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Pairs of commutative rings in which almost all intermediate rings are zero dimensional

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Abstract In this paper, we deal with ring extensions $R \subset S$ with a few non zero-dimensional intermediate rings. We prove that the set of intermediate rings of such extensions is finite. If moreover, R is integrally closed in S, then $R \subset S$ is a Prüfer extension. We show that $R \subset S$ has a unique non zero-dimensional intermediate ring if and only if $R \subset S$ is a minimal extension, $\dim(R) = 1$ and $\dim(S) = 0$. We also characterize ring extensions $R \subset S$ with exactly two non zero-dimensional intermediate rings. In particular, we show that $|[R, S]| \leq 4$.

1 Introduction

Throughout this paper all rings are commutative with nonzero identity. We assume that all ring homomorphisms preserve the units. If R is a subring of S, then we let $[R, S]$ denote the *set of intermediate rings* in the ring extension $R \subseteq S$. We also set $[R, S[:= [R, S] \setminus \{S\}$ and $]R, S] := [R, S] \setminus \{R\}$. In the particular case where R is an integral domain with quotient field K, then $[R, K]$ is called the *set of overrings* of R. Let $\mathcal{P}$ be a ring-theoretic property. Recall from [31, p. 34] that a pair of rings (R, S) is said to be a *$\mathcal{P}$-pair* if $R \subseteq S$ and each ring in $[R, S]$ satisfies $\mathcal{P}$. A ring R is said to be a *maximal non-$\mathcal{P}$ subring of S* if R is a proper subring of S, R does not satisfy $\mathcal{P}$, while each ring in $]R, S]$ satisfies $\mathcal{P}$ (cf. [52]). It is worth noting that among the difficult problems to solve in commutative algebra are the characterizations of $\mathcal{P}$-pairs of rings and maximal non-$\mathcal{P}$ subrings. In fact, the structures of the bottom and the top rings forming the extension have a subtle influence on the intermediary rings in between. We point out that several researchers in the area of commutative algebra have studied pairs of rings and maximal non-$\mathcal{P}$ subrings for some properties $\mathcal{P}$ such as Jaffard (cf. [15, 16, 28]), valuation (cf. [18, 36, 37, 38, 46]), Prüfer (cf. [5, 35, 39]), Universally catenarian (cf. [6, 14, 17]), ACCP [8, 44], Noetherian (cf. [10, 52]), Artinian (cf. [1, 3, 42]), Dedekind [2], PVD [47, 48], Mori [45], Almost valuation [43], integrally closed [40, 41, 46], prime ideally equal [4], PID [7, 46], treed [11, 12], etc.

In [42] (resp., [1, 3]), the authors have studied ring extensions with at most two (resp., with exactly three) non-Artinian intermediate rings. Our aim here is to complete this circle of ideas by dealing with pairs of rings with a few non-zero-dimensional intermediate rings.

Recall that a ring R is called *zero-dimensional* if each prime ideal of R is maximal. Any Artinian ring is zero-dimensional. It is worth mentioning that zero-dimensional rings, and especially local Artinian rings, play an important role in algebraic geometry, for instance in deformation theory. Among the reasons that prompted us to do this study, we mention three. First, the growing interest in $\mathcal{P}$-pairs and maximal non-$\mathcal{P}$ subrings as described above. Second, the nice paper [32], written by Gilmer and Heinzer, concerning zero-dimensional pairs of rings. In fact, in [32, Corollary 4.2] Gilmer and Heinzer have completely characterized zero-dimensional pairs of rings. Third, the paper [42] where the authors have considered the problem, but for the property "Artinian". In this context, we would like to point out that we cannot apply directly [42] in our study, even though the properties "Artinian" and "zero-dimensional" appear to be closely related. In fact, when the ring extension $R \subset S$ has exactly n non zero-dimensional intermediate rings, nothing guarantees that it has exactly n non-Artinian intermediate rings.

Before presenting our results, we need to mention some definitions, tools and results that we will use throughout this research. Let $R \subseteq S$ be a ring extension. We let $\overline{R}_S$ denote the integral closure of R in S and R' denote the integral closure of R (in its total quotient ring). Any ring extension $R \subset S$ such that $[R, S] = \{R, S\}$ is called *minimal* (cf. [27]). As $\overline{R}_S$ lies between R and S, it results that if $R \subset S$ is minimal, then it is either *closed* (that is, $\overline{R}_S = R$); or integral (that is, $\overline{R}_S = S$). A pair of rings (R, S) is called *normal* if $T \subseteq S$ is an integrally closed extension (in the sense that T is integrally closed in S) for each ring $T \in [R, S]$ (cf. [20]). Davis was the first who introduced the concept of normal pairs, but for integral domains. Normal pairs generalize Prüfer domains. In fact, if R is an integral domain with quotient field K, then (R, K) is a normal pair if and only if R is a Prüfer domain (cf. [20, Theorem 1] or [29, Theorems 23.4(1) and 26.1(1)]). Other interesting results about normal pairs of integral domains have been established in [9]. The study of normal pairs of rings with zero divisors has attracted attention of many researchers (see for instance, [19], [25] and [50]). This led to the generalization of many results from the domain-theoretic case to arbitrary rings. We draw the reader's attention that, on the other hand, Knebusch and Zhang [50] have introduced the concept of Prüfer extensions. Recall that $R \subseteq S$ is called a *Prüfer* extension if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$. It was proved [50, Theorem 5.2] that $R \subseteq S$ is a Prüfer extension if and only if (R, S) is a normal pair. Any ring extension $R \subseteq S$ has a greatest Prüfer subextension $R \subseteq \widetilde{R}^S$, called the *Prüfer hull* of R in S (cf. [50]) and a ring extension $R \subseteq S$ is called *Prüfer-closed* if $R = \widetilde{R}^S$.

We have organized our paper as follows: In Section 2, we study ring extensions $R \subset S$ with only one non zero-dimensional intermediate ring. We show in Theorem 2.7 that the number of non zero-dimensional intermediate rings is 1 if and only if R is a maximal non zero-dimensional subring of S if and only if $R \subset S$ is a closed minimal extension with $\dim(R) = 1$ and $\dim(S) = 0$. As a consequence, if S is an integral domain, then R is a maximal non zero-dimensional subring of S if and only if R is a rank one valuation domain with quotient field S (see Corollary 2.8). In Section 3, we present some interesting properties of pairs of rings with a few non zero-dimensional intermediate rings. A key result is Theorem 3.2, which states that if $R \subset S$ is a ring extension with $n \geq 1$ non zero-dimensional intermediate rings, then $[R, S]$ is finite. If moreover $R \subset S$ is integrally closed, then (R, S) is a normal pair and all maximal chains of rings in $[R, S]$ have the same finite length. Section 4 is devoted to a complete description of ring extensions having exactly two non zero-dimensional intermediate rings (see Theorems 4.2 and 4.3). In Corollary 4.5, we treat the particular case of integral domains.

We use "$\subseteq$" for inclusion and "$\subset$" for strict inclusion. Most of our notation is standard and can for instance be found in [29] and [49].

2 Maximal non zero-dimensional subrings

In 1992, Gilmer and Heinzer have studied and characterized zero-dimensional pairs of rings (cf. [32, Corollary 4.2]). We label their result as Theorem 2.1.

Theorem 2.1. *Let $R \subset S$ be a ring extension. Then the following conditions are equivalent:*

(i) (R, S) *is a zero-dimensional pair.*

(ii) $R \subset S$ *is an integral extension and* $\dim(R) = 0$.

(iii) $R \subset S$ *is an integral extension and* $\dim(S) = 0$.

Our investigation for this work was motivated by the following nice result due to Gilmer and Heinzer (cf. [31, Theorem 1]).

Lemma 2.2. *Let $R \subset S$ be a ring extension and $n \geq 0$ an integer. Assume that* $\dim(T) \leq n$ *for each ring T in $[R, S[$ that is finitely generated as an R-algebra. Then* $\dim(S) \leq n$. *In particular, if* $\dim(T) = 0$ *for each ring T in $[R, S[$, then* $\dim(S) = 0$.

The following proposition is an easy consequence of Lemma 2.2.

Proposition 2.3. *Let $R \subset S$ be a ring extension and n a positive integer. Assume that $T_1, T_2, \cdots, T_n$ are the only intermediate rings between R and S such that* $\dim(T_i) \geq 1$ *for any i. Then* $\dim(R) \geq 1$.

The next result follows immediately from Proposition 2.3.

Corollary 2.4. *Let $R \subset S$ be a ring extension. Then the following statements are equivalent:*

(i) The number of non zero-dimensional rings in $[R, S]$ is 1.

(ii) R is a maximal non zero-dimensional subring of S.

Before leaving this train of thoughts, we provide in the next corollary a new and short proof of [13, Proposition 3.21].

Corollary 2.5. *Let $R \subset S$ be a minimal ring extension. Then* $\dim(R) = 0$ *if and only if* $\dim(S) = 0$ *and $R \subset S$ is an integral extension.*

Proof. If $\dim(R) = 0$, then $\dim(S) = 0$ by Lemma 2.2. Hence, (R, S) is a zero-dimensional pair. Thus, $R \subset S$ is an integral extension by virtue of Theorem 2.1. Conversely, assume that $\dim(S) = 0$ and $R \subset S$ is an integral extension. Then, by integrality, $\dim(R) = 0$. □

Our main purpose here is to provide a characterization of maximal non zero-dimensional subrings. Recall from [49, p. 28] that a ring extension $R \subseteq S$ is said to satisfy incomparability (in short, INC) if and only if distinct comparable prime ideals of S contract to distinct prime ideals of R. Dobbs [23] has introduced the concept of INC-pairs of rings. More precisely, given a ring extension $R \subseteq S$, we say that (R, S) is an *INC-pair* if $R \subseteq T$ satisfies INC for any ring $T \in [R, S]$. We recall from [33] that a ring extension $R \subseteq S$ is called a *P-extension* if any element of S is a root of a polynomial in $R[X]$ with unit content. It was demonstrated in [22, Theorem] that (R, S) is an INC pair if and only if $R \subseteq S$ is a P-extension. Normal pairs of rings and P-extensions are closely related. In fact, (R, S) is a normal pair if and only if $R \subseteq S$ is an integrally closed P-extension (see [19, Theorem 1]).

Lemma 2.6. *If R is a maximal non zero-dimensional subring of S, then (R, S) is a normal pair.*

Proof. One can easily check that (R, S) is a INC pair. Indeed, if $A \in]R, S]$, then $\dim(A) = 0$. Therefore, $R \subset A$ satisfies INC. On the other hand, R is integrally closed in S. To see this, assume by way of contradiction that $\overline{R}_S \neq R$. Since R is a maximal non zero-dimensional subring of S, then necessarily $\dim(\overline{R}_S) = 0$. Hence, by integrality, $\dim(R) = \dim(\overline{R}_S) = 0$, which is a contradiction. An application of [19, Theorem 1] ensures that (R, S) is a normal pair. □

Theorem 2.7. *Let $R \subset S$ be a ring extension. Then the following statements are equivalent:*

(i) R is a maximal non zero-dimensional subring of S.

(ii) $R \subset S$ is a closed minimal extension and $\dim(S) = 0$.

(iii) $R \subset S$ is a minimal extension, $\dim(R) = 1$ *and* $\dim(S) = 0$.

(iv) $R/(R : S)$ is a one-dimensional valuation domain with quotient field $S/(R : S)$ and $\dim(S) = 0$.

Proof. (i)⇒(ii) The fact that R is integrally closed in S follows from Lemma 2.6. It remains to show that $R \subset S$ is a minimal ring extension. To this end, let $B \in]R, S]$. Then (B, S) is a zero-dimensional pair. Thus, $B \subseteq S$ is an integral extension by virtue of Theorem 2.1. But, B is integrally closed in S according to Lemma 2.6. This yields $B = S$.
(ii)⇒(iii) Follows from [13, Proposition 3.2].
(iii)⇔(iv) Follows from [21, Corollary 2.14].
(iii)⇒(i) Trivial. □

Applying Theorem 2.7 to integral domains, we derive immediately the following corollary.

Corollary 2.8. *Let $R \subset S$ be an extension of integral domains. Then the following statements are equivalent:*

(i) R is a maximal non zero-dimensional subring of S.

(ii) R is a rank one valuation domain with quotient field S.

3 Some properties of Ring extensions with a few non zero-dimensional intermediate rings

We start this section with some general results concerning ring extensions $R \subset S$ with $n \geq 1$ non zero-dimensional intermediate rings.

Lemma 3.1. *Let $R \subset S$ be a ring extension with (exactly) $n \geq 2$ non zero-dimensional intermediate rings. If $T \in]R, S]$ is minimal with respect to being non zero-dimensional, then $R \subset T$ is a minimal extension.*

Proof. Assume the contrary and let A be a ring such that $R \subset A \subset T$. As T is minimal with respect to being non zero-dimensional, then each ring in $[A, T[$ is zero-dimensional. Therefore, Lemma 2.2 ensures that $\dim(T) = 0$, the desired contradiction completing the proof. □

Recall that a ring extension $R \subseteq S$ is said to be an *FIP extension* (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. Recall also that $R \subseteq S$ is said to be an *FCP extension* if each chain in $[R, S]$ is finite. Clearly, each FIP extension is an FCP extension. In [24], the authors have characterized FCP and FIP extensions. Following Dobbs and Shapiro [26], an FCP ring extension is called *catenarian* if all maximal chains between R and S have the same length (equivalently, the Jordan-Hölder chain condition holds true).

Theorem 3.2. *Let $R \subset S$ be a ring extension with (exactly) $n \geq 1$ non zero-dimensional intermediate rings. Then the following conditions hold true:*

(i) $R \subset S$ is an FIP extension.

(ii) If $R \subset A \subseteq S$ and $\dim(A) = 0$, then (A, S) is a zero-dimensional pair. In particular, $A \subseteq S$ is an integral extension.

(iii) If moreover $R \subset S$ is integrally closed, then the following hold true:

a. (R, S) is a normal pair.

b. $R \subset S$ is catenarian.

c. We have:

$$|[R, S]| = \begin{cases} n+1 & \text{if } \dim(S) = 0 \\ n & \text{if } \dim(S) \geq 1 \end{cases}$$

Proof. (i) We argue by induction on n. If $n = 1$, then $R \subset S$ is a closed minimal extension according to Theorem 2.7. Thus, $R \subset S$ is an FIP extension. Assume now that the result holds for any ring extension $A \subset B$ with k non zero-dimensional intermediate rings, where $k \leq n$ and let $R \subset S$ be a ring extension with $n + 1$ non zero-dimensional intermediate rings. Using Proposition 2.3, $\dim(R) \geq 1$. So, let $T \in]R, S]$ be minimal with respect to being non zero-dimensional. According to Lemma 3.1, $R \subset T$ is a minimal extension. Thus, $R \subset T$ is an FIP extension. On the other hand, since the number of non zero-dimensional intermediate rings between T and S is $\leq n$, then the induction hypothesis asserts that $T \subset S$ is an FIP extension. It follows that $R \subset T$ and $T \subseteq S$ are both P-extensions. Thus, $R \subset S$ is also a P-extension (cf. [19, Theorem 2]). We will discuss the following two cases:
case 1. R is integrally closed in S.
In this case, $R \subset T$ is an integrally closed extension. Moreover, (R, S) is a normal pair. Thus, $T \subseteq S$ is also an integrally closed extension. On the other hand, it follows from the above comments that $R \subset T$ and $T \subseteq S$ are both P-extensions. Therefore, $R \subset S$ is an FIP extension by virtue of [24, Corollary 6.5].
case 2. R is not integrally closed in S.
In this case, the number of non zero-dimensional intermediate rings between $\overline{R}_S$ and S is $\leq n$. Thus, the induction hypothesis asserts that $\overline{R}_S \subseteq S$ is an FIP extension. As $\dim(R) \geq 1$, then $\dim(D) \geq 1$ for each ring D in $[R, \overline{R}_S]$. Thus, $|[R, \overline{R}_S]| \leq n + 1$. If $|[R, \overline{R}_S]| \leq n$, then the induction hypothesis ensures that $R \subset \overline{R}_S$ is an FIP extension. Hence, it follows from [24, Theorem 3.13] that $R \subset S$ is also an FIP extension. Now, if $|[R, \overline{R}_S]| = n+1$, then two subcases

can occur.
subcase 2.1. $\overline{R}_S \neq S$.
In this subcase, $\overline{R}_S$ would be a maximal non-zero-dimensional subring of S. Hence, $\overline{R}_S \subset S$ would be a minimal extension. Thus, $\overline{R}_S \subset S$ is an FIP extension. An appeal to [24, Theorem 3.13] guarantees that $R \subset S$ is an FIP extension.
subcase 2.2. $\overline{R}_S = S$.
In this subcase, $|[R, S]| = n + 1$. Thus, $R \subset S$ is an FIP extension.
(ii) Let $A \in]R, S]$ such that $\dim(A) = 0$ and let $B \in [A, S]$. Our task is to show that $\dim(B) = 0$. Note that the ring extension $A \subseteq B$ inherits the FIP property from $R \subset S$. Thus, there exists a (finite) chain of rings $A = A_0 \subset A_1 \subset ... \subset A_l = B$. As $A \subset A_1$ is minimal and $\dim(A) = 0$, then $\dim(A_1) = 0$ accordingly to Corollary 2.4. Again, as $A_1 \subset A_2$ is minimal and $\dim(A_1) = 0$, then $\dim(A_2) = 0$. Proceed along the same lines, one can easily check that $\dim(B) = 0$. Hence, (A, S) is a zero-dimensional pair. It follows from Theorem 2.1 that $A \subseteq S$ is an integral extension.
(iii) a. Since $R \subset S$ is an integrally closed extension satisfying FIP, then [24, Theorem 6.3] ensures that (R, S) is a normal pair.
b. As $R \subset S$ is an integrally closed extension satisfying FCP, then $R \subset S$ is catenarian by virtue of [34, Theorem 1, p. 172].
c. Assume first that $\dim(S) = 0$ and let $A \in [R, S]$ such that $\dim(A) = 0$. Then, $A \neq R$ by Proposition 2.3. It follows from (ii) that $A \subseteq S$ is an integral extension. But, A is integrally closed in S since (R, S) is a normal pair. This yields that $A = S$. So S is the unique zero-dimensional intermediate ring between R and S. Therefore, $|[R, S]| = n + 1$. Suppose now that $\dim(S) \geq 1$ and let $A \in [R, S]$. If $\dim(A) = 0$, then $A \in]R, S]$ (since $\dim(R) \geq 1$) and $A \subseteq S$ is an integral extension by using (ii), which is impossible since A and S have distinct (Krull) dimensions. It follows that $\dim(A) \geq 1$. Therefore, each intermediate ring between R and S is non-zero-dimensional. This implies that $|[R, S]| = n$. □

4 When is the number of non zero-dimensional intermediate rings 2?

Our main goal in this section is to completely describe the structure of ring extensions with exactly two non zero-dimensional intermediate rings. We start our investigation with the following lemma.

Lemma 4.1. *Let $R \subset S$ be a ring extension with (exactly) two non zero-dimensional intermediate rings. Then (exactly) one of the following conditions holds true:*

(i) $R \subset S$ is a minimal extension and $\dim(S) \geq 1$.

(ii) There exists an intermediate ring T such that $R \subset T$ and $T \subset S$ are minimal extensions with $\dim(S) = 0$ *and* $\dim(T) = 1$.

Proof. According to Proposition 2.3, $\dim(R) \geq 1$. Let T be the second non zero-dimensional ring such that $R \subset T \subseteq S$. Then $R \subset T$ is a minimal ring extension by virtue of Lemma 3.1. If $T = S$, then $R \subset S$ is a minimal extension with both R and S non zero-dimensional. Suppose now that $T \neq S$. Then, T would be a maximal non zero-dimensional subring of S. Thus, Theorem 2.7 guarantees that $T \subset S$ is a minimal extension, $\dim(S) = 0$ and $\dim(T) = 1$. □

The next result provides a classification of integrally closed ring extensions with exactly two non zero-dimensional intermediate rings.

Theorem 4.2. *Let $R \subset S$ be a ring extension such that R is integrally closed in S. Then the following statements are equivalent:*

(i) The number of non zero-dimensional intermediate rings is 2.

(ii) (Exactly) one of the following conditions holds:

a. $R \subset S$ is a closed minimal extension and $\dim(S) \geq 1$.

b. $[R, S]$ ordered by the usual set inclusion is a chain of length 2 *and* $\dim(S) = 0$.

Proof. (i)⇒(ii) If $\dim(S) \geq 1$, then $|[R, S]| = 2$ by virtue of Theorem 3.2. Hence, $R \subset S$ is a closed minimal extension. Now, if $\dim(S) = 0$, then using again Theorem 3.2 we get $|[R, S]| = 3$. Therefore, $[R, S]$ ordered by the usual set inclusion is a chain of length 2.
(ii)⇒(i) If $R \subset S$ is a minimal extension with $\dim(S) \geq 1$, then according to Lemma 2.2, $\dim(R) \geq 1$. Hence, we are done. Assume now that $[R, S]$ is a chain of length 2 and $\dim(S) = 0$. Then $[R, S] = \{R, T, S\}$, where $R \subset T$ and $T \subset S$ are closed minimal extensions. We infer that $\dim(T) \geq 1$, since otherwise (T, S) would be a zero-dimensional pair and so S would be integral over T according to Theorem 2.1, which is impossible since T is integrally closed in S. Moreover, $\dim(R) \geq 1$ since otherwise T would be zero-dimensional by virtue of Lemma 2.2. Therefore, there are exactly two non zero-dimensional intermediate rings between R and S, namely R and T. The proof is complete. □

In the following theorem, we determine all ring extensions $R \subset S$ with exactly two non zero-dimensional intermediate rings in case R is not integrally closed in S.

Theorem 4.3. *Let $R \subset S$ be a ring extension such that R is not integrally closed in S. Then the following statements are equivalent:*

(i) The number of non zero-dimensional intermediate rings is 2.

(ii) (Exactly) one of the following conditions holds:

a. $R \subset S$ is a minimal integral extension with $\dim(S) \geq 1$.

b. $\dim(S) = 0$, $\dim(R) = 1$ *and $[R, S]$ ordered by the usual set inclusion is either a chain of length 2 or consists of two chains of length 2.*

Proof. (i)⇒(ii) If $R \subset S$ is a minimal extension, then it must be integral since R is not integrally closed in S. Moreover, as there are exactly two non zero-dimensional intermediate rings between R and S, then R and S should be non zero-dimensional. Now, assume that $R \subset S$ is not a minimal extension. Then according to Lemma 4.1, there exists an intermediate ring T such that $R \subset T$ and $T \subset S$ are minimal extensions, $\dim(S) = 0$ and $\dim(T) = 1$. Thus, $T \subset S$ should be a closed minimal extension. It follows that the minimal ring extension $R \subset T$ is integral. In this case $\overline{R}_S \in [T, S]$ and hence $T = \overline{R}_S$. On the other hand, Theorem 3.2 (i) ensures that $R \subset S$ is a P-extension. Now, we will discuss the following two cases:
case 1. $R \subset S$ is a Prüfer-closed extension.
We show in this case that $[R, S]$ is a chain of length 2. More precisely, $[R, S] = \{R, \overline{R}_S, S\}$. To this end, let $A \in [R, S]$. As $R \subset \overline{R}_S$ is a minimal ring extension, then either $A \cap \overline{R}_S = R$ or $A \cap \overline{R}_S = \overline{R}_S$. If $A \cap \overline{R}_S = R$, then R is integrally closed in A and as $R \subseteq A$ is a P-extension, then (R, A) is a normal pair. Thus $A = R$ since $R \subset S$ is a Prüfer-closed extension. If $A \cap \overline{R}_S = \overline{R}_S$, then $A \in [\overline{R}_S, S]$. Therefore, $A = \overline{R}_S$ or $A = S$ because $\overline{R}_S \subset S$ is a minimal ring extension.
case 2. $R \subset S$ is not a Prüfer-closed extension.
In this case, we show that $[R, S]$ consists of two chains of length 2.
First claim. $R \subset \widetilde{R}^S$ is a (closed) minimal extension.
For, let $A \in]R, \widetilde{R}^S]$. Then A and $\overline{R}_S$ are incomparable under inclusion. This yields that $A \in [R, S] \setminus \{R, \overline{R}_S, S\}$. Hence, $\dim(A) = 0$. It follows that R is a maximal non zero-dimensional subring of $\widetilde{R}^S$. Therefore, $R \subset \widetilde{R}^S$ is a closed minimal extension accordingly to Theorem 2.7.
Second claim. $[R, S] = \{R, \overline{R}_S, S, \widetilde{R}^S\}$.
Indeed, let $B \in [R, S]$. If $B \cap \overline{R}_S = R$, then R is integrally closed in B and as $R \subseteq B$ is a P-extension, then (R, B) is a normal pair. Thus $B \in [R, \widetilde{R}^S]$. So $B = R$ or $B = \widetilde{R}^S$ because $R \subset \widetilde{R}^S$ is minimal by the first claim. If $B \cap \overline{R}_S = \overline{R}_S$, then B contains $\overline{R}_S$ and so $B = \overline{R}_S$ or $B = S$, which completes the proof of our claim.
(ii)⇒(i) If $R \subset S$ is a minimal extension and $\dim(S) \geq 1$, then we are done, by using Proposition 2.3. Suppose now that $[R, S]$ is a chain of length 2, $\dim(S) = 0$ and $\dim(R) = 1$. We claim that $[R, S] = \{R, \overline{R}_S, S\}$. Indeed, as R is not integrally closed in S, then $R \neq \overline{R}_S$. Moreover, as $\dim(\overline{R}_S) = \dim(R) = 1$ and $\dim(S) = 0$, then $\overline{R}_S \neq S$. It follows that $[R, S] = \{R, \overline{R}_S, S\}$ as claimed. Clearly, there are exactly two non zero-dimensional intermediate rings between R and S, namely R and $\overline{R}_S$. Now, assume that $[R, S]$ consists of two chains of length 2 such that

$\dim(S) = 0$ and $\dim(R) = 1$. Then there exist two incomparable rings A_1 and A_2 distinct from R and from S such that $[R, S] = \{R, A_1, A_2, S\}$. Since $\dim(\overline{R}_S) = \dim(R) = 1 \neq \dim(S)$, then clearly $\overline{R}_S \neq S$. Without loss of generality, we can suppose that $\overline{R}_S = A_1$ and hence $\dim(A_1) = \dim(\overline{R}_S) = \dim(R) = 1$. One can check that $\dim(A_2) = 0$. Indeed, assume the contrary, then A_2 would be a maximal non zero-dimensional subring of S and hence $A_2 \subset S$ would be a closed minimal extension accordingly to Theorem 2.7. Thus $R \subset A_2$ is a minimal integral extension, for otherwise R would be integrally closed in S, which contradicts the assumption made on R. Hence $A_2 \subseteq \overline{R}_S$, which is a contradiction with the incomparability of A_2 and $A_1 = \overline{R}_S$. Therefore, there are exactly two non zero-dimensional intermediate rings between R and S, namely R and $\overline{R}_S$. The proof is complete. □

Corollary 4.4. *If the number of non zero-dimensional intermediate rings in a ring extension $R \subset S$ is 2, then $|[R, S]| \leq 4$.*

The next corollary classifies ring extensions $R \subset S$ with exactly two non zero-dimensional intermediate rings in the particular case where S is an integral domain.

Corollary 4.5. *Let $R \subset S$ be an extension of integral domains. Then the following statements are equivalent:*

(i) The number of non zero-dimensional intermediate rings is 2.

(ii) (Exactly) one of the following conditions holds:

a. $R \subset S$ is a minimal extension and S is not a field.

b. R is a rank two valuation domain with quotient field S.

c. $R \subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S.

Proof. (i)⇒(ii) If $R \subset S$ is a minimal extension, then we are done. Suppose now that $R \subset S$ is not a minimal extension. It follows from Theorems 4.2 and 4.3 that S is a field and R is not a field. If R is integrally closed in S, then $[R, S]$ is a chain of length 2 by virtue of Theorem 4.2. According to [51, page 1738, lines 8-13], S is the quotient field of R. Thus, R is a rank two valuation domain with quotient field S. If R is not integrally closed in S, then [51, page 1738, lines 8-13] and Theorem 4.3 guarantee that S is the quotient field of R and $R \subset R' \subset S$ is a chain of length 2. Hence, $R \subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S.
(ii)⇒(i) If $R \subset S$ is a minimal extension and S is not a field, then R cannot be a field (see [27, Théorème 2.2]). If R is a rank two valuation domain with quotient field S, then $[R, S] = \{R, V, S\}$, where V is the rank one valuation overring of R. Finally, if $R \subset R'$ is a minimal extension and R' is a rank one valuation domain with quotient field S, then [30, Theorem 2.4] ensures that $[R, S] = \{R, R', S\}$. Therefore, in all cases, there are exactly two non zero-dimensional intermediate rings between R and S. This completes the proof. □

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