Palestine Journal of Mathematics

Vol 13(3)(2024) , 593–610

Complex bivariate Fibonacci and Lucas polynomials and new generating functions with some special numbers

Amina Daoui, Ali Boussayoud and Khadidja Boubellouta

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99,13F10; Secondary 13A15,13M05.

Keywords and phrases: Generating functions, Bivariate complex Fibonacci polynomials, k-Fibonacci numbers, k-Jacobsthal numbers, k-Balancing numbers.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Ali. Boussayoud

Abstract In this paper, we use certain operators for symmetric functions in order to derive new generating functions for the products of k-Fibonacci numbers, k-Pell numbers, k-Jacobsthal numbers, k-Balancing numbers and k-Mersenne numbers with bivariate complex Fibonacci and Lucas polynomials.

1 Introduction

There have been many papers over several years concerned with identities of bivariate complex polynomials and numbers. In particular, in [5], M.Asci and E.Gurel studied bivariate complex Fibonacci and Lucas polynomials, providing their generating functions, explicit formulas, Binet's formulas and other properties. In another paper [6], they defined and studied the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, giving the generating functions, Binet's formulas, explicit formulas, Q matrices, determinantal representations, and partial derivatives of these polynomials. In [26], A.F Horadam defined the complex Fibonacci polynomials and the complex Fibonacci numbers, and gave the divisibility properties of bivariate complex Fibonacci polynomials.

Recently, many other relevant numbers and polynomials have been studied, such as Pell numbers and k-Pell Lucas numbers and many other polynomials, and a numerous papers have been published in this context, defining operators to derive new symmetric properties and determine their generating functions.

The k-Fibonacci numbers $\{F_{k,n}\}_{n\in\mathbb{N}}$ were introduced by Falcon and Plaza in [23], they defined them recurrently by

$$\begin{cases} F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad k \geq 1, \quad n \leq 2, \\ F_{k,0} = 1 \quad and \quad F_{k,1} = k. \end{cases}$$

The bivariate complex Fibonacci polynomials $\{F_n(x,y)\}_{n=0}^{\infty}$ were defined in [5] as follows

$$\begin{cases} F_{n+1}(x,y) = ixF_n(x,y) + yF_{n-1}(x,y), \quad n \geq 1, \\ F_0(x,y) = 0, \quad F_1(x,y) = 1. \end{cases}$$

Binet's formula and the explicit formulas of the bivariate complex Fibonacci polynomials are respectively given by

$$F_n(x,y) = \frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)},$$

and

$$F_n(x,y) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{1} (ix)^{n-2j-1} y^j.$$

Note that $\alpha(x,y)$, $\beta(x,y)$ are the roots of the characteristic equation $t^2 - ixt - y = 0$.

The k-Lucas numbers $\{L_{k,n}\}_{n\in\mathbb{N}}$ were introduced by Falcon in [24], they defined them recurrently by

$$\begin{cases} L_{k,n} = kL_{k,n-1} + L_{k,n-2}, & k \geq 1, \quad n \leq 2, \\ L_{k,0} = 2, \quad L_{k,1} = k. \end{cases}$$

The bivariate complex Lucas polynomials $\{L_n(x,y)\}_{n=0}^{\infty}$ were defined in [5] as

$$\begin{cases} L_{n+1}(x,y) = ixL_n(x,y) + yL_{n-1}(x,y), & n \geq 1, \\ L_0(x,y) = 2, \quad L_1(x,y) = ix. \end{cases}$$

Binet's formula and the explicit formulas of bivariate complex Lucas polynomials are respectively given by

$$L_n(x,y) = \alpha^n(x,y) - \beta^n(x,y),$$

and

$$L_n(x,y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (ix)^{n-2j} y^j.$$

For any positive integer k, the k-Pell sequence, denoted by $(P_{k,n})_{n\in\mathbb{N}}$ is defined recursively in [19] as follows

$$\begin{cases} P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, & n \geq 1 \\ P_{k,0} = 0, \quad P_{k,1} = 1. \end{cases}$$

The following characteristic equation is associated with the recurrence relation of k-Pell numbers

$$r^2 - 2r - k = 0.$$

Note that the roots of this equation are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Since $\sqrt{1+k} > 1$, then $r_2 < 0$ and hence, $r_2 < 0 < r_1$.

Also, we have that $r_1 + r_2 = 2, r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$.

For $k = 1$, we obtain $r_1 = 1 + \sqrt{2}$ which is known as the silver ratio and is related to the Pell number sequence, as it represents the limiting ratio for the consecutive Pell numbers.

The Binet's formula for the k-Pell numbers is given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where r_1, r_2 are the roots of the characteristic equation $r^2 - 2r - k = 0$ with $r_1 > r_2$.

In a similar manner to the k-Pell numbers, we define the k-Jacobsthal numbers using the recurrence relation given in [29].

For any positive integer k, the k-Jacobsthal sequence say $(J_{k,n})_{n\in\mathbb{N}}$ is defined recurrently by

$$\begin{cases} J_{k,n+1} = 2J_{k,n} + kJ_{k,n-1}, & n \geq 1, \\ J_{k,0} = 0, \quad J_{k,1} = 1. \end{cases}$$

The following characteristic equation is associated with the recurrence relation of k-Jacobsthal numbers

$$r^2 - r - k = 0,$$

such that the roots of this equation are $r_1 = \dfrac{1+\sqrt{1+4k}}{2}$ and $r_2 = \dfrac{1-\sqrt{1+4k}}{2}$.

Also, the Binet's formula for the k-Jacobsthal number is given by

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2},$$

where r_1, r_2 are the roots of the characteristic equation $r^2 - 2r - k = 0$ with $r_1 > r_2$.

We are also interested in Balancing numbers, which were first defined by R.P. Finkelstein [25], who referred to them numerical centres. The definition is as follows

A positive integer n is called a Balancing number if

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+r),$$

holds for some positive integer r. Then they are the solutions of this Diophantine equation, and r is a balancer corresponding to a Balancing number n. they satisfy the following recurrence relation

$$\begin{cases} B_{n+1} = 6B_n - B_{n-1}, \quad n \geqslant 1, \\ B_0 = 1, \quad B_1 = 1 \end{cases}$$

The generalized Balancing numbers are called k-Balancing numbers. They were introduced by Ray in [33]. The n^{th} k-Balancing numbers $B_{k,n}$ are recursively defined by

$$\begin{cases} B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}, \ k \geqslant 1, \\ B_{k,0} = 1, \quad B_{k,1} = 1 \end{cases}.$$

The explicit formula of k-Balancing numbers is given by

$$\alpha_1^{n+2} = 6k\alpha_1^{n+1} - \alpha_1^n,$$

and

$$\alpha_2^{n+2} = 6k\alpha_2^{n+1} - \alpha_2^n,$$

where α_1 and α_2 are the roots of the equation $\alpha_1^2 = 6k\alpha_1 - 1$ and $\alpha_2^2 = 6k\alpha_2 - 1$. Note that the roots of the equation $\alpha^2 = 6k\alpha - 1$ are $3k + \sqrt{9k^2 - 1}$ and $3k - \sqrt{9k^2 - 1}$.

The Binet's formula for the n^{th} k-Balancing numbers is given by

$$B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

We are also mainly interested in the Mersenne numbers, so we will introduce some of their properties.

Definition 1.1. For $n \in \mathbb{N}$, the Mersenne sequence, denoted by $\{M_n\}_{n\in\mathbb{N}}$, is defined recursively by

$$\begin{cases} M_n = 3M_{n-1} - 2M_{n-2}, \quad \forall n \geqslant 2, \\ M_0 = 0, \quad M_1 = 1. \end{cases}$$

Definition 1.2. For $n \in \mathbb{N}$, the k-Mersenne sequence, denoted by $\{M_{k,n}\}_{n\in\mathbb{N}}$, is defined recursively by

$$\begin{cases} M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2}, \quad \forall n \geqslant 2 \\ M_{k,0} = 0, \quad M_{k,1} = 1. \end{cases}$$

The Binet's formula is given by

$$M_{k,n} = \tfrac{r_1^n - r_2^n}{r_1 - r_2},$$

such that $r_1 = \frac{3k+\sqrt{9k^2-8}}{2}$ and $r_2 = \frac{3k-\sqrt{9k^2-8}}{2}$ are the roots of the characteristic equation of the sequence $M_{k,n}$.

2 Definitions and Properties

In this section, we present symmetric functions along with some of their properties and useful notions from the literature. We will also introduce a set of definitions to clarify their use in the new results.

Definition 2.1. [1] A function $f(x_1, x_2, \ldots, x_n)$ in n variables is symmetric if, for all permutations of the index set $(1, 2, \ldots, n)$, the following equality holds

$$f(x_1, x_2, \ldots, x_n) = f(x_{s(1)}, x_{s(2)}, \ldots, x_{s(n)}).$$

This means a function of several variables is symmetric if its values does not change when we swap variables.

Definition 2.2. [14] Let k and n be two positive integers and let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of given variables. Then the elementary symmetric function $e_k(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is defined by

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i_1+i_2+\ldots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_n^{i_n}, \ \ 0 \leq k \leq n, \tag{2.1}$$

with $i_1, i_2 \ldots i_n = 0 \vee 1$.

Definition 2.3. [14] Let k and n be two positive integers and let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a set of given variables. Then the complete symmetric functions $h_k(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is defined by

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i_1+i_2+\ldots+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_n^{i_n}, \tag{2.2}$$

with $i_1, i_2 \ldots i_n \geq 0$.

Definition 2.4. [14] Let n be a positive integer and $A = \{a_1, a_2\}$ be a set of given variables. Then the symmetric function S_n is defined by

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

with

$$S_0(a_1 + a_2) = 1, \ \ S_1(a_1 + a_2) = a_1 + a_2, \ \ S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2.$$

Remark 2.5. We have $S_n(a_1 + a_2) = 0$, for $n < 0$.

Remark 2.6. Let $A = \{a_1, a_2\}$ be an alphabet. We have

$$S_n(a_1 + a_2,) = h_n(a_1, a_2).$$

Definition 2.7. [14] The symmetrizing operator $\delta_{a_1 a_2}^k$ is defined by

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \quad \forall k \in \mathbb{N}. \tag{2.3}$$

Remark 2.8. If $f(a_i) = a_i$ in formula (2.3), then we obtain

$$\delta_{a_1 a_2}^k f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2}.$$

Remark 2.9. Let $A = \{a_1, a_2\}$ be an alphabet. We have

$$\delta_{a_1 a_2}^k (a_1) = S_k(a_1 + a_2).$$

3 Main Theorem

Our results are based on the following theorem witch was previously proven in [10].

Theorem 3.1. *Let A, B and C be three alphabets,* $\{a_1, a_2\}$, $\{b_1, b_2\}$ *and* $\{c_1, c_2\}$ *respectively. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)S_{n+k-1}(C)z^n =$$

$$\frac{b_1^k b_2^k \begin{pmatrix} \left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-k-1}(B)C_2^{n+k}z^n \\ -\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_2^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-k-1}(B)C_1^{n+k}z^n \end{pmatrix}}{(c_1-c_2)\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)\prod\limits_{a\in A}(1-ab_1c_2z)\prod\limits_{a\in A}(1-ab_2c_2z)}, \tag{3.1}$$

for all $k \in \mathbb{N}_0$.

Proof. By applying the operator $\delta_{c_1c_2}^k \delta_{b_1b_2}^k$ to the series $f(b_1c_1z) = \sum\limits_{n=0}^{\infty} S_n(A)b_1^n c_1^n z^n$, we obtain

$$\begin{aligned}
\delta_{c_1c_2}^k \delta_{b_1b_2}^k (f(b_1c_1z)) &= \delta_{c_1c_2}^k \delta_{b_1b_2}^k \left(\sum_{n=0}^{\infty} S_n(A)b_1^n c_1^n z^n\right) \\
&= \delta_{c_1c_2}^k \left(\frac{b_1^k \sum\limits_{n=0}^{\infty} S_n(A)b_1^n c_1^n z^n - b_2^k \sum\limits_{n=0}^{\infty} S_n(A)b_2^n c_1^n z^n}{b_1 - b_2}\right) \\
&= \delta_{c_1c_2}^k \left(\sum_{n=0}^{\infty} S_n(A)\frac{b_1^{n+k} - b_2^{n+k}}{b_1 - b_2} c_1^n z^n\right) \\
&= \delta_{c_1c_2}^k \left(\sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_1^n z^n\right) \\
&= \frac{c_1^k \sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_1^n z^n - c_2^k \sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_2^n z^n}{c_1 - c_2} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_1^{n+k} z^n - \sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)c_2^{n+k} z^n}{c_1 - c_2} \\
&= \sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)\left(\frac{c_1^{n+k} - c_2^{n+k}}{c_1 - c_2}\right) z^n \\
&= \sum_{n=0}^{\text{inf}} S_n(A)S_{n+k-1}(B)S_{n+k-1}(C)z^n.
\end{aligned}$$

On the other hand, since

$$f(b_1c_1z) = \frac{1}{\prod\limits_{a\in A}(1-ab_1c_1z)},$$

$$\delta_{c_1c_2}^{k}\delta_{b_1b_2}^{k}\left(\frac{1}{\prod\limits_{a\in A}(1-ab_1c_1z)}\right) = \delta_{c_1c_2}^{k}\left(\frac{\frac{b_1^k}{\prod\limits_{a\in A}(1-ab_1c_1z)}-\frac{b_2^k}{\prod\limits_{a\in A}(1-ab_2c_1z)}}{b_1-b_2}\right)$$

$$= \delta_{c_1c_2}^{k}\left(\frac{b_1^k\prod\limits_{a\in A}(1-ab_2c_1z)-b_2^k\prod\limits_{a\in A}(1-ab_1c_1z)}{(b_1-b_2)\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)}\right).$$

Using the fact that

$$\sum_{n=0}^{\infty}S_n(-A)b_1^nc_1^nz^n=\prod_{a\in A}(1-ab_1c_1z),$$

then

$$\delta_{c_1c_2}^{k}\delta_{b_1b_2}^{k}\left(\frac{1}{\prod\limits_{a\in A}(1-ab_1c_1z)}\right) = \delta_{c_1c_2}^{k}\left(\frac{\sum\limits_{n=0}^{\infty}S_n(-A)b_1^kb_2^nc_1^nz^n-\sum\limits_{n=0}^{\infty}S_n(-A)b_2^kb_1^nc_1^nz^n}{(b_1-b_2)\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)}\right)$$

$$= \delta_{c_1c_2}^{k}\left(\frac{-b_1^kb_2^k\sum\limits_{n=0}^{\infty}S_n(-A)\left(\frac{b_1^{n-k}-b_2^{n-k}}{b_1-b_2}\right)c_1^nz^n}{\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)}\right)$$

$$= \delta_{c_1c_2}^{k}\left(\frac{-b_1^kb_2^k\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_1^nz^n}{\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)}\right)$$

$$=\frac{1}{c_1-c_2}\left(\frac{-c_1^kb_1^kb_2^k\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_1^nz^n}{\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)}+\frac{c_2^kb_1^kb_2^k\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)c_2^nz^n}{\prod\limits_{a\in A}(1-ab_1c_2z)\prod\limits_{a\in A}(1-ab_2c_1z)}\right)$$

$$=\frac{b_1^kb_2^k\left(\begin{array}{l}\left(\sum\limits_{n=0}^{\infty}S_n(-A)b_2^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)b_1^nc_1^nz^n\right)\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)C_2^{n+k}z^n\\ -\left(\sum\limits_{n=0}^{\infty}S_n(-A)b_2^nc_2^nz^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-A)b_1^nc_2^nz^n\right)\sum\limits_{n=0}^{\infty}S_n(-A)S_{n-k-1}(B)C_1^{n+k}z^n\end{array}\right)}{(c_1-c_2)\prod\limits_{a\in A}(1-ab_1c_1z)\prod\limits_{a\in A}(1-ab_2c_1z)\prod\limits_{a\in A}(1-ab_1c_2z)\prod\limits_{a\in A}(1-ab_2c_2z)}.$$

This completes the proof. □

If $k=0,1,2$ in Theorem 3.1, we deduce the following lemmas.

Lemma 3.2. *[10] Let A, B and C be three alphabets defined as* $\{a_1, a_2\}$, $\{b_1, b_2\}$ *and* $\{c_1, c_2\}$ *respectively. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(B)S_{n-1}(C)z^n = \frac{M_1}{D}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

with

$$\begin{aligned} M_1 = {} & (a_1 + a_2)z - a_1a_2(b_1 + b_2)(c_1 + c_2)z^2 + b_1b_2c_1c_2(a_1 + a_2)(2a_1a_2 - (a_1 + a_2)^2)z^3 \\ & + a_1a_2b_1b_2c_1c_2(b_1 + b_2)(c_1 + c_2)(a_1 + a_2)^2z^4 - b_1b_2c_1c_2a_1^2a_2^2(a_1 + a_2)(b_1b_2(c_1 + c_2)^2 + \\ & c_1c_2(b_1 + b_2)^2 - c_1c_2b_1b_2)z^5 + a_1^3a_2^3b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2)z^6. \end{aligned}$$

$$\begin{aligned} D = {} & 1 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z + (b_1b_2(a_1 + a_2)^2(c_1 + c_2)^2 + ((b_1 + b_2)^2 - 2b_1b_2) \\ & ((a_1 + a_2)^2c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2(c_1 + c_2)^2))z^2 - (a_1 + a_2)(b_1 + b_2)(c_1 + c_2) \\ & (b_1b_2c_1c_2(a_1 + a_2)^2 + b_1b_2a_1a_2(c_1 + c_2)^2 + a_1a_2c_1c_2(b_1 + b_2)^2 - 5a_1a_2c_1c_2b_1b_2)z^3 \\ & + (a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^4 + c_1^2c_2^2b_1^2b_2^2(a_1 + a_2)^4 + a_1^2a_2^2b_1^2b_2^2(c_1 + c_2)^4 - a_1a_2b_1b_2c_1c_2 \\ & (4b_1b_2c_1c_2(a_1 + a_2)^2 + 4a_1a_2c_1c_2(b_1 + b_2)^2 + 4a_1a_2b_1b_2(c_1 + c_2)^2 - (a_1 + a_2)^2(b_1 + b_2)^2 \\ & (c_1 + c_2)^2) + 6a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4 - a_1a_2b_1b_2c_1c_2(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)(b_1b_2c_1c_2(a_1 + a_2)^2 \\ & + a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 - 5a_1a_2b_1b_2c_1c_2)z^5 + (a_1^2a_2^2b_1^3b_2^3c_1^2c_2^2(a_1 + a_2)^2 \\ & (c_1 + c_2)^2 + a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2((b_1 + b_2)^2 - 2b_1b_2)((a_1 + a_2)^2c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2(c_1 + c_2)^2))z^6 \\ & - a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^7 + a_1^4a_2^4b_1^4b_2^4c_1^4c_2^4z^8. \end{aligned}$$

Lemma 3.3. *[10] Let A, B and C be three alphabets,* $\{a_1, a_2\}$, $\{b_1, b_2\}$ *and* $\{c_1, c_2\}$ *respectively. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+1}(B)S_{n+1}(C)z^n = \frac{M_2}{D}, \quad n \in \mathbb{N}_0, \tag{3.3}$$

with

$$\begin{aligned} M_2 = {} & (c_1 + c_2)(b_1 + b_2) - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z \\ & + c_1c_2b_1b_2(a_1 + a_1)^2(b_1 + b_1)(c_1 + c_1)z^2 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^3 \\ & - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2)z^4 + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^5. \end{aligned}$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_n(C)z^n = \frac{M_3}{D}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

with

$$\begin{aligned} M_3 = {} & (c_1 + c_2)(b_1 + b_2)z - (a_1 + a_2)(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2)z^2 \\ & + c_1c_2b_1b_2(a_1 + a_1)^2(b_1 + b_1)(c_1 + c_1)z^3 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_1)((a_1 + a_1)^2 - 2a_1a_2)z^4 \\ & - a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2)z^5 + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^6. \end{aligned}$$

Lemma 3.4. *[10] Let A, B and C be three alphabets,* $\{a_1, a_2\}$, $\{b_1, b_2\}$ *and* $\{c_1, c_2\}$ *respectively. Then we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(B)S_n(C)z^n = \frac{M_4}{D}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

with

$$\begin{aligned} M_4 = {} & 1 - (a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 + b_1b_2c_1c_2(a_1 + a_2)^2 - 3a_1a_2c_1c_2b_1b_2)z^2 \\ & + 2a_1a_2b_1b_2c_1c_2(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)z^3 - (b_1b_2a1^2a_2^2c_1^2c_2^2(b_1 + b_2)^2 + c_1c_2a_1^2a_2^2b_1^2b_2^2 \\ & (c_1 + c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1 + a_2)^2 - 3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4 + a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^6. \end{aligned}$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n = \frac{M_5}{D}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

with

$$M_5 = z - (a_1a_2c_1c_2(b_1+b_2)^2 + a_1a_2b_1b_2(c_1+c_2)^2 + b_1b_2c_1c_2(a_1+a_2)^2 - 3a_1a_2c_1c_2b_1b_2)z^3 + 2a_1a_2b_1b_2c_1c_2(a_1+a_2)(b_1+b_2)(c_1+c_2)z^4 - (b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1+b_2)^2 + c_1c_2a_1^2a_2^2b_1^2b_2^2 (c_1+c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1+a_2)^2 - 3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^5 + a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^7.$$

4 Some new results

In this section, we use the aforementioned theorem to derive new generating functions of products of some well-known numbers and polynomials.

First step

Firstly, by replacing a_2 by $[-a_2]$ and b_2 by $[-b_2]$ and c_2 by $[-c_2]$ and by making the substitutions $c_1 - c_2 = k$, $b_1 - b_2 = k$, $a_1 - a_2 = ix$, $c_1c_2 = 1$, $b_1b_2 = 1$, and $a_1a_2 = y$ in (3.5) and (3.4), we obtain the following results

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_n(b_1+[-b_2])S_n(c_1+[-c_2]) = \frac{F^2F(x,y)}{D_1}, \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(b_1+[-b_2])S_n(c_1+[-c_2]) = \frac{N_1}{D_1}, \tag{4.2}$$

with

$$F^2F(x,y) = k^2z + ix(2k^2+1)z^2 - x^2k^2z^3 - ix(-x^2+2y)z^4 - y^2k^2z^5 + y^2ixz^6,$$

$$N_1 = 1 - (2k^2y - x^2 + 3y)z^2 - 2yixk^2z^3 + (2k^2y^2 - x^2y + 3y^2)z^4 - y^3z^6,$$

and

$$D_1 = 1 - ixk^2z - (k^4y - 2k^2x^2 + 4k^2y - 2x^2 + 4y)z^2 - (2ik^4xy - ik^2x^3 + 5ik^2xy)z^3 + (k^4x^2y + 2k^4y^2 + 8k^2y^2 + x^4 - 4x^2y + 6y^2)z^4 + (2ik^4xy^2 - ik^2x^3y + 5ik^2xy^2)z^5 - (k^4y^3 - 2k^2x^2y^2 + 4k^2y^3 - 2x^2y^2 + 4y^3)z^6 + y^3ixk^2z^7 + y^4z^8.$$

From the previous results, we conclude the following

Theorem 4.1. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Fibonacci numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2F_n(x,y)z^n = \frac{k^2z + ix(2k^2+1)z^2 - x^2k^2z^3 - ix(-x^2+2y)z^4 - y^2k^2z^5 + y^2ixz^6}{D_1}.$$

Corollary 4.2. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Fibonacci numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} F_n^2F_n(x,y)z^n = \frac{z + 3ixz^2 - x^2z^3 - ix(-x^2+2y)z^4 - y^2z^5 + y^2ixz^6}{D_{F_n^2F(x,y)}},$$

with

$$D_{F_n^2F(x,y)} = 1 - ixz - (-4x^2+9y)z^2 - (-ix^3+7ixy)z^3 + (x^4 - 3x^2y + 16y^2)z^4 + (-ix^3y + 7ixy^2)z^5 - (-4x^2y^2 + 9y^3)z^6 + y^3ixz^7 + y^4z^8.$$

Theorem 4.3. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Fibonacci numbers and bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} F_{k,n}^2 L_n(x,y) z^n = \frac{F^2 L(x,y)}{D_1},$$

with

$$\begin{aligned} F^2L(x,y) = 2 &- ixk^2z + (x^2y^2 - 2y^3)z^6 + ixk^2y^2z^5 \\ &+ (4k^2y^2 - 2x^2y + 6y^2 - x^2(-x^2+2y))z^4 \\ &+ (ik^2x^3 - 4ik^2xy)z^3 + (-4k^2y + 2x^2 - 6y + x^2(2k^2+1))z^2. \end{aligned}$$

Proof. We have

$$L_n(x,y) = 2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]), \text{ (see[12])},$$

then

$$\begin{aligned} \sum_{n=0}^{\infty} F_{k,n}^2 L_n(x,y) z^n &= \sum_{n=0}^{\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])S_n(b_1 + [-b_2])S_n(c_1 + [-c_2]))z^n \\ &= 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(b_1 + [-b_2])S_n(c_1 + [-c_2])z^n \\ &\quad -ix\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_n(b_1 + [-b_2])S_n(c_1 + [-c_2])z^n \\ &= 2\frac{1 - (2k^2y - x^2 + 3y)z^2 - 2yixk^2z^3 + (2k^2y^2 - x^2y + 3y^2)z^4 - y^3z^6}{D_1} \\ &\quad -ix\frac{k^2z + ix(2k^2+1)z^2 - x^2k^2z^3 - ix(-x^2+2y)z^4 - y^2k^2z^5 + y^2ixz^6}{D_1}, \end{aligned}$$

so

$$\sum_{n=0}^{\infty} F_{k,n}^2 L_n(x,y) z^n = \frac{F^2 L(x,y)}{D_1},$$

with

$$\begin{aligned} F^2L(x,y) = 2 - ixk^2z + (x^2y^2 - 2y^3)z^6 + ixk^2y^2z^5 + (4k^2y^2 - 2x^2y + 6y^2 - x^2(-x^2+2y))z^4 \\ + (ik^2x^3 - 4ik^2xy)z^3 + (-4k^2y + 2x^2 - 6y + x^2(2k^2+1))z^2. \end{aligned}$$

This completes the proof. □

Corollary 4.4. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Fibonacci numbers and bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} F_n^2 L_n(x,y) z^n = \frac{N_{F_n^2 L_n(x,y)}}{D_{F_n^2 L(x,y)}},$$

with

$$\begin{aligned} N_{F_n^2 L_n(x,y)} = 2 - ixz + (x^2y^2 - 2y^3)z^6 + y^2ixz^5 + (10y^2 - 2x^2y - x^2(-x^2+2y))z^4 \\ + (ix^3 - 4ixy)z^3 + (5x^2 - 10y)z^2. \end{aligned}$$

$$D_{F_n^2 L(x,y)} = D_{F_n^2 F(x,y)}.$$

Second step

Now, we will make the following substitutions a_2 by $[-a_2]$ and b_2 by $[-b_2]$ and c_2 by $[-c_2]$. We also set $c_1 - c_2 = 2$, $b_1 - b_2 = 2$, $a_1 - a_2 = ix$, $c_1c_2 = k$, $b_1b_2 = k$, and $a_1a_2 = y$ in (3.6) and (3.2). We then obtain the following

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{P^2F(x,y)}{D_2}, \tag{4.3}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{N_2}{D_2}. \tag{4.4}$$

where

$$P^2F(x,y) = z - (-k^2x^2 + 3k^2y + 8ky)z^3 - 8yk^2ixz^4 + (-k^4x^2y + 3k^4y^2 + 8k^3y^2)z^5 - y^3k^6z^7,$$

$$N_2 = ixz + 4yz^2 - k^2ix(-x^2 + 2y)z^3 + 4yk^2x^2z^4 + k^2y^2ix(k^2 + 8k)z^5 - 4y^3k^4z^6,$$

and

$$\begin{aligned} D_2 = 1 &- 4ixz - (-2k^2x^2 + 4k^2y - 8kx^2 + 16ky + 16y)z^2 - (-4ik^2x^3 + 20ik^2xy + 32ikxy)z^3 \\ &+ (k^4x^4 - 4k^4x^2y + 6k^4y^2 + 32k^3y^2 + 16k^2x^2y + 32k^2y^2)z^4 + (-4ik^4x^3y + 20ik^4xy^2 \\ &+ 32ik^3xy^2)z^5 - (-2k^6x^2y^2 + 4k^6y^3 - 8k^5x^2y^2 + 16k^5y^3 + 16k^4y^3)z^6 \\ &+ 4y^3k^6ixz^7 + y^4k^8z^8. \end{aligned}$$

From the previous results, we conclude the following theorems and corollaries

Theorem 4.5. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Pell numbers with bivariate complex Fibonacci polynomials is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} P_{k,n}^2 F(x,y)z^n &= \frac{z - (-k^2x^2 + 3k^2y + 8ky)z^3 - 8yk^2ixz^4}{D_2} \\ &+ \frac{(-k^4x^2y + 3k^4y^2 + 8k^3y^2)z^5 - y^3k^6z^7}{D_2}. \end{aligned}$$

Corollary 4.6. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Pell numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} P_n^2 F(x,y)z^n = \frac{z - (-x^2 + 11y)z^3 - 8yixz^4 + (-x^2y + 11y^2)z^5 - y^3z^7}{D_{P_n^2F(x,y)}},$$

where

$$\begin{aligned} D_{P_n^2F_n(x,y)} = 1 &- 4ixz - (-10x^2 + 36y)z^2 - (-4ix^3 + 52ixy)z^3 + (x^4 + 12x^2y + 70y^2)z^4 \\ &+ (-4ix^3y + 52ixy^2)z^5 - (-10x^2y^2 + 36y^3)z^6 + 4y^3ixz^7 + y^4z^8. \end{aligned}$$

Theorem 4.7. *For $n \in \mathbb{N}$, the new generating function of the product of k-Pell numbers and bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} P_{k,n}^2 L_n(x,y)z^n = \frac{P^2L(x,y)}{D_2},$$

where

$$\begin{aligned} P^2L(x,y) = ixy^3k^6z^7 &- 8y^3k^4z^6 + (2k^2y^2ix(k^2 + 8k) - ix(-k^4x^2y + 3k^4y^2 + 8k^3y^2))z^5 \\ &+ (-2k^2ix(-x^2 + 2y) - ix(k^2x^2 - 3k^2y - 8ky))z^3 + 8yz^2 + ixz. \end{aligned}$$

Proof. We have $L_n(x,y) = 2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])$ (see [12]), then

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_{k,n}^2 L_n(x,y) z^n \\
=\ & \sum_{n=0}^{\infty} S_{n-1}(b_1+[-b_2]S_{n-1}(c_1+[-c_2])[2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])]z^n \\
=\ & 2\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2])z^n \\
& -ix\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2])z^n \\
=\ & 2\frac{ixz + 4yz^2 - k^2ix(-x^2+2y)z^3 + 4yk^2x^2z^4 + k^2y^2ix(k^2+8k)z^5 - 4y^3k^4z^6}{D_2} \\
& -ix\frac{z - (-k^2x^2+3k^2y+8ky)z^3 - 8yk^2ixz^4 + (-k^4x^2y+3k^4y^2+8k^3y^2)z^5 - y^3k^6z^7}{D_2} \\
=\ & \frac{P^2L(x,y)}{D_2},
\end{aligned}$$

where

$$\begin{aligned}
P^2L(x,y) &= ixy^3k^6z^7 - 8y^3k^4z^6 + (2k^2y^2ix(k^2+8k) - ix(-k^4x^2y+3k^4y^2+8k^3y^2))z^5 \\
&\quad + (-2k^2ix(-x^2+2y) - ix(k^2x^2-3k^2y-8ky))z^3 + 8yz^2 + ixz.
\end{aligned}$$

This completes the proof. □

Corollary 4.8. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Pell numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} P_n^2 F(x,y) z^n = \frac{N_{P_n^2 L_n(x,y)}}{D_{P_n^2 F_n(x,y)}},$$

where

$$\begin{aligned}
N_{P_n^2 L_n(x,y)} &= ixy^3z^7 - 8y^3z^6 + (18y^2ix - ix(-x^2y+11y^2))z^5 \\
&\quad + (-2ix(-x^2+2y) - ix(x^2-11y))z^3 + 8yz^2 + ixz, \\
D_{P_n^2 L(x,y)} &= D_{P_n^2 F(x,y)}.
\end{aligned}$$

Third step

Thirdly, by replacing a_2 by $[-a_2]$ and b_2 by $[-b_2]$ and c_2 by $[-c_2]$ and by making the substitutions $c_1 - c_2 = k$, $b_1 - b_2 = k$, $a_1 - a_2 = ix$, $c_1c_2 = 2$, $b_1b_2 = 2$, and $a_1a_2 = y$ in (3.6) and (3.2), we obtain the following

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]) = \frac{J^2F(x,y)}{D_3}, \tag{4.5}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]) = \frac{N_3}{D_3}, \tag{4.6}$$

where

$$J^2F(x,y) = z - (4k^2y - 4x^2 + 12y)z^3 - 8yixk^2z^4 + (16k^2y^2 - 16x^2y + 48y^2)z^5 - 64y^3z^7,$$

$$N_3 = ixz + yk^2z^2 - 4ix(-x^2+2y)z^3 + 4yk^2x^2z^4 + 4y^2ix(4k^2+4)z^5 - 16y^3k^2z^6.$$

and

$$\begin{aligned} D_3 = {} & 1 - ixk^2z - (k^4y - 4k^2x^2 + 8k^2y - 8x^2 + 16y)z^2 - (4ik^4xy - 4ik^2x^3 + 20ik^2xy)z^3 \\ & + (4k^4x^2y + 8k^4y^2 + 64k^2y^2 + 16x^4 - 64x^2y + 96y^2)z^4 + (16ik^4xy^2 - 16ik^2x^3y \\ & + 80ik^2xy^2)z^5 - (16k^4y^3 - 64k^2x^2y^2 + 128k^2y^3 - 128x^2y^2 + 256y^3)z^6 \\ & + 64y^3ixk^2z^7 + 256y^4z^8. \end{aligned}$$

From our previous results, we obtain the following theorems and corollaries

Theorem 4.9. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Jacobsthal numbers with bivariate complex Fibonacci polynomials is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} J_{k,n}^2 F_n(x,y) z^n &= \frac{z + (4x^2 - 4k^2y - 12y)z^3 - 8ik^2xyz^4}{D_3} \\ &+ \frac{(-16x^2y + 16k^2y^2 + 48y^2)z^5 - 64y^3z^7}{D_3}, \end{aligned}$$

Corollary 4.10. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Jacobsthal numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} J_n^2 F_n(x,y) z^n = \frac{z + (4x^2 - 16y)z^3 - 8ixyz^4 + (-16x^2y + 64y^2)z^5 - 64y^3z^7}{D_{J_n^2 F_n(x,y)}},$$

where

$$\begin{aligned} D_{J_n^2F_n(x,y)} = {} & 1 - ixz - (-12x^2 + 25y)z^2 - (-4ix^3 + 24ixy)z^3 + (16x^4 - 60x^2y + 168y^2)z^4 \\ & + (-16ix^3y + 96ixy^2)z^5 - (-192x^2y^2 + 400y^3)z^6 + 64y^3ixz^7 + 256y^4z^8. \end{aligned}$$

Theorem 4.11. *For $n \in \mathbb{N}$, the new generating function of the product of squares of the product of k-Jacobsthal numbers and bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} J_{k,n}^2 L_n(x,y) z^n = \frac{J^2L(x,y)}{D_3},$$

where

$$\begin{aligned} J^2L(x,y) = {} & 64ixy^3z^7 - 32y^3k^2z^6 + (8y^2ix(4k^2+4) - ix(16k^2y^2 - 16x^2y + 48y^2))z^5 \\ & + (-8ix(-x^2+2y) - ix(-4k^2y + 4x^2 - 12y))z^3 + 2yk^2z^2 + ixz. \end{aligned}$$

Proof. we will consistently use the expression that we have $L_n(x,y) = 2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])$ (see [12]), then

$$\begin{aligned} & \sum_{n=0}^{\infty} J_{k,n}^2 L_n(x,y) z^n \\ = {} & \sum_{n=0}^{\infty} S_{n-1}(b_1 + [-b_2] S_{n-1}(c_1 + [-c_2])[2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])] z^n \\ = {} & 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(c_1 + [-c_2]) z^n \\ & -ix\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(b_1 + [-b_2]) S_{n-1}(c_1 + [-c_2]) z^n \\ = {} & 2\frac{ixz + yk^2z^2 - 4ix(-x^2+2y)z^3 + 4yk^2x^2z^4 + 4y^2ix(4k^2+4)z^5 - 16y^3k^2z^6}{D_3} \\ & -ix\frac{z - (4k^2y - 4x^2 + 12y)z^3 - 8yixk^2z^4 + (16k^2y^2 - 16x^2y + 48y^2)z^5 - 64y^3z^7}{D_3} \end{aligned}$$

hence,

$$\sum_{n=0}^{\infty} J_{k,n}^2 L_n(x,y)z^n = \frac{J^2L(x,y)}{D_3},$$

where

$$J^2L(x,y) = 64ixy^3z^7 - 32y^3k^2z^6 + (8y^2ix(4k^2+4) - ix(16k^2y^2 - 16x^2y + 48y^2))z^5 \\ + (-8ix(-x^2+2y) - ix(-4k^2y + 4x^2 - 12y))z^3 + 2yk^2z^2 + ixz.$$

This completes the proof. □

Corollary 4.12. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Jacobsthal numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} J_n^2 L_n(x,y)z^n = \frac{N_{J_n^2 L(x,y)}}{D_{J_n^2 L(x,y)}},$$

where

$$N_{J_n^2 L_n(x,y)} = 64ixy^3z^7 - 32y^3z^6 + (64y^2ix - ix(-16x^2y + 64y^2))z^5 \\ + (-8ix(-x^2+2y) - ix(4x^2 - 16y))z^3 + 2yz^2 + ixz.$$

$$D_{J_n^2 F_n(x,y)} = 1 - ixz + (12x^2 - 25y)z^2 + (4ix^3 - 24ixy)z^3 + (16x^4 - 60x^2y + 168y^2)z^4 \\ - (16ix^3y - 96ixy^2)z^5 + (192x^2y^2 - 400y^3)z^6 + 64y^3ixz^7 + 256y^4z^8.$$

$$D_{J_n^2 L(x,y)} = D_{J_n^2 F(x,y)}.$$

Fourth step

In order to derive new generating functions of the product of squares of k-Balancing numbers and complex bivariate Fibonacci polynomials, as well as the product of squares of k-Balancing numbers with complex bivariate Lucas polynomials, we make the following substitutions a_2 by $[-a_2]$, b_2 by $[-b_2]$, c_2 by $[-c_2]$ and we set $c_1 - c_2 = 6k$, $b_1 - b_2 = 6k$, $a_1 - a_2 = ix$, $c_1c_2 = -1$, $b_1b_2 = -1$, and $a_1a_2 = y$ in (3.6) and (3.2), we then obtain

$$\sum_{n=0}^{\infty} S_{n-1}(c_1 + [-c_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(a_1 + [-a_2]) = \frac{B^2F(x,y)}{D_4}, \tag{4.7}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{N_4}{D_4}, \tag{4.8}$$

with

$$B^2F(x,y) = z - (-72k^2y - x^2 + 3y)z^3 - 72yixk^2z^4 + (-72k^2y^2 - x^2y + 3y^2)z^5 - y^3z^7,$$

$$N_4 = ixz + 36yk^2z^2 - ix(-x^2 + 2y)z^3 + 36yk^2x^2z^4 + y^2ix(-72k^2 + 1)z^5 - 36y^3k^2z^6,$$

and

$$D_4 = 1 - 36ixk^2z - (1296k^4y + 72k^2x^2 - 144k^2y - 2x^2 + 4y)z^2 \\ - (-2592ik^4xy - 36ik^2x^3 + 180ik^2xy)z^3 \\ + (1296k^4x^2y + 2592k^4y^2 - 288k^2y^2 + x^4 - 4x^2y + 6y^2)z^4 \\ + (-2592ik^4xy^2 - 36ik^2x^3y + 180ik^2xy^2)z^5 \\ - (1296k^4y^3 + 72k^2x^2y^2 - 144k^2y^3 - 2x^2y^2 + 4y^3)z^6 + 36y^3ixk^2z^7 + y^4z^8.$$

From previous results, we conclude the following theorems and corollaries

Theorem 4.13. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Balancing numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} B_{k,n}^2 F_n(x,y) z^n = \frac{z - (-72k^2y - x^2 + 3y)z^3 - 72yixk^2z^4}{D_4} + \frac{(-72k^2y^2 - x^2y + 3y^2)z^5 - y^3z^7}{D_4}.$$

Corollary 4.14. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Balancing numbers with bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} B_n^2 F_n(x,y) z^n = \frac{z - (-x^2 - 69y)z^3 - 72yixz^4 + (-x^2y - 69y^2)z^5 - y^3z^7}{D_{B_n^2 F(x,y)}},$$

where

$$\begin{aligned} D_{B_n^2 F(x,y)} &= 1 - 36ixz - (70x^2 + 1156y)z^2 - (-36ix^3 - 2412ixy)z^3 \\ &+ (x^4 + 1292x^2y + 2310y^2)z^4 + (-36ix^3y - 2412ixy^2)z^5 \\ &- (70x^2y^2 + 1156y^3)z^6 + 36y^3ixz^7 + y^4z^8. \end{aligned}$$

Theorem 4.15. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Balancing numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} B_{k,n}^2 L_n(x,y) z^n = \frac{B^2L(x,y)}{D_4},$$

with

$$\begin{aligned} B^2L(x,y) &= ixy^3z^7 - 72y^3k^2z^6 + (2y^2ix(-72k^2+1) - ix(-72k^2y^2 - x^2y + 3y^2))z^5 \\ &+ (-2ix(-x^2+2y) - ix(72k^2y + x^2 - 3y))z^3 + 72yk^2z^2 + ixz. \end{aligned}$$

Proof. We have

$$L_n(x,y) = 2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2]), \text{ (see[12])},$$

then

$$\begin{aligned} &\sum_{n=0}^{\infty} B_{k,n}^2 L_n(x,y) z^n \\ = &\sum_{n=0}^{\infty} (2S_n(a_1 + [-a_2]) - ixS_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]))z^n \\ = &\ 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2])z^n \\ &-ix\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2])z^n \\ = &\ 2\frac{ixz + 36yk^2z^2 - ix(-x^2+2y)z^3 + 36yk^2x^2z^4 + y^2ix(-72k^2+1)z^5 - 36y^3k^2z^6}{D_4} \\ &-ix\frac{z - (-72k^2y - x^2 + 3y)z^3 - 72yixk^2z^4 + (-72k^2y^2 - x^2y + 3y^2)z^5 - y^3z^7}{D_4}, \end{aligned}$$

so,

$$\sum_{n=0}^{\infty} B_{k,n}^2 L_n(x,y) z^n = \frac{B^2L(x,y)}{D_4},$$

where

$$B^2L(x,y) = ixy^3z^7 - 72y^3k^2z^6 + (2y^2ix(-72k^2+1) - ix(-72k^2y^2 - x^2y + 3y^2))z^5$$
$$+ (-2ix(-x^2+2y) - ix(72k^2y + x^2 - 3y))z^3 + 72yk^2z^2 + ixz.$$

This completes the proof. □

Corollary 4.16. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Balancing numbers with bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} B_n^2 F_n(x,y) z^n = \frac{N_{B_n^2 L(x,y)}}{D_{B_n^2 L(x,y)}},$$

where

$$N_{B_n^2 L(x,y)} = ixy^3z^7 - 72y^3z^6 + (-142y^2ix - ix(-x^2y - 69y^2))z^5$$
$$+ (-2ix(-x^2+2y) - ix(x^2+69y))z^3 + 72yz^2 + ixz,$$
$$D_{B_n^2 L(x,y)} = D_{B_n^2 F(x,y)}.$$

Fifth Step

Now, we will make the following substitutions in order to derive new generating functions for the product of squares of k-Mersenne numbers with complex bivariate Fibonacci polynomials, and for the product of squares of k-Mersenne numbers with complex bivariate Lucas polynomials, by replacing a_2 by $[-a_2]$, b_2 by $[-b_2]$, c_2 by $[-c_2]$ and setting $c_1 - c_2 = 3k$, $b_1 - b_2 = 3k$, $a_1 - a_2 = ix$, $c_1c_2 = -2$, $b_1b_2 = -2$, and $a_1a_2 = y$ in (3.2) and (3.6), we then obtain the following results

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{M^2F(x,y)}{D_5}, \tag{4.9}$$

and

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{n-1}(b_1 + [-b_2])S_{n-1}(c_1 + [-c_2]) = \frac{N_5}{D_5}, \tag{4.10}$$

where

$$M^2F(x,y) = z - (-36k^2y - 4x^2 + 12y)z^3 - 72yixk^2z^4 + (-144k^2y^2 - 16x^2y + 48y^2)z^5 - 64y^3z^7,$$
$$N_5 = ixz + 9k^2yz^2 - 4ix(-x^2+2y)z^3 + 36yk^2x^2z^4 + 4y^2ix(-36k^2+4)z^5 - 144y^3k^2z^6,$$

and

$$D_5 = 1 - 9ik^2xz + (-36k^2x^2 + 8x^2 - 81k^4y + 72k^2y - 16y)z^2$$
$$+ (36ik^2x^3 + 324ik^4xy - 180ik^2xy)z^3$$
$$+ (16x^4 + 324k^4x^2y - 64x^2y + 648k^4y^2 - 576k^2y^2 + 96y^2)z^4$$
$$+ (-144ik^2x^3y - 1296ik^4xy^2 + 720ik^2xy^2)z^5$$
$$+ (-576k^2x^2y^2 + 128x^2y^2 - 1296k^4y^3 + 1152k^2y^3 - 256y^3)z^6$$
$$+ 576ik^2xy^3z^7 + 256y^4z^8.$$

From the above results, we obtain the following theorems and corollaries

Theorem 4.17. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Mersenne numbers and complex bivariate Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} M_{k,n}^2 F_n(x,y) z^n = \frac{z + (36k^2y + 4x^2 - 12y)z^3 - 72yixk^2z^4}{D_5} - \frac{(144k^2y^2 + 16x^2y - 48y^2)z^5 - 64y^3z^7}{D_5}.$$

Corollary 4.18. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Mersenne numbers and complex bivariate Fibonacci polynomials is given by*

$$\sum_{n=0}^{\infty} M_n^2 F_n(x,y) z^n = \frac{z-(-4x^2-24y)z^3-72yixz^4+(-16x^2y-96y^2)z^5-64y^3z^7}{D_{M_n^2 F(x,y)}},$$

and we have

$$D_{M_n^2 F_n(x,y)} = 1-9ixz+(-28x^2-25y)z^2+(36ix^3+144ixy)z^3+(16x^4+260x^2y+168y^2)z^4 + (-144ix^3y-576ixy^2)z^5+(-448x^2y^2-400y^3)z^6+(576ixy^3)z^7+256y^4z^8.$$

Theorem 4.19. *For $n \in \mathbb{N}$, the new generating function of the product of squares of k-Mersenne numbers and complex bivariate Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} M_n^2 L_n(x,y) z^n = \frac{M^2L(x,y)}{D_5},$$

where

$$M^2L(x,y) = 64ixy^3z^7-288y^3k^2z^6+(8y^2ix(-36k^2+4)-ix(-144k^2y^2-16x^2y+48y^2))z^5 + (-8ix(-x^2+2y)-ix(36k^2y+4x^2-12y))z^3+18k^2yz^2+ixz.$$

Proof. We have

$$L_n(x,y) = 2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2]), \text{ (see[12])},$$

then

$$\begin{aligned}
&\sum_{n=0}^{\infty} M_{k,n}^2 L_n(x,y) z^n \\
=\ &\sum_{n=0}^{\infty} (2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2]))z^n \\
=\ &2\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2])z^n \\
&-ix\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(b_1+[-b_2])S_{n-1}(c_1+[-c_2])z^n \\
=\ &2\frac{ixz+9k^2yz^2-4ix(-x^2+2y)z^3+36yk^2x^2z^4+4y^2ix(-36k^2+4)z^5-144y^3k^2z^6}{D_5} \\
&-ix(\frac{z-(-36k^2y-4x^2+12y)z^3-72yixk^2z^4+(-144k^2y^2-16x^2y+48y^2)z^5-64y^3z^7}{D_5},
\end{aligned}$$

hence,

$$\sum_{n=0}^{\infty} M_{k,n}^2 L_n(x,y) z^n = \frac{M^2L(x,y)}{D_5},$$

where

$$M^2L(x,y) = 64ixy^3z^7-288y^3k^2z^6+(8y^2ix(-36k^2+4)-ix(-144k^2y^2-16x^2y+48y^2))z^5 + (-8ix(-x^2+2y)-ix(36k^2y+4x^2-12y))z^3+18k^2yz^2+ixz.$$

This completes the proof. □

Corollary 4.20. *For $n \in \mathbb{N}$, the new generating function of the product of squares of Mersenne numbers and complex bivariate Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} M_n^2 L(x,y) z^n = \frac{N_{6M_n^2 F_n(x,y)}}{D_{M_n^2 L(x,y)}},$$

where

$$N_{6M_n^2 L_n(x,y)} = 64ixy^3z^7 - 288y^3z^6 + (-256y^2ix - ix(-16x^2y - 96y^2))z^5 + (-8ix(-x^2 + 2y) - ix(4x^2 + 24y))z^3 + 18yz^2 + ixz,$$

and

$$D_{M_n^2 L(x,y)} = D_{M_n^2 F(x,y)}.$$

5 Conclusion

In this paper, the use of symmetric functions, specifically Eqs (3.1), (3.2), (3.3), (3.4), and (3.5), allows us to derive new theorems and corollaries for obtaining new generating functions of the products of bivariate complex Fibonacci and Lucas polynomials with k-Fibonacci numbers, k-Pell numbers, k-Jacobsthal numbers, k-Balancing numbers, and k-Mersenne numbers.

References

[1] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle, Adv. Math, 103(2), 180–195, (1994).

[2] K. Adegoke, R. Frontczak, T. Goy, Some Special Sums With Squared Horadam Numbers And Generalized Tri-bonacci Numbers, Palestine Journal of Mathematics, 11(1), 66–73, (2022).

[3] K. Adegoke, R. Frontczak, T. Goy, New Binomial Fibonacci Sums, Palestine Journal of Mathematics, 13(1), 323–339, (2024).

[4] B. Aloui, A. Boussayoud, Generating functions of the product of the k-Fibonacci and k-Pell numbers and Chebyshev polynomials of the third and fourth kind, Math.Eng.Sci.Aerosp. MESA. 12(1), 245–257, (2021).

[5] M. Asci, E. Gurel, On bivariate complex Fibonacci and Lucas polynomials, Conference on Mathematical Sciences ICM 2012, March 11–14, (2012).

[6] M. Asci, E. Gurel, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers, Notes on Number Theory and Discrete Mathematics, Vol. 19(1), 25–36, (2013).

[7] A. Behera, G.K. Panda, On the Square Roots of Triangular Numbers, Fibonacci Quart. 37(2), 98–105, (1999).

[8] A. Berczes, K. Liptai, I. Pink, On Generalized Balancing Sequences, Fibonacci Quart. 48(2), 121–128, (2010).

[9] Kh. Boubellouta, Fonctions symétriques et leurs applications à certains nombres et polynômes, (Doctoral dissertation). Mohamed Seddik Ben Yahia University , Jijel, Algeria. (2020).

[10] Kh. Boubellouta, A. Boussayoud, M. Kerada, Symmetric functions for second-order recurrence sequences, (2020).

[11] Kh. Boubellouta, A. Boussayoud, S. Araci, M. Kerada, Some Theorems On Generating Functions and Their Applications, Advanced studies in contemporary mathematics. 30(3), 307–324, (2020).

[12] S. Boughaba, A. Boussayoud, Kh. Boubellouta, Generating functions of modified, Pell numbers and bivariate complex Fibonacci polynomials, Turkish J. Anal. Number 7, 113—116, (2019).

[13] S. Boughaba , A. Boussayoud, On Some Identities And Generating Function of Both k-jacobsthal Numbers and Symmetric Functions in Several Variables, Konuralp J. Math. 7(2), 235–242, (2019).

[14] A. Boussayoud, M. Kerada, A. Abderrezzak, A Generalization of Some Orthogonal Polynomials, Springer Proc. Math. Stat. 41, 229-235, (2013).

[15] A. Boussayoud, M. Kerada, R. Sahali, Symmetrizing Operations on Some Orthogonal Polynomials, Int. Electron. J. Pure Appl. Math. 9, 191–199, (2015).

[16] A. Boussayoud, A. Abderrezzak, M.Kerada, Some applications of symmetric functions. Integers, 15, (A48), 1–7, (2015).

[17] A. Boussayoud, S. Boughaba, On some identities and generating functions for k-Pell sequences and Chebyshev polynomials, Online J. Anal. Comb, 14(3), 1–13, (2019).

[18] A. Boussayoud, S. Boughaba, M. Kerada, S. Araci, M.Acikgoz, Generating functions of binary products of k-Fibonacci and orthogonal polynomials, Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat., RACSAM., 113(3), 2575–2586, (2019).

[19] P. Catarino, On some identities and generating functions for k-Pell numbers, Int.J. Math. Anal, 7(37-40), 1877–1884, (2013).

[20] P. Catarino, On generating Matrices of the k-Pell, k-Pell-Lucas and modified k-Pell sequences, Pure Math. Sci., 3(2), 71–77, (2014).

[21] B.G.S. Doman, J.K. Williams, Fibonacci and Lucas polynomials, Math. Proc.Cambridge Philos. Soc, 90(3), 385–387, (1981).

[22] M.R. Eslahchi, M. Dehghan, S. Amani, The third and fourth kinds of Chebyshev polynomials and best uniform approximation, Math. Comput. Modelling, 55(5-6), 1746–1762, (2012).

[23] S. Falcon, A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, Chaos Solitons Fractals., 33(1), 38–49, (2007).

[24] S. Falcon, On the k-Lucas numbers of arithmetic indexes, Appl. Math, 3(10), 1202–1206, (2012).

[25] R.P. Finkelstein, The house problem. Amer. Math. Monthly, 72(10), 1082-–1088, (1965).

[26] A.F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, Duke Math. J. 32, 437-6446, (1965).

[27] A. F.Horadam, J.M. Mahon, Pell and Pell-Lucas polynomials, Fibonacci Quart, 23(1), 7–20, (1985).

[28] A.F. Horadam, P. Filipponi, Derivative sequences of Jacobsthal and Jacobsthal-Lucas polynomials, Fibonacci Quart, 35(4), 352–357, (1997).

[29] D. Jhala, K. Sisodiya, G.P.S. Rathore, On some identities for k-Jacobsthal numbers, Int. J. Math. Anal, 7(9-12), 551–556, (2013).

[30] D. Jhala, Some properties of the k-Jacobsthal Lucas sequence, International Journal of Modern Sciences and Engineering Technology, 1(3), 87–92, (2014).

[31] I.G. Macdonald, Symmetric functions, Hall polynomias, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (1979).

[32] H. Merzouk, A. Boussayoud, M. Chelgham, Symmetric functions of generalized polynomials of second order, Turkish Journal of Analysis and Number Theory, 7(5), 135–139, (2019).

[33] P.K. Ray, On the properties of k-balancing numbers, Ain Shams Engineering Journal, 9, 395-402, (2018).

[34] N. Saba, A. Boussayoud, M. Ferkioui, S. Boughaba, Symmetric functions1 of binary products of Gaussian Jacobsthal Lucas polynomials and Chebyshev polynomials, Palestine Journal of Mathematics, 10(2), 452–464, (2021).

[35] N. Saba, A. Boussayoud, A new class of ordinary generating functions of binary products of Mersenne Lucas numbers with several numbers, Palestine Journal of Mathematics, 12(2), 450–463, (2023).

[36] M. Shah, D. Shah, F-Trinomial numbers, Palestine Journal of Mathematics, 13(2), 359–368, (2024).

Author information

Amina Daoui, Department of Mathematics,University Ferhat Abbas Setif1.
Laboratory of Mathematics and their Interactions, Abd el Hafidh Boussof University center of Mila., Algeria.
E-mail: a.daoui@centre-univ-mila.dz

Ali Boussayoud, LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel., Algeria.
E-mail: aboussayoud@yahoo.fr

Khadidja Boubellouta, LMAM Laboratory and Department of Basic Education, Mohamed Seddik Ben Yahia University, Jijel., Algeria.
E-mail: khadidjaboubellouta@gmail.com

Received: 2024-02-09
Accepted: 2024-07-08