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A CUBIC B-SPLINE FINITE ELEMENT METHOD FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION

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Abstract This work proposes a numerical solution for the Volterra integro-differential equation (VIDE). The method proposed in this study combines the cubic B-spline method with the finite element method to effectively solve this type of equation. To handle the integral part of the equation. Gauss rules are employed to approximate the integral of the differential equations. These rules ensure the accuracy employed to approximate the integral of differential equations. These rules ensure the accuracy and precision of the results obtained. In addition, the study investigates the coercivity and continuity of the the suggested problems to address challenges in deriving a priori error estimates. Through this analysis, the convergence of the method is proven. To demonstrate the effectiveness of the proposed method, several examples are presented. These examples serve to illustrate the applicability of the method and also allow for a comparison between the approximate results and analytical solutions.

1 Introduction

Initial- and boundary-value problems with integro-differential equations (IEDs) are abundant in physical and biological modeling as well as in respective (bio-)engineering applications. In recent years, the finite element method (FEM) has gained significant attention as a numerical technique for approximating solutions to ordinary and partial differential equations that do not involve memory effects [1, 2, 3, 3, 5].

However, the implementation of FEM with cubic B-spline functions for solving VIDE has seen slower progress compared to other methods. As a result, researchers have dedicated considerable effort to developing efficient numerical schemes for approximating solutions to these problems. Numerical solutions for such equations have been explored using various techniques, Galerkin finite element method [6, 7, 8, 9]. Various methods have been applied to solve IEDs. For more details see [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

The aim of this study is to explore the use of modified cubic B-spline finite element methods to approximate solutions for VIDE. However, there are challenges involved in constructing a linear system that can yield highly accurate results. One of the challenges is integrating the integro parts of the equation. To address this, a quadrature rule can be employed to obtain exact approximations for the integral parts. Another challenge is dealing with Dirichlet boundary conditions using B-spline functions. This requires finding appropriate techniques to incorporate these boundary conditions into the framework of the B-spline finite element method

The present study is prepared as follows: In Section 2, the numerical method of the presented problem was given. Convergence analysis of suggested schemes is proved in Section 3. Some numerical experiments are shown in Section 4. Finally, conclusions are given in Section 5.

2 Second-order linear VIDE

Consider the model problem as

$$-z''(s)+b(s)z'(s)+c(s)z(s)=f(s)+\int_0^s k(s,t)z(t)dt, \tag{2.1}$$

with boundary conditions

$$z(a)=z(b)=0,\quad s,t\in(0,1), \tag{2.2}$$

where $b(s), c(s), f(s)$ and $k(s,t)$ are continuous real-valued functions on the interval $[0,1]$. The weak form of (2.1) can be done multiplying both sides of the (2.1) by a test function $z\in H_0^1(0,1)$ and integrating by parts of (2.1), we obtain

$$\int_0^1 z'(x)v'(x)ds+\int_0^1 b(s)z'(s)v(s)ds+\int_0^1 c(s)z(s)v(s)ds$$
$$=\int_0^1 f(s)v(s)ds+\int_0^1 v(s)\int_0^1 k(s,t)z(t)dtds,$$

such that

$$a(z,v)=F(v),\quad v\in H_0^1(0,1), \tag{2.3}$$

where $a(z,v)$ is a symmetric bilinear form and $F(v)$ is a linear functional given by:

$$a(z,v)=\int_0^1 z'(s)v'(s)ds+\int_0^1 b(s)z'(s)v(s)ds+\int_0^1 c(s)z(s)v(s)ds$$
$$-\int_0^1 v(s)\int_0^s k(s,t)z(t)dtds$$
$$F(v)=\int_0^1 f(s)v(s)ds.$$

2.1 Finite element approximation

To process with finite element approximation, let S^h be a finite dimensional space such that $S^h\subset H_0^1(\Omega)$. To develop the numerical method for approximating the solution of a boundary value problem (2.1), the interval $[0,1]$ is partitioned into $N+1$ uniformly spaced points x_i such that $0<x_0<x_1<\cdots<x_{N-1}<x_N=1$ and $h=\frac{1}{N+1}$. Setting

$$S^h=\left\{v\in S_3\in C^2[0,1]: v(0)=v(1)=0\right\}, \tag{2.4}$$

where S_3 is the space of all polynomials of degree ≤ 3. We seek to find an approximation $Z_h\in S^h$ such that

$$a(Z_h,v_h)=F(v_h), v_h\in S^h, \tag{2.5}$$

where

$$a(Z_h,v)=\int_0^1 Z_h'(s)v'(s)ds+\int_0^1 b(s)Z_h'(s)v(s)ds+\int_0^1 c(s)Z_h(s)v(s)ds$$
$$-\int_0^1 v(s)\int 0^s k(s,t)Z_h(t)dtds,$$
$$F(v_h)=\int_0^1 f(s)v_h(s)ds.$$

2.2 Calculation and assembly of stiffness and mass matrices

Let N be a positive integer number and $a = 0 < s_0 < s_1 < \cdots < s_{N-1} < s_N = b$ and $h = 1/(N+1)$ be a uniform partition of $[a, b]$ and $h = \frac{b-a}{N}$. The typical third-degree B-spline basis functions defined as

$$B_i^3(s) = \begin{cases} h^{-3} g_1(s - s_{i-2}), & s \in [s_{i-2}, s_{i-1}] \\ g_2\left(\frac{s - s_{i-1}}{h}\right) & s \in [s_{i-1}, s_i] \\ g_2\left(\frac{s_{i+1} - s}{h}\right) & s \in [s_i, s_{i+1}] \\ h^{-3} g_1(s_{i+2} - s), & s \in [s_{i+1}, s_{i+2}] \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

where

$$g_1(s) = s^3,\ g_2(s) = 1 + 3s + 3s^2 - 3s^3, \text{ for } i = -1, 0, \ldots, N+1.$$

For a sufficiently smooth function $z(s)$ there always exists a unique third-degree spline Z_h such that

$$Z_h = \sum_{m=-1}^{N+1} \alpha_i B_i^3(s), \tag{2.7}$$

where α_i are unknown quantities to be determined from (2.6). For the sake of simplicity, using $z_i = Z_h(s_i, t)$

$$\begin{cases} Z_i = \alpha_{i-1} + 4\alpha_i + \alpha_{i+1} \\ Z_i' = \frac{3}{h}\left(\alpha_{i-1} - \alpha_{i+1}\right) \\ Z_i'' = \frac{6}{h^2}\left(\alpha_{i-1} - 2\alpha_i + \alpha_{i+1}\right). \end{cases} \tag{2.8}$$

Using (2.7) and boundary conditions (2.2), gives

$$\begin{cases} a_1 = -a_{-1} - 4a_0 \\ a_{N+1} = -a_{N-1} - 4a_N. \end{cases} \tag{2.9}$$

Substituting (2.8) in (2.7), we have

$$Z_h = a_0\left[B_0(s) - 4B_{-1}(s)\right] + a_1\left[B_1(s) - B_{-1}(s)\right] + \cdots + \\ + a_{N-1}\left[B_{N-1}(s) - B_{N+1}(s)\right] + a_N\left[B_N(s) - 4B_N(s)\right],$$

so that

$$Z_h = a_0\overline{B_0}(s) + a_1\overline{B_1}(s) + \cdots + a_{N-1}\overline{B_{N-1}}(s) + a_N\overline{B_N}(s),$$

where

$$\begin{cases} \overline{B_0}(s) = B_0(s) - 4B_{-1}(s) \\ \overline{B_1}(s) = B_1(s) - B_{-1}(s) \\ \overline{B_i}(s) = B_i(s),\ i = 2, 3, \ldots, N-2 \\ \overline{B_{N-1}}(s) = B_{N-1}(s) - B_{N+1}(s) \\ \overline{B_N}(s) = B_N(s) - 4B_{N+1}(s). \end{cases} \tag{2.10}$$

Table 1. The values of $B_i(s)$ and their derivatives.

	s_i	s_{i-1}	s_{i+1}	$else$
$B_i(s)$	1/6	2/3	1/6	0
$B_i'(s)$	$\frac{-1}{2h}$	0	$\frac{1}{2h}$	0
$B_i''(s)$	$\frac{1}{h^2}$	$\frac{-2}{h^2}$	$\frac{1}{h^2}$	0

Inserting (2.10) in (2.5) with choosing $v_h = \overline{B}_m(s)$, gives

$$\sum_{i=0}^{N}\alpha_i\Big(\int_0^1 \overline{(B_i)'}(s)(B_m)'(s) + \int_0^1 b(s)\overline{(B_i)'}(s)(B_m)(s)ds + \int_0^1 c(s)\overline{(B_i)'}(s)(B_m)'(s)ds$$
$$-\int_0^1 \overline{B_m(s)}\int_0^s k(s,t)\overline{B_i(t)}dtds\Big) = \int_0^1 f(s)\overline{B_m(s)}ds,$$

or

$$\left(A^e + bB^e + cC^e - D^e\right)\alpha = F^e, \tag{2.11}$$

where

$$A^e = \int_0^1 \overline{(B_i)'}(s)\overline{(B_m)'}(s)ds = \frac{3}{10h}\begin{bmatrix} 80 & 43 & -20 & -1 \\ 43 & 104 & -14 & -24 \\ -20 & -14 & 80 & -15 \\ -1 & -24 & -15 & 80 \end{bmatrix} \tag{2.12}$$

$$B^e = \int_0^1 \overline{(B_i)'}(s)\overline{(B_m)}(s)ds = \frac{1}{20}\begin{bmatrix} 0 & 133 & 52 & 1 \\ -133 & 0 & 244 & 56 \\ -52 & -244 & 0 & 245 \\ -1 & -56 & -245 & 0 \end{bmatrix} \tag{2.13}$$

$$C^e = \int_0^1 \overline{(B_i)}(s)\overline{(B_m)}(s)ds = \frac{h}{40}\begin{bmatrix} 496 & 733 & 116 & 1 \\ 733 & 2296 & 1190 & 120 \\ 116 & 1190 & 2416 & 1191 \\ 1 & 120 & 1191 & 2416 \end{bmatrix} \tag{2.14}$$

are element matrices. The latter integral F^e and D^e will be handled numerically using the composite 3-point Guass quadrature given by:

$$\int_0^1 g(s)ds = \sum_{k=0}^{N}\int_{s_k}^{s_{k+1}} g(s)ds = \frac{h}{2}\sum_{k=0}^{N}\int_{-1}^{1} g\left(s_k + \frac{h}{2}(1+t)\right)ds \approx \frac{h}{2}\sum_{k=0}^{N}\sum_{j=0}^{3} C_j g(\xi_{k,j}) \tag{2.15}$$

$\xi_{k,j} = \frac{h}{2}\xi_j + \frac{S_k+S_{k+1}}{2}$, $k = 0, 1, \ldots, N$, $\quad i = 1, 2, 3$.
Therefore, using (2.15) we obtain

$$F^e = \int_0^1 f(s)\overline{(B_i)}(s)ds \approx \frac{h}{2}\sum_{k=0}^{N}\sum_{j=0}^{3} C_j f(\xi_{k,j})\overline{(B_i)}(\xi_{k,j}),\ i = 1, 2, \ldots, N+1. \tag{2.16}$$

$$D^e = \int_0^1 k(s,t)\overline{(B_i)}(s)dtds \approx \sum_{k=0}^{N}\sum_{j=0}^{3} C_j K_{k,j}\overline{(B_i)}(\xi_{k,j}),\ i = 1, 2, \ldots, N+1. \tag{2.17}$$

To find the coefficient $\alpha_0, \alpha_1, \ldots.., \alpha_N$, we assemble all the matrices in (2.11), imply that

$$(A + B + bC + cD)\alpha = F,$$

for each matrix has A^e, B^e, C^e and D^e assembled from the element matrices $A, B, C,$ and D such that
$A = \frac{3}{10h}(-1, -44, 14, 344, 14, -44, -1)$, $B = \frac{1}{20}(1, -108, -622, 0, -622, -108, 1)$, and

$$C = \frac{3}{10h}(-1, -24, -15, 80, -15, -24, -1).$$

3 Convergence analysis

The aim of this section is to present the error analysis theorems for the proposed method. The lemma below will show (2.3) satisfying the Lax–Milgram theorem.

Lemma 3.1. *(Existence, uniqueness) Let $a(z, v)$ be bilinear form defined by (2.3). Suppose that $N_1 \leq c(s) \leq N_2$, $Q_1 \leq b(s) \leq Q_2$ and $0 \leq b^{'}(s) \leq T_2$, then $a(z, v)$ is a V-ellipticity, (2.3) has a unique solution.*

$$a(z, z) \geq C_{covr}\|z\|_a^2, \quad \forall w \in H_0^1(0, 1), \tag{3.1}$$

$$a(z, v) \leq C_{cont}\|z\|_a.\|v\|_a, \quad \forall z, v \in H_0^1(0, 1). \tag{3.2}$$

Proof. Recalling weak form in (2.3) along with Cauchy–Schwarz inequality, we have

$$a(z, v) \leq \|z(s)\|_a\|v(s)\|_a + Q_2\|z(s)\|_a\|v(s)\|_{L^2(0,1)} + N_2\|z(s)\|_{L^2(0,1)}\|v(s)\|_{L^2(0,1)} + KR\|z(s)\|_a\|v(s)\|_a = (1 + Q_2 + N_2 + KR)\|z(s)\|_a\|v(s)\|_a = C_{cont}\|z\|_a.\|v\|_a.$$

Using Poincaré-Friedrichs inequality

$$\|v(s)\|_{L^2(0,1)} \leq C_p\|v(s)\|_{H_0^1(0,1)} = C_p\|v(s)\|_a,$$

and setting

$$C_{\text{cont}} = (1 + Q_2 + N_2 + KR), \ K = \max|K(s, t)|, x \in [0, 1], t \in [0, x], \text{and } R = \|1\|_{L^2(0,1)}.$$

The second part of (3.2) of above lemma is proved. To prove the V-ellipticity of $a(z, v)$, we have

$$\int_0^1 z'(s)z'(s)ds + \int_0^1 c(s)z(s)z(s)ds \geq \int_0^1 (z')^2(s)ds \geq \frac{1}{1+c}\|z\|_{H^1(0,1)}^2, \tag{3.3}$$

and

$$-\int_0^1 z(s)\int_0^s k(s, t)z(t)dsds = -\frac{1}{2}\int_0^1 b'(s)(z(s))^2ds \geq -\frac{T_2}{2}\int_0^1 (z(s))^2ds \geq -\frac{T_2}{2}\|z\|_{H^1(0,1)}^2, \tag{3.4}$$

also

$$-\int_0^1 z(s)\int_0^s k(s, t)z(t)dtds \geq -|\int_0^1 z(s)\int_0^s k(s, t)z(t)dtds| \tag{3.5}$$

$$\geq -KR\|z\|_{L^2(0,1)}^2 \geq KR\|z\|_{H^1(0,1)}^2. \tag{3.6}$$

$$\tag{3.7}$$

Combining (3.3)-(3.5), we have

$$a(z, z) \geq \left(\frac{1}{1+c} - \frac{T_2}{2} - KR\right)\|z\|_{H^1(0,1)}^2, \tag{3.8}$$

or

$$a(z, z) \geq C_{\text{covr}}\|z\|_{H^1(0,1)}^2, \tag{3.9}$$

where $C_{\text{covr}} = \left(\frac{1}{1+c} - \frac{T_2}{2} - KR\right)$, c is a Poincare's constant. Thus $a(z, v)$ is a $V-$elliptic if $C_{\text{covr}} \geq 0$. Therefore, by the Lax-Milgram theorem and the $V-$ ellipticity of $a(z, v)$, equation (2.3) has a unique solution. □

3.1 Error Estimates in the Energy Norm

This section aims to employ Cea's lemma to derive an optimal error bound for the finite element approximation for a one-dimensional model problem. To start with, the energy norm $\|v\|_a$ defined by

$$\|z\|_a^2 = \|z'\|^2 + \|z\|^2 + \int_0^s k(s,t)\|z(t)\|^2 dt = \|z\|^2_{H^1(0,1)}.$$

3.2 Definition

Let $H^1(0,1)$ be a Hilbert space, and suppose a be a symmetric and $H_0^1(0,1)$-elliptic bilinear form. We define an inner product as follows

$$(\cdot,\cdot) : H^1(0,1) \times H^1(0,1) \longrightarrow \mathbb{R},$$

$$(z,v)_a = a(z,v),$$

which is called the inner product energy. Also we define energy norm as follows

$$\|z\|_a^2 = a(z,z).$$

Suppose that z is the exact solution of the problem and Z_h be its approximate solution, then, we have

$$a(z,v) = (f(x),v), \ \forall v \in S^h$$

$$a(Z_h,v_h) = (f(x),v_h), \ \forall v_h \in S^h.$$

If $e = z - Z_h$, then

$$a(e,v_h) = 0, \ \forall v_h \in S^h. \tag{3.10}$$

The error can be decomposed as

$$e = z - Z_h = z - \Pi_h z + \Pi_h z - Z_h = \epsilon_1 - \epsilon_2,$$

where

$$\epsilon_1 = z - \Pi_h z, \ \epsilon_2 = \Pi_h z - Z_h,$$

and

$$\Pi_h z : H^1(0,1) \longrightarrow S^h$$

is the interpolant of the exact solution $z \in H^1(0,1)$. Applying energy norm in above, gives

$$\|e\|_a = \|z - Z_h\|_a \le \|z - \Pi_h z\|_a + \|\Pi_h z - Z_h\|_a \le \|\epsilon_1\|_a + \|\epsilon_2\|_a.$$

Lemma 3.2. *Let $S^h \subseteq H_0^1(0,1)$, where $0 \le h \le 1$. For $z \in H_0^1(0,1) \cap H^4(0,1)$, there exists a constant C independent of h such that*

$$\inf_{v_h \in s_h} \|\Pi_h z - Z_h\|_a \le C h^3.$$

Proof. See [26]. □

Theorem 3.3. Assume that z be the exact solution of (2.3) and Z_h be the approximate solution of the variational formulation (2.5), then

$$\|z - Z_h\|_a \le C_1 h^3,$$

where $C_1 = \frac{C C_{cont}}{C_{covr}} + C.$

Proof.

$$\|e\|_a = z - Z_h = z - \Pi_h z + \Pi_h z - Z_h = \epsilon_1 + \epsilon_2,$$

where $\epsilon_1 = z - \Pi_h z$, $\epsilon_2 = \Pi_h z - Z_h$. Plugging $e = \epsilon_1 + \epsilon_2$ in (3.10), this becomes

$$a(\epsilon_1 + \epsilon_2, v) = a(\epsilon_1, v) + a(\epsilon_2, v) = 0.$$

Testing $v = \epsilon_2$, gives

$$a(\epsilon_1, \epsilon_2) + a(\epsilon_2, \epsilon_2) = 0.$$

Therefore

$$a(\epsilon_1, \epsilon_2) = -a(\epsilon_2, \epsilon_2),$$

using the (3.1) and (3.2), imply that

$$C_{\text{covr}} \|\epsilon_2\|_a^2 \leq C_{\text{cont}} \|\epsilon_1\|_a \; \|\epsilon_2\|_a$$

which leads to

$$\|\epsilon_2\|_a \;\leq \frac{C_{\text{cont}}}{C_{\text{covr}}} \|\epsilon_1\|_a$$

from Lemma (3.2), gives

$$\|\epsilon_1\|_a = Ch^3.$$

Finally, we obtain

$$\|e\|_a \leq \|\epsilon_1\|_a + \|\epsilon_2\|_a = \frac{CC_{\text{cont}}}{C_{\text{covr}}} h^3 + Ch^3 = C_1 h^3.$$

□

4 Numerical Experiments

The section illustrates the performance of a presented method, through an implementation based on Mathematica programming. The error norms are used to measure the error between the numerical and exact solution. The error is defined as

$$E(s) = z(s) - Z_h(s).$$

Then, the pointwise error is

$$\varepsilon_A(s) = |E(s_i)|.$$

Example 4.1. Given the following first order VIDE

$$z'(s) + z(s) = 1 - s - \frac{2}{3} s^3 - 2 \int_0^s z(t) dt,$$

with initial condition

$$z(0) = 0,$$

and exact solution $z(s) = s - s^2$.

Table 2. Pointwise error $\varepsilon_A(s)$ for Example 4.1.

s	$Z_h(s)$	$z(s)$	$\varepsilon_A(s)$
0.1	$9.000e^{-02}$	$9.e^{-02}$	$1.544e^{-06}$
0.2	$1.600e^{-02}$	$1.6e^{-01}$	$2.487e^{-06}$
0.3	$2.100e^{-01}$	$2.1e^{-01}$	$7.297e^{-06}$
0.4	$2.399e^{-01}$	$2.4e^{-01}$	$1.603e^{-05}$
0.5	$2.500e^{-01}$	$2.5e^{-01}$	$6.851e^{-06}$
0.6	$2.399e^{-01}$	$2.4e^{-01}$	$1.564e^{-05}$
0.7	$2.100e^{-01}$	$2.1e^{-01}$	$8.723e^{-06}$
0.8	$1.600e^{-01}$	$1.6e^{-01}$	$2.841e^{-06}$
0.9	$9.000e^{-02}$	$9.e^{-02}$	$1.470e^{-06}$

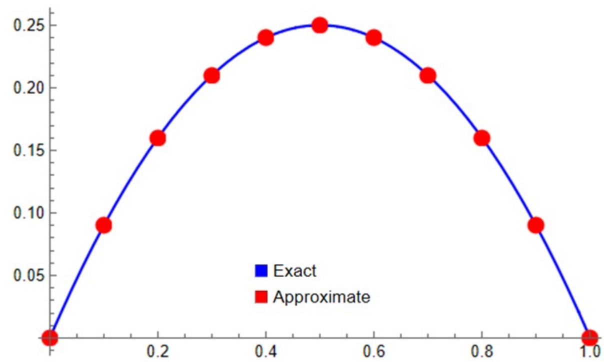

Figure 1. Numerical and exact solution of Example 4.1.

Example 4.2. Consider the VIDE

$$z''(s) - z(s) = -f(s) - \int_0^s (s-t)z(t)dt,\ z(0) = z(1) = 0,$$

where

$$f(s) = s - \frac{s^3}{6} - se^{s-1} - e^{-1}(2+s),$$

and the exact solution $z(s) = s - se^{s-1}$.

Table 3. Pointwise error $\varepsilon_A(s)$ for Example 4.2.

s	$Z_h(s)$	$z(s)$	$\varepsilon_A(s)$
0.1	$5.930e^{-02}$	$5.934e^{-02}$	$3.526e^{-05}$
0.2	$1.100e^{-01}$	$1.101e^{-01}$	$4.026e^{-05}$
0.3	$1.509e^{-01}$	$1.510e^{-01}$	$9.038e^{-05}$
0.4	$1.804e^{-01}$	$1.804e^{-01}$	$6.642e^{-05}$
0.5	$1.966e^{-01}$	$1.967e^{-01}$	$8.194e^{-05}$
0.6	$1.977e^{-01}$	$1.978e^{-01}$	$7.125e^{-05}$
0.7	$1.813e^{-01}$	$1.814e^{-01}$	$9.604e^{-05}$
0.8	$1.449e^{-01}$	$1.450e^{-01}$	$4.870e^{-05}$
0.9	$8.560e^{-02}$	$8.564e^{-02}$	$4.292e^{-05}$

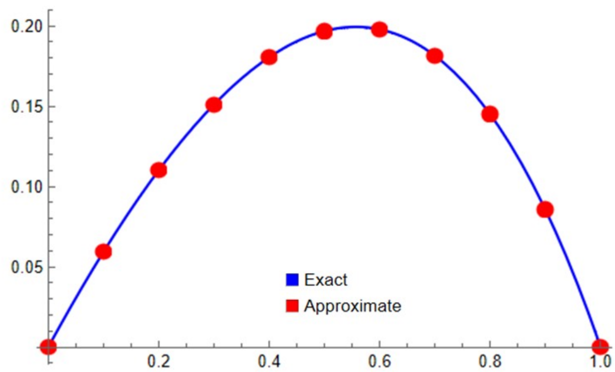

Figure 2. Numerical and exact solution of Example 4.2.

Example 4.3. Consider the VIDE

$$-z''(s) - 3z(s) = f(s) - \int_0^s \sin(s+t)z(t)dt,\ z(0) = z(1) = 0,$$

where

$$f(s) = (2 - 3s + 3s^2) + (s^2 - s - 2)\cos(2s) - (2s - 1)\sin(2s) - \sin(2s) + 2\cos(s),$$

and the exact solution $z(s) = -s^2 + s$.

Table 4. Pointwise error $\varepsilon_A(s)$ for Example 4.3.

s	$Z_h(s)$	$z(s)$	$\varepsilon_A(s)$
0.1	$8.993e^{-02}$	$9.e^{-02}$	$6.640e^{-05}$
0.2	$1.599e^{-01}$	$1.6e^{-01}$	$8.855e^{-05}$
0.3	$2.098e^{-01}$	$2.1e^{-01}$	$1.633e^{-04}$
0.4	$2.398e^{-01}$	$2.4e^{-01}$	$1.390e^{-04}$
0.5	$2.498e^{-01}$	$2.5e^{-01}$	$1.578e^{-04}$
0.6	$2.398e^{-01}$	$2.4e^{-01}$	$1.378e^{-04}$
0.7	$2.098e^{-01}$	$2.1e^{-01}$	$1.611e^{-04}$
0.8	$1.599e^{-01}$	$1.6e^{-01}$	$8.617e^{-05}$
0.9	$8.993e^{-02}$	$9e^{-02}$	$6.469e^{-05}$

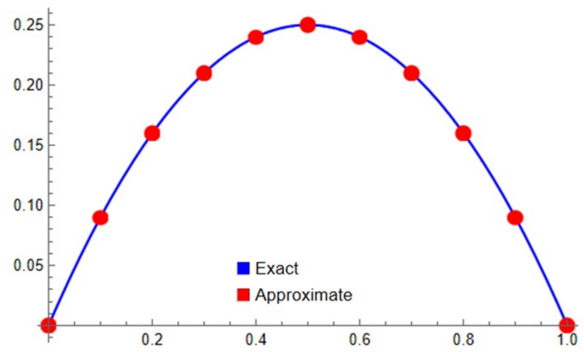

Figure 3. Numerical and exact solution of Example 4.3.

Example 4.4. Consider the second order VIDE

$$-z''(s) + 4z(s) = -f(s) - \int_0^s stz(t)dt, \; z(0) = z(1) = 0,$$

where

$$f(s) = \frac{s^3}{2}\cosh(1) - \frac{s^2}{2}\sinh(2s-1) + \frac{s}{4}\cosh(2s-1) - \frac{s}{4}\cosh(1) - 4\cosh(1),$$

and the exact solution $z(s) = \cosh(2s-1) - \cosh(1)$.

Table 5. Pointwise error $\varepsilon_A(s)$ for Example 4.3.

s	$Z_h(s)$	$z(s)$	$\varepsilon_A(s)$
0.1	$8.993e^{-02}$	$9.e^{-02}$	$6.640e^{-05}$
0.2	$1.599e^{-01}$	$1.6e^{-01}$	$8.855e^{-05}$
0.3	$2.098e^{-01}$	$2.1e^{-01}$	$1.633e^{-04}$
0.4	$2.398e^{-01}$	$2.4e^{-01}$	$1.390e^{-04}$
0.5	$2.498e^{-01}$	$2.5e^{-01}$	$1.578e^{-04}$
0.6	$2.398e^{-01}$	$2.4e^{-01}$	$1.378e^{-04}$
0.7	$2.098e^{-01}$	$2.1e^{-01}$	$1.611e^{-04}$
0.8	$1.599e^{-01}$	$1.6e^{-01}$	$8.617e^{-05}$
0.9	$8.993e^{-02}$	$9e^{-02}$	$6.469e^{-05}$

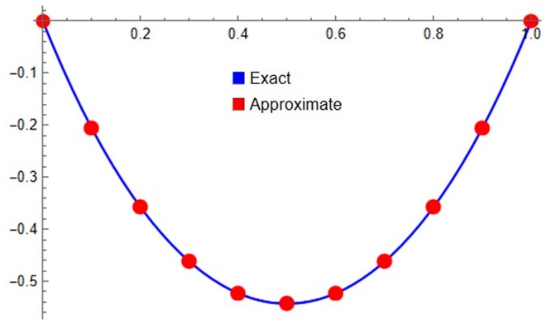

Figure 4. Numerical and exact solution of Example 4.4.

This pointwise error norm of $\varepsilon_A(s)$ is presented in Tables $2-5$. These tables provide information about the accuracy of the numerical solutions obtained using the cubic B-spline finite element method. Based on the results presented in these tables, it can be concluded that the method is reliable and produces results that are consistent with analytical solutions. This suggests that the numerical method is effective in approximating the solution to the given problem. Furthermore, the physical behavior of both the exact and approximate solutions at different levels of spatial discretization $h \leq 1$ is depicted graphically in Figures $1-4$. These figures visually represent how well the numerical solutions capture the true behavior of the system, thereby providing additional evidence of the method's reliability and accuracy.

5 Conclusion

This study combines a finite element method with a cubic B-spline function to solve second-order VIDE. The work not only focuses on developing the numerical method but also addresses the existence and uniqueness of solutions and verifies the accuracy of error estimates. To evaluate the precision of the proposed scheme, several test problems are considered, and error norms are calculated for different levels of spatial discretization. The numerical experiments demonstrate that the results obtained from the method are efficient, reliable, fruitful, and powerful in approximating the solutions to the given problems. Moreover, the performance for the method shows good agreement between the exact solution and the approximate solution. However, it would also be interesting to explore the application of a compact finite difference method for solving such problems. For more detailed information, refer to the references [29, 30, 31, 32].

References

[1] D. Sharma, R. Jiwari, and S. Kumar, *Numerical solution of two point boundary value problems using Galerkin-finite element method,* International journal of nonlinear science, **13**, (2), 204-210, (2012).

[2] Y. A. Sabawi, *A Posteriori $L_\infty(H^1)$ Error Bound in Finite Element Approximation of Semdiscrete Semilinear Parabolic Problems*, first international conference of computer and applied sciences, 102-106, (2019).

[3] Y. A. Sabawi, *A posteriori error analysis in finite element approximation for fully discrete semilinear parabolic problems*, Finite Element Methods and Their Applications, (2020).

[4] A. Cangiani, E. Georgoulis, and Y. Sabawi, *Adaptive discontinuous Galerkin methods for elliptic interface problems*. Mathematics of Computation. **87** (314), 2675-707, (2018).

[5] A. Cangiani, E.H. Georgoulis,and Y.A. Sabawi Cangiani, *Convergence of an adaptive discontinuous Galerkin method for elliptic interface problems*. Journal of Computational and Applied Mathematics, **367**, 112397, (2020).

[6] A. K. Pani and T. E. Peterson, *Finite element methods with numerical quadrature for parabolic integro-differential equations* . SIAM Journal on Numerical Analysis, **3**, (3), 1084-1105, (1996).

[7] H. O. Al-Humedi, *Combining cubic B-spline Galerkin method with quadratic weight function for solving partial integro-differential equations*. Journal of Al-Qadisiyah for computer science and mathematics,**12**, (1), 9–20, (2020).

[8] Y. A. Sabawi, *Posteriori error bound for fully-discrete semi-linear parabolic integro-differential equations*, Journal of physics **1999**(1) 012085: (2021).

[9] B. O. Hussen and Y. A. Sabawi, *B-Spline Cubic Finite Element Method for Solving Ordinary Differential Equations*. Eurasian Journal of Science and Engineering, **8** (3), 39-53, (2023).

[10] K. Styś and T. Styś, *A higher-order finite difference method for solving a system of integro-differential equations*. Journal of Computational and Applied Mathematics, **126** (1-2), 33-46, (2000).

[11] H. O. Al-Humedi and Z. A. Jameel, *Cubic B-spline least-square method combine with a quadratic weight function for solving integro-differential equations*. Earthline Journal of Mathematical Sciences, **4** (1), 99-113, (2020).

[12] M. Erfanian and H. Zeidabadi, *Approximate solution of linear Volterra integro-differential equation by using cubic B-spline finite element method in the complex plane*. Advances in Difference Equations, vol. **2019** (1), 62, (2019).

[13] W. Long, D. Xu, and X. Zeng, *Quasi wavelet based numerical method for a class of partial integro-differential equation*. Applied Mathematics and Computation, **218**, (24), 11842-11850, (2012).

[14] M. Gachpazan, A. Kerayechian, and H. Zeidabadi, *Finite element method for solving linear Volterra integro-differential equations of the second kind*. Journal of Information and Computing Science, **9**, (2014).

[15] R Saber S, Sabawi Y, Hasso MS. *Numerical Solution of the Fredholm Integro-Differential Equations using High-Order Compact Finite Difference Method*. Journal of Education and Science. 2023 Sep 1;32(3): 9-22.

[16] N.A. Mohammad, Y.A. Sabawi and M.S. Hasso, *Haar Wavelet Method For the Numerical Solution of Nonlinear Fredholm Integro-Differential Equations*, J. Appl. Computat. Math, **32**(4), 10-25, (2023).

[17] A. Mohsen and M. El-Gamel, *On the numerical solution of linear and nonlinear Volterra integral and integro-differential equations* , Applied Mathematics and Computation, **217**, (7), 3330-3337, (2010).

[18] J. R. Loh and C. Phang, *A new numerical scheme for solving system of Volterra integro-differential equation*, Alexandria Engineering Journal, **57** (2), 1117-1124, (2018).

[19] S. A. Hamasalih, M. R. Ahmed, and S. S. Ahmed, *Solution Techniques Based on Adomian and Modified Adomian Decomposition for Nonlinear Integro-Fractional Differential Equations of the Volterra-Hammerstein Type*, Journal of University of Babylon for Pure and Applied Sciences, vol. **28** (1), 194-216, (2020).

[20] Y. Jafarzadeh and B. Keramati, *Numerical method for a system of integro-differential equations and convergence analysis by Taylor collocation*, Ain Shams Engineering Journal, **9**, (4), 1433-1438, (2018).

[21] G. Gripenberg, *Stability of the Expected Value of the Solution of a Random Volterra Integral Equation, The Journal of Integral Equations*, 159-173, (1983).

[22] S. Kumar, R. K. Pandey, H. Srivastava, and G. Singh, *A convergent collocation approach for generalized fractional integro-differential equations using Jacobi poly-fractonomials*, Mathematics, **9**, (9), 979, (2021).

[23] P. Vasconcelos, J. Matos, and M. Trindade, *Spectral Lanczos' Tau method for systems of nonlinear integro-differential equations*, Integral Methods in Science and Engineering: Theoretical Techniques, **1**, 305-314, (2017).

[24] A. Daraghmeh, N. Qatanani and H. Jarar'a , *Computational Methods for Solving System of Linear Fredholm Integral Equations*, Palestine Journal of Mathematics., **12** (3),

[25] U.P. Singh, *Applications of Orthonormal Bernoulli polynomials For Approximate Solution of Some Volterra Integral Equations*, Palestine Journal of Mathematics., **11**, 162–170, (2022).

[26] I. A. Malik and T. Jalal, *Solvability of Hammerstein Integral Equation In Banach Space Lp*, Palestine Journal of Mathematics. **14** (4), 205–214, (2022).

[27] M. Amirfakhrian and K. Shakibi, *Solving integro-differential equation by using b-spline interpolation*, International Journal of Mathematical Modelling and Computations, **3** (3), 237-244, (2013).

[28] P. G. Ciarlet, *The finite element method for elliptic problems*. SIAM, (2002).

[29] C. Yao et al., *Development of a fourth-order compact finite difference scheme for simulation of simulated-moving-bed process*, Scientific reports,**10** (1), 7820, (2020).

[30] M. A. Pirdawood and Y. A. Sabawi, *High-order solution of Generalized Burgers–Fisher Equation using compact finite difference and DIRK methods*, Journal of Physics **1999** (1), 012088: IOP Publishing, (2021).

[31] Y. A. Sabawi, M. A. Pirdawood, and M. I. Sadeeq, *A compact fourth-order implicit-explicit Runge-Kutta type method for solving diffusive Lotka–Volterra system*, Journal of Physics, **1999**, (2021), 012103, (2021).

[32] Biazar and M. B. Mehrlatifan, *A compact finite difference scheme for reaction-convection-diffusion equation*, Chiang Mai J. Sci, **45**, (3), 1559-1568, (2018).

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