w_1^2},\ -\frac{1}{4},\ \frac{2 - w_2}{4w_1},\ \frac{w_2}{4w_1}\Big)$$

or

$$\Big(\frac{1}{w_1^2}\Big(1 - \frac{2}{n}\Big),\ -\frac{1}{w_2(2 - w_2)n},\ \frac{1}{2w_1 w_2 n},\ \frac{1}{2w_1 w_2 n}\Big) \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \mathrm{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

Suppose that $\frac{1}{2} \leq w_2 < 1$.

**Claim.** $T = (0,\ 2w_2 - 1,\ w_1,\ w_1) \in \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let

$$f = \Big(0,\ \frac{2w_1}{(2w_2 - 1)n},\ \frac{1}{2w_1} - \frac{1}{n},\ \frac{1}{2w_1} - \frac{1}{n}\Big) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})^* \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (0, -(2w_2 - 1), w_1, w_1) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let

$$f = \Big(0,\ -\frac{2w_1}{(2w_2-1)n},\ \frac{1}{2w_1} - \frac{1}{n},\ \frac{1}{2w_1} - \frac{1}{n}\Big) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^* \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2, -(w_2^2 - w_2 + 1), w_1 w_2, w_1(1 - w_2)) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let

$$f = \Big(\frac{2-w_2}{4w_1^2},\ -\frac{1}{2},\ \frac{3-2w_2}{4w_1},\ 0\Big) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ -w_2(2 - w_2),\ w_1(1 - w_2),\ w_1(1 - w_2)) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let

$$f = \Big(\frac{(2-w_2)^2}{4w_1^2},\ -\frac{1}{4},\ \frac{1}{4w_1},\ \frac{1}{4w_1}\Big)$$

or

$$\Big(\frac{1}{w_1^2}\Big(1 - \frac{2}{n}\Big),\ -\frac{1}{w_2(2-w_2)n},\ \frac{1}{4w_1 w_2 n},\ \frac{1}{4w_1 w_2 n}\Big) \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ -(3w_2^2 - 4w_2 + 2),\ w_1 w_2,\ w_1 w_2) \in \exp B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let

$$f = \Big(\frac{2-w_2^2}{4w_1^2},\ -\frac{1}{4},\ \frac{2-w_2}{4w_1},\ \frac{2-w_2}{4w_1}\Big) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

By Theorem 3.2, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

(b) It is enough to show that $\text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})} \subseteq \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$. Let $T = (a, b, c) \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$. By Theorem 3.3, $T = (a, b, c) \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ if and only if $\pm(a, -b, -c) \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$. Thus, we may assume that $a \geq 0$ and $c \geq 0$.

**Claim.** $T = (0,\ \pm 1,\ 0) \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let $f = (0,\ 1,\ 0) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*$. By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ w_2^2,\ w_1 w_2) \in \exp B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_1, w_2 < 1$

Let

$$f = \Big(\frac{(2-w_2)^2}{4w_1^2},\ \frac{1}{4},\ \frac{2-w_2}{4w_1}\Big) \in \mathcal{L}_s(^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$

By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ -w_2(2-w_2),\ w_1w_2) \in \exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$ for $0 < w_2 < \frac{1}{2}$

Let
$$f = \Big(\frac{(2-w_2)^2}{4w_1^2},\ -\frac{1}{4},\ \frac{1}{2w_1}\Big)$$
or
$$\Big(\frac{1}{w_1^2}\Big(1-\frac{2}{n}\Big),\ -\frac{1}{w_2(2-w_2)n},\ \frac{1}{2w_1w_2n}\Big) \text{ for a sufficiently large } n \in \mathbb{N}.$$
By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

Suppose that $\frac{1}{2} \leq w_2 < 1$.

**Claim.** $T = (w_1^2,\ -w_2(2-w_2),\ w_1(1-w_2)) \in \exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let
$$f = \Big(\frac{1}{3w_1^2},\ -\frac{2}{3},\ \frac{2(1-w_2)}{3w_1}\Big) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$
By Theorem 2.9, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 2.6, $T$ is an exposed point.

**Claim.** $T = (0,\ 2w_2 - 1,\ w_1) \in \exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let
$$f = \Big(0,\ \frac{2w_1}{(2w_2-1)n},\ \frac{1}{2w_1} - \frac{1}{n}\Big) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^* \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (0,\ -(2w_2 - 1),\ w_1) \in \exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let
$$f = \Big(0,\ -\frac{2w_1}{(2w_2-1)n},\ \frac{1}{2w_1} - \frac{1}{n}\Big) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^* \text{ for a sufficiently large } n \in \mathbb{N}.$$

By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

**Claim.** $T = (w_1^2,\ -(3w_2^2 - 4w_2 + 2),\ w_1w_2) \in \exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})}$

Let
$$f = \Big(\frac{2-w_2^2}{4w_1^2},\ -\frac{1}{4},\ \frac{2-w_2}{4w_1}\Big) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})^*.$$
By Theorem 3.4, $\|f\| = 1$. Note that $f(T) = 1$ and $|f(S)| < 1$ for every $S \in \text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} \backslash \{\pm T\}$. By Theorem 3.1, $T$ is an exposed point.

This completes the proof. □

Theorems 2.4-5 and 3.5 show the following:

**Remark 3.6.** It is true that $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})} = \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w_1,w_2)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{h(w_1,w_2)})$ for $0 < w_1, w_2 < 1$.

## 4 Extreme and exposed points of $\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})$

**Theorem 4.1.** *Let* $0 < w_1, w_2 < 1$ *and* $T = (a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})$. *Let* $\tilde{T} = (b, a, d, c) \in \mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})$. *Then,*

$(a)$ $\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} = \|\tilde{T}\|_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}$;

$(b)$ $\text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{(a, b, c, d) \in \mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)}) : (b, a, d, c) \in \text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}\Big\}$.

*Proof.* (a). Note that for $(x, y) \in \mathbb{R}^2$, $\|(x, y)\|_{h'(w_1,w_2)} = \|(y, x)\|_{h(w_2,w_1)}$. It follows that

$$\begin{aligned}
\|T\|_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} &= \sup_{\|(x_j,y_j)\|_{h'(w_1,w_2)}=1, j=1,2} |T((x_1, y_1),\ (x_2, y_2))| \\
&= \sup_{\|(x_j,y_j)\|_{h'(w_1,w_2)}=1, j=1,2} |\tilde{T}((y_1, x_1),\ (y_2, x_2))| \\
&= \sup_{\|(y_j,x_j)\|_{h(w_2,w_1)}=1, j=1,2} |\tilde{T}((y_1, x_1),\ (y_2, x_2))| \\
&= \|\tilde{T}\|_{\mathcal{L}(^2\mathbb{R}^2_{h(w_2,w_1)})}.
\end{aligned}$$

(b) follows from (a). □

**Theorem 4.2.** *Let* $0 < w_1, w_2 < 1$.
$(a)$ *Let* $w_1 < \frac{1}{2}$. *Then,*

$$\begin{aligned}
&\text{ext}\, B_{\mathcal{L}(^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{(\pm 1,\ 0,\ 0,\ 0), \pm(w_1,\ 0,\ 0,\ \pm w_2), \pm(w_1,\ 0,\ \pm w_2,\ 0), \\
&\pm(w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(-w_1^2,\ w_2^2,\ -w_1w_2,\ w_1w_2), \\
&\pm(-w_1^2,\ w_2^2,\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ w_2^2,\ -w_1w_2,\ -w_1w_2), \\
&\pm(-w_1(2 - w_1),\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(w_1(2 - w_1),\ w_2^2,\ -w_1w_2,\ w_1w_2), \\
&\pm(w_1(2 - w_1),\ w_2^2,\ w_1w_2,\ -w_1w_2), \pm(-w_1(2 - w_1),\ w_2^2,\ -w_1w_2,\ -w_1w_2)\Big\}.
\end{aligned}$$

$(b)$ *Let* $\frac{1}{2} \leq w_1$.

*Then,*

$$\begin{aligned}
\operatorname{ext} B_{\mathcal{L}({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{&(\pm 1,\ 0,\ 0,\ 0), \pm(w_1,\ 0,\ 0,\ \pm w_2), \pm(w_1,\ 0,\ \pm w_2,\ 0),\\
&\pm(w_1^2,\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(-w_1^2,\ w_2^2,\ -w_1w_2,\ w_1w_2),\\
&\pm(-w_1^2,\ w_2^2,\ w_1w_2,\ -w_1w_2), \pm(w_1^2,\ w_2^2,\ -w_1w_2,\ -w_1w_2),\\
&\pm(2w_1-1,\ 0,\ w_2,\ w_2), \pm(-(2w_1-1),\ 0,\ -w_2,\ w_2),\\
&\pm(2w_1-1,\ 0,\ w_2,\ -w_2), \pm(2w_1-1,\ 0,\ -w_2,\ -w_2),\\
&\pm(-(w_1^2-w_1+1),\ w_2^2,\ w_2(1-w_1),\ w_1w_2),\\
&\pm(w_1^2-w_1+1,\ w_2^2,\ -w_1w_2,\ w_2(1-w_1)),\\
&\pm(w_1^2-w_1+1,\ w_2^2,\ -w_2(1-w_1),\ w_1w_2),\\
&\pm(-(w_1^2-w_1+1),\ w_2^2,\ -w_2(1-w_1),\ -w_1w_2),\\
&\pm(-(w_1^2-w_1+1),\ w_2^2,\ w_1w_2,\ w_2(1-w_1)),\\
&\pm(w_1^2-w_1+1,\ w_2^2,\ -w_1w_2,\ w_2(1-w_1)),\\
&\pm(w_1^2-w_1+1,\ w_2^2,\ w_1w_2,\ -w_2(1-w_1)),\\
&\pm(-(w_1^2-w_1+1),\ w_2^2,\ -w_1w_2,\ -w_2(1-w_1)),\\
&\pm(-w_1(2-w_1),\ w_2^2,\ w_2(1-w_1),\ w_2(1-w_1)),\\
&\pm(w_1(2-w_1),\ w_2^2,\ w_2(1-w_1),\ -w_2(1-w_1)),\\
&\pm(w_1(2-w_1),\ w_2^2,\ -w_2(1-w_1),\ w_2(1-w_1)),\\
&\pm(-w_1(2-w_1),\ w_2^2,\ -w_2(1-w_1),\ -w_2(1-w_1)),\\
&\pm(-(3w_1^2-4w_1+2),\ w_2^2,\ w_1w_2,\ w_1w_2), \pm(3w_1^2-4w_1+2,\ w_2^2,\ w_1w_2,\ -w_1w_2),\\
&\pm(3w_1^2-4w_1+2,\ w_2^2,\ -w_1w_2,\ w_1w_2), \pm(-(3w_1^2-4w_1+2),\ w_2^2,\ -w_1w_2,\ -w_1w_2)\Big\}.
\end{aligned}$$

*Proof.* It follows from Theorems 2.4 and 4.1. □

**Theorem 4.3.** *Let* $0 < w_1, w_2 < 1$.
$(a)$ *Let* $w_1 < \frac{1}{2}$. *Then,*

$$\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{\pm(1,\ 0,\ 0), \pm(w_1^2,\ w_2^2,\ \pm w_1w_2), \pm(w_1(2-w_1),\ -w_2^2,\ \pm w_1w_2)\Big\}.$$

$(b)$ *Let* $w_1 = \frac{1}{2}$.
*Then,*

$$\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{\pm(1,\ 0,\ 0), \pm(1/4,\ w_2^2,\ \pm w_2/2),\ \pm(3/4,\ -w_2^2,\ \pm w_2/2)\Big\}.$$

$(c)$ *Let* $\frac{1}{2} < w_1$.
*Then,*

$$\begin{aligned}
\operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \Big\{&\pm(0,\ 1,\ 0), \pm(w_1^2,\ w_2^2,\ \pm w_1w_2),\ \pm(2w_1-1,\ 0,\ \pm w_2),\\
&\pm(w_1(2-w_1),\ -w_2^2,\ \pm w_2(1-w_1)), \pm(3w_1^2-4w_1+2,\ -w_2^2,\ \pm w_1w_2)\Big\}.
\end{aligned}$$

*Proof.* It follows from Theorems 2.5 and 4.1. □

**Corollary 4.4.** *Let* $0 < w_1, w_2 < 1$. *Then,*
$(a)$ $\exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \operatorname{ext} B_{\mathcal{L}({}^2\mathbb{R}^2_{h'(w_1,w_2)})}$;
$(b)$ $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \operatorname{ext} B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})}$.

*Proof.* It follows from Theorems 3.5 and 4.1. □

**Remark 4.5.** Let $0 < w_1, w_2 < 1$.

(a) $\text{ext}\, B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \text{ext}\, B_{\mathcal{L}({}^2\mathbb{R}^2_{h'(w_1,w_2)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})$.

(b) $\exp B_{\mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})} = \exp B_{\mathcal{L}({}^2\mathbb{R}^2_{h'(w_1,w_2)})} \cap \mathcal{L}_s({}^2\mathbb{R}^2_{h'(w_1,w_2)})$.

**Theorem 4.6.** *Let $\mathbb{R}^2_{\|\cdot\|}$ be a normed space such that $\text{ext}\, B_{\mathbb{R}^2_{\|\cdot\|}}$ is finite. Let $n \geq 2$. Then,*

$$\begin{aligned}
&\text{ext}\, B_{\mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|})} \\
\supseteq\ &\{T \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) : \textit{there is } A_T \subseteq (\text{ext}\, B_{\mathbb{R}^2_{\|\cdot\|}})^n \textit{ such that } |A_T| = 2^n \\
&\textit{and } \|T\| = 1 = |T(X_1, \ldots, X_n)| \textit{ for every } (X_1, \ldots, X_n) \in A_T \textit{ and that} \\
&\textit{if } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n} \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) \textit{ satisfying } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n) = 0 \\
&\textit{for every } (X_1, \ldots, X_n) \in A_T,\ \textit{then } a_{k_1\cdots k_n} = 0 \textit{ for all } k_j = 1, 2,\ 1 \leq j \leq n\}.
\end{aligned}$$

*Proof.* Let

$$\begin{aligned}
\mathcal{F}\ =\ &\{T \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) : \text{there is } A_T \subseteq (\text{ext}\, B_{\mathbb{R}^2_{\|\cdot\|}})^n \text{ such that } |A_T| = 2^n \\
&\text{and } \|T\| = 1 = |T(X_1, \ldots, X_n)| \text{ for every } (X_1, \ldots, X_n) \in A_T \text{ and that} \\
&\text{if } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n} \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) \text{ satisfying } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n) = 0 \\
&\text{for every } (X_1, \ldots, X_n) \in A_T,\ \text{then } a_{k_1\cdots k_n} = 0 \text{ for all } k_j = 1, 2,\ 1 \leq j \leq n\}.
\end{aligned}$$

Let $T \in \mathcal{F}$.

**Claim.** $T \in \text{ext}\, B_{\mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|})}$

Suppose not.

There is nonzero $(b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n} \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|})$ such that

$$\|T \pm (b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}\| \leq 1.$$

For every $(X_1, \ldots, X_n) \in A_T$, it follows that

$$\begin{aligned}
1\ &\geq\ |T \pm (b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n)| \\
&\geq\ \max\Big\{|1 + (b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n)|, \\
&\qquad |1 - (b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n)|\Big\} \\
&=\ 1 + |(b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n)|,
\end{aligned}$$

which shows that $(b_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n) = 0$ for every $(X_1, \ldots, X_n) \in A_T$. By the hypothesis for $A_T$, $b_{k_1\cdots k_n} = 0$ for all $k_j = 1, 2,\ 1 \leq j \leq n$. This is a contradiction. □

**Question.** *Is it true that*

$$\begin{aligned}
&\text{ext}\, B_{\mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|})} \\
=\ &\{T \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) : \textit{there is } A_T \subseteq (\text{ext}\, B_{\mathbb{R}^2_{\|\cdot\|}})^n \textit{ such that } |A_T| = 2^n \\
&\textit{and } \|T\| = 1 = |T(X_1, \ldots, X_n)| \textit{ for every } (X_1, \ldots, X_n) \in A_T \textit{ and that} \\
&\textit{if } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n} \in \mathcal{L}({}^n\mathbb{R}^2_{\|\cdot\|}) \textit{ satisfying } (a_{k_1\cdots k_n})_{k_j=1,2,\ 1\leq j\leq n}(X_1, \ldots, X_n) = 0 \\
&\textit{for every } (X_1, \ldots, X_n) \in A_T,\ \textit{then } a_{k_1\cdots k_n} = 0 \textit{ for all } k_j = 1, 2,\ 1 \leq j \leq n\}?
\end{aligned}$$

Note that if $n = 2$ and $\mathbb{R}^2_{\|\cdot\|} = \ell^2_1, \ell^2_\infty, \mathbb{R}^2_{h(w_1,w_2)}$, then the question is true (see [15]).

## References


[1] R. M. Aron, Y. S. Choi, S. G. Kim and M. Maestre, *Local properties of polynomials on a Banach space,* Illinois J. Math., **45**, 25–39, (2001).

[2] Y. S. Choi, H. Ki and S. G. Kim, *Extreme polynomials and multilinear forms on* $l_1$*,* J. Math. Anal. Appl., **228**, 467–482, (1998).

[3] Y. S. Choi and S. G. Kim, *The unit ball of* $\mathcal{P}(^2\ell_2^2)$*,* Arch. Math. (Basel), **71**, 472–480, (1998)

[4] Y. S. Choi and S. G. Kim, *Extreme polynomials on* $c_0$*,* Indian J. Pure Appl. Math., **29**, 983–989, (1998).

[5] Y. S. Choi and S. G. Kim, *Exposed points of the unit balls of the spaces* $\mathcal{P}(^2\ell_p^2)$ $(p = 1, 2, \infty)$*,* Indian J. Pure Appl. Math., **35**, 37–41, (2004).

[6] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London 1999.

[7] S. Dineen, *Extreme integral polynomials on a complex Banach space,* Math. Scand., **92**, 129–140, (2003),

[8] B. C. Grecu, *Geometry of 2-homogeneous polynomials on* $\ell_p$ *spaces,* $1 < p < \infty$, J. Math. Anal. Appl., **273**, 262–282, (2002).

[9] B. C. Grecu, G. A. Munoz-Fernandez and J. B. Seoane-Sepulveda, *Unconditional constants and polynomial inequalities,* J. Approx. Theory, **161**, 706–722, (2009).

[10] S. G. Kim, *Exposed 2-homogeneous polynomials on* $\mathcal{P}(^2\ell_p^2)$ $(1 \leq p \leq \infty)$*,* Math. Proc. R. Ir. Acad., **107A**, 123–129, (2007).

[11] S. G. Kim, *The unit ball of* $\mathcal{L}_s(^2\ell_\infty^2)$*,* Extracta Math., **24**, 17–29, (2009).

[12] S. G. Kim, *The unit ball of* $\mathcal{P}(^2d_*(1,w)^2)$*,* Math. Proc. R. Ir. Acad., **111A**, 79–94, (2011).

[13] S. G. Kim, *Exposed symmetric bilinear forms of* $\mathcal{L}_s(^2d_*(1,w)^2)$*,* Kyungpook Math. J., **54**, 341–347, (2014).

[14] S. G. Kim, *Exposed 2-homogeneous polynomials on the 2-dimensional real predual of Lorentz sequence space,* Mediterr. J. Math. **13** (2016), 2827–2839.

[15] S. G. Kim, *Geometry of multilinear forms on* $\mathbb{R}^m$ *with a certain norm,* Acta Sci. Math. (Szeged), **87** (1–2), 233–245, (2021).

[16] S. G. Kim and S. H. Lee, *Exposed 2-homogeneous polynomials on Hilbert spaces,* Proc. Amer. Math. Soc., **131**, 449–453, (2003).

[17] G. A. Munoz-Fernandez, S. Revesz and J. B. Seoane-Sepulveda, *Geometry of homogeneous polynomials on non symmetric convex bodies,* Math. Scand., **105**, 147–160, (2009).

[18] G. A. Munoz-Fernandez and J. B. Seoane-Sepulveda, *Geometry of Banach spaces of trinomials,* J. Math. Anal. Appl., **340**, 1069–1087, (2008).


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