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GENERAL DEGREE SUBTRACTION SPECTRA AND ENERGY

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Abstract The study of spectral graph theory is concerned with the relationships between the spectra of certain matrices associated with a graph and the structural properties of that graph. The energy of a graph is related to the total π-electron energy in molecule represented by a molecular graph. In this paper, we introduce a matrix called as a general degree subtraction of a graph and compute its characteristic polynomial and spectra in terms of the first general Zagreb index. We explore its bounds for spectral radius. Further we observe the effects on the general degree subtraction energy when some operations are applied. We also give an algorithm with time complexity $O(n)$ to find the general degree subtraction energy of a graph.

1 Introduction

The characteristics polynomial, spectrum, eigenvalues, and energy of a graph frequently appear in mathematical sciences, chemistry, physics, etc. One of the application of the characteristics polynomial is to get the information about the structural properties of a graph [5, 8, 9]. Eigenvalues are used in H$\ddot{u}$ckel molecular orbital [7, 8, 11, 18]. Many researches introduced many different matrices like Laplacian Matrix[1, 10], Distance matrix [3], degree product matrix[22], Degree Exponent Subtraction matrix[25], degree subtraction matrix[21], degree square subtraction matrix[17] and many more and also studied their eigenvalues and energy. Inspired by this we introduced a general degree subtraction matrix of a simple connected graph G.

Let G be a simple connected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \cdots, v_j, \cdots, v_n\}$ be a vertex set, $N(v)$ be the first neighbor vertex set of v, $d_j = deg_G(v_j)$ be the degree of a vertex v_j of G.

The general degree subtraction of a graph G of order n is an $n \times n$ matrix which is denoted by ${}^{\alpha}DS(G)$ and defined as ${}^{\alpha}DS(G) = [ds_{jk}]$ where

$$ds_{jk} = \begin{cases} {d_j}^{\alpha} - {d_k}^{\alpha} & if \;\; v_j \neq v_k \\ 0 & \text{otherwise,} \end{cases}$$

where α is a nonzero real number.

Let I_n be an identity matrix of order n. Then ${}^{\alpha}DS(G)$-eigenvalues of G are the roots of the characteristic polynomial $\phi(G, \zeta) = 0$, and they are labeled as $\zeta_1, \zeta_2, \ldots, \zeta_n$. Since ${}^{\alpha}DS(G)$ is a skew symmetric matrix, its eigenvalues are either purely imaginary or zero. The collection of ${}^{\alpha}DS(G)$-eigenvalues is called as ${}^{\alpha}DS(G)$-spectra. The general degree subtraction energy of G is denoted by ${}^{\alpha}DSE$ and defined as

$${}^{\alpha}DSE(G) = \sum_{j=1}^{n} |\zeta_j|. \tag{1.1}$$

Consider δ a minimum degree of G, Δ a maximum degree of G and $\pi(G) = (d_1, d_2, \ldots, d_n)$ is the graphic sequence of G. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ are distinct eigenvalues of adjacency matrix of G, and $m_1, m_2, \cdots, m_k$, where $k \leq n$ are multiplicity of the adjacency eigenvalues of G respectively. Then the spectra of G [4] is defined as

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}.$$

Li and Zheng [27] introduced the general Zagreb index as $M_1^\alpha(G) = \sum_{j=1}^{n} d_j^\alpha$, where $\alpha \in \mathbb{R} - \{0, 1\}$ and general Randić index as $M_2^\alpha(G) = \sum_{uv \in E} (d_u d_v)^\alpha$, where α is a nonzero real number and E is the edge set of G.

In [28], Li and Zhao characterized all trees with the first three smallest and largest values of the first general Zagreb index when α is an integer or a fraction $\frac{1}{p}$ for a nonzero integer p. In [30] H. Zhang and S. Zhang identified all the unicyclic graphs with the first three smallest and largest values of the first general Zagreb index.

In this paper, we obtain the characteristic polynomial and spectra of general degree subtraction of a graph G in terms of the first general Zagreb index. We explore its bounds for spectral radius. Further we observe the effects on the general degree subtraction energy when some operations are applied. We also give an algorithm to find the general degree subtraction energy of a graph with time complexity $O(n)$.

2 Spectra of ${}^\alpha DS(G)$

Theorem 2.1. *Let G be a graph of order n, size m and $M_1^\alpha(G)$ is first general Zagreb index. Then the ${}^\alpha DS(G)$-spectra is*

$$Spec(G) = \begin{pmatrix} 0 & i\sqrt{n\sum_{j=1}^{n} d_j^{2\alpha} - (M_1^\alpha(G))^\alpha} & -i\sqrt{n\sum_{j=1}^{n} d_j^{2\alpha} - (M_1^\alpha(G))^2} \\ n-2 & 1 & 1 \end{pmatrix},$$

where $i = \sqrt{-1}$.

Proof. The characteristic polynomial of ${}^\alpha DS(G)$ is

$$\phi(G:\zeta) = det(\zeta I - {}^\alpha DS(G))$$

$$= \begin{vmatrix} \zeta & -d_1^\alpha + d_2^\alpha & -d_1^\alpha + d_3^\alpha & \cdots & -d_1^\alpha + d_n^\alpha \\ -d_2^\alpha + d_1^\alpha & \zeta & -d_2^\alpha + d_3^\alpha & \cdots & -d_2^\alpha + d_n^\alpha \\ -d_3^\alpha + d_1^\alpha & -d_3^\alpha + d_2^\alpha & \zeta & \cdots & -d_3^\alpha + d_n^\alpha \\ \vdots & & & \vdots & \\ -d_n^\alpha + d_1^\alpha & -d_n^\alpha + d_2^\alpha & -d_n^\alpha + d_3^\alpha & \cdots & \zeta \end{vmatrix}. \quad (2.1)$$

To obtain (2.2) from (2.1) subtract first row from all the succeeding rows

$$= \begin{vmatrix} \zeta & -d_1^\alpha + d_2^\alpha & -d_1^\alpha + d_3^\alpha & \cdots & -d_1^\alpha + d_n^\alpha \\ -d_2^\alpha + d_1^\alpha - \zeta & \zeta + d_1^\alpha - d_2^\alpha & -d_2^\alpha + d_1^\alpha & \cdots & -d_2^\alpha + d_1^\alpha \\ -d_3^\alpha + d_1^\alpha - \zeta & -d_3^\alpha + d_1^\alpha & \zeta + d_1^\alpha - d_3^\alpha & \cdots & -d_3^\alpha + d_1^\alpha \\ \vdots & & & \vdots & \\ -d_n^\alpha + d_1^\alpha - \zeta & -d_n^\alpha + d_1^\alpha & -d_n^\alpha + d_1^\alpha & \cdots & \zeta + d_1^\alpha - d_n^\alpha \end{vmatrix}. \quad (2.2)$$

To obtain (2.3) from (2.2) subtract first column from all the succeeding columns

$$= \begin{vmatrix} \zeta & -d_1^\alpha + d_2^\alpha - \zeta & -d_1^\alpha + d_3^\alpha - \zeta & \cdots & -d_1^\alpha + d_n^\alpha - \zeta \\ -d_2^\alpha + d_1^\alpha - \zeta & 2\zeta & \zeta & \cdots & \zeta \\ -d_3^\alpha + d_1^\alpha - \zeta & \zeta & 2\zeta & \cdots & \zeta \\ \vdots & & & \vdots & \\ -d_n^\alpha + d_1^\alpha - \zeta & \zeta & \zeta & \cdots & 2\zeta \end{vmatrix}. \quad (2.3)$$

To obtain (2.4) from (2.3) subtract second column from all the succeeding columns

$$= \begin{vmatrix} \zeta & -d_1^\alpha + d_2^\alpha - \zeta & -d_2^\alpha + d_3^\alpha & \cdots & -d_2^\alpha + d_n^\alpha - \zeta \\ -d_2^\alpha + d_1^\alpha - \zeta & 2\zeta & -\zeta & \cdots & -\zeta \\ -d_3^\alpha + d_1^\alpha - \zeta & \zeta & \zeta & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n^\alpha + d_1^\alpha - \zeta & \zeta & 0 & \cdots & \zeta \end{vmatrix}. \tag{2.4}$$

To obtain (2.5) from (2.4) add rows $3, 4, \cdots, n$ to the second row

$$= \begin{vmatrix} \zeta & -d_1^\alpha + d_2^\alpha - \zeta & -d_2^\alpha + d_3^\alpha & \cdots & -d_2^\alpha + d_n^\alpha - \zeta \\ -M_1^\alpha(G) + nd_1^\alpha - (n-1)\zeta & n\zeta & 0 & \cdots & 0 \\ -d_3^\alpha + d_1^\alpha - \zeta & \zeta & \zeta & \cdots & 0 \\ \vdots & & & \vdots & \\ -d_n^\alpha + d_1^\alpha - \zeta & \zeta & 0 & \cdots & \zeta \end{vmatrix}_{n\times n}. \tag{2.5}$$

$$\text{Let } M = \begin{bmatrix} \zeta & -d_1^\alpha + d_2^\alpha - \zeta \\ -M_1^\alpha(G) + nd_1^\alpha - (n-1)\zeta & n\zeta \end{bmatrix}_{2\times 2}$$

$$N = \begin{bmatrix} -d_2^\alpha + d_3^\alpha & \cdots & -d_2^\alpha + d_n^\alpha - \zeta \\ 0 & \cdots & 0 \end{bmatrix}_{2\times (n-2)}$$

$$P = \begin{bmatrix} -d_3^\alpha + d_1^\alpha - \zeta & \zeta \\ \vdots & \vdots \\ -d_n^\alpha + d_1^\alpha - \zeta & \zeta \end{bmatrix}_{(n-2)\times 2}$$

$$Q = \begin{bmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & \zeta \end{bmatrix}_{(n-2)\times (n-2)}.$$

$$\begin{aligned} \phi(G:\zeta) &= \zeta^{n-2} \left| M - N \tfrac{1}{\zeta} I_{(n-2)} P \right|_{2\times 2} \\ &= \zeta^{n-2} \left| \begin{bmatrix} \zeta & -d_1^\alpha + d_2^\alpha - \zeta \\ -M_1^\alpha(G) + nd_1^\alpha - (n-1)\zeta & n\zeta \end{bmatrix} - \tfrac{1}{\zeta} \begin{bmatrix} K & L \\ 0 & 0 \end{bmatrix} \right|, \end{aligned}$$

where

$$K = -\sum_{j=1}^{n} d_j^{2\alpha} + M_1^\alpha(G)(d_1^\alpha + d_2^\alpha) - n(d_1 d_2)^\alpha - M_1^\alpha(G)\zeta + \zeta(d_1^\alpha + d_2^\alpha) + (n-2)d_2^\alpha \zeta,$$

$$L = (M_1^\alpha(G) - d_1^\alpha - d_2^\alpha)\zeta - (n-2)d_2^\alpha \zeta.$$

Therefore,

$$\phi(G:\zeta) \quad = \zeta^{n-2} \left(\zeta^2 + n \textstyle\sum_{j=1}^n d_j^{2\alpha} - \left(\sum_{j=1}^n d_j^\alpha \right)^2 \right). \tag{2.6}$$

From (2.6), the ${}^\alpha DS(G)$-eigenvalues are 0 with multiplicity $(n-2)$ times and

$$\pm i \sqrt{ n \sum_{j=1}^{n} d_j^{2\alpha} - \left(\sum_{j=1}^{n} d_j^\alpha \right)^2 }. \tag{2.7}$$

If α is an arbitrary real number, not 0 and 1, then

$$\phi(G:\zeta) \quad = \zeta^{n-2}\left(\zeta^2 + n\sum_{j=1}^n d_j^{2\alpha} - (M_1^\alpha(G))^2\right). \tag{2.8}$$

From (2.8), the ${}^{\alpha}DS(G)$-eigenvalues are 0 with multiplicity $(n-2)$ and $\pm i\sqrt{n\sum_{j=1}^n d_j^{2\alpha} - (M_1^\alpha(G))^2}$. □

Corollary 2.2. *If G is a regular graph, then the characteristic polynomial of general degree subtraction of G is*

$$\phi(G:\zeta) = \zeta^n. \tag{2.9}$$

3 Bounds for spectral radius $|\zeta_1|$ of ${}^{\alpha}DS(G)$

Theorem 3.1. *Let G be a graph of order n, and $|\zeta_j|, j = 1,2,\ldots,n$ are the non ascending absolute ${}^{\alpha}DS(G)$-eigenvalues. Then they satisfy the following relations*

$$(i) \qquad \zeta_2 = -\zeta_1 \tag{3.1}$$

$$(ii) \qquad \zeta_1 = i\sqrt{Z} \text{ or } \zeta_2 = -i\sqrt{Z} \tag{3.2}$$

$$(iii) \qquad |\zeta_1| = |\zeta_2| = \sqrt{Z}, \tag{3.3}$$

where $Z = \sum_{1\le j<k\le n} ({d_j}^\alpha - {d_k}^\alpha)^2$ and $i = \sqrt{-1}$.

Proof. Since we have two nonzero ${}^{\alpha}DS(G)$-eigenvalues ζ_1 and ζ_2 remaining eigenvalues are zero and

$$\sum_{j=1}^n \zeta_j = trace({}^{\alpha}DS(G)) = 0.$$

Therefore,

$$\zeta_2 = -\zeta_1.$$

Since,

$$\sum_{j=1}^n \zeta_j^2 = trace({}^{\alpha}DS(G)^2) = -2\sum_{1\le j<k\le n} ({d_j}^\alpha - {d_k}^\alpha)^2 = -2Z.$$

Which implies

$$\zeta_1 = i\sqrt{Z} \text{ or } \zeta_2 = -i\sqrt{Z}.$$

Hence,

$$|\zeta_1| = |\zeta_2| = \sqrt{Z}.$$

□

Theorem 3.2. *Let G be a graph of order n with the maximum degree Δ and minimum degree δ and let $|\zeta_1|$ be the largest absolute ${}^{\alpha}DS(G)$-eigenvalue. Then*

$$|\zeta_1| \le |\Delta^\alpha - \delta^\alpha|\sqrt{\frac{n(n-1)}{2}}. \tag{3.4}$$

Equality holds if G is a regular graph or $\alpha = 0$.

Proof. Since,

$$Z = \sum_{1\le j<k\le n} \left({d_j}^{\alpha} - {d_k}^{\alpha}\right)^2 .$$

Then

$$Z \le \binom{n}{2} \left(\Delta^{\alpha} - \delta^{\alpha}\right)^2 . \tag{3.5}$$

From (3.3) and (3.5) we get the required result. □

Theorem 3.3. *Let G be a graph of order n with the maximum degree Δ and minimum degree δ. Let $|\zeta_j|$, $j = 1, 2, \cdots, n$ be the non ascending absolute ${}^{\alpha}DS(G)$-eigenvalues and let α be an arbitrary real number, not 0 and 1.*

(i) If $\alpha < 0$, then

$$0 \le |\zeta_1| \le n\delta^{\alpha} - M_1^{\alpha}(G).$$

(ii) If $\alpha > 0$, then

$$0 \le |\zeta_1| \le n\Delta^{\alpha} - M_1^{\alpha}(G).$$

Equality holds if G is a regular graph.

Proof. Let $v_1, v_2, \cdots, v_n$ are the vertices of G and d_j be the degree of v_j. Since the sum of the elements of j^{th} row in ${}^{\alpha}DS(G)$ is $nd_j^{\alpha} - M_1^{\alpha}(G)$. Therefore we get the following

$$min\{nd_j^{\alpha} - M_1^{\alpha}(G)\} \le |\zeta_j| \le max\{nd_j^{\alpha} - M_1^{\alpha}(G)\}.$$

Since ${}^{\alpha}DS(G)$-eigenvalues are either purely imaginary or zeros so we get the following equation.

(i) If $\alpha < 0$, then

$$0 \le |\zeta_j| \le n\delta^{\alpha} - M_1^{\alpha}(G). \tag{3.6}$$

Since $|\zeta_1|$ and $|\zeta_2|$ are nonzero ${}^{\alpha}DS(G)$-eigenvalues and remaining eigenvalues are zero, then from (3.6) we get the required result.

(ii) If $\alpha > 0$, then

$$0 \le |\zeta_j| \le n\Delta^{\alpha} - M_1^{\alpha}(G). \tag{3.7}$$

Since $|\zeta_1|$ and $|\zeta_2|$ are nonzero ${}^{\alpha}DS(G)$-eigenvalues and remaining eigenvalues are zero, then from (3.7) we get the required result.

□

Corollary 3.4. *Let G be a graph of order n with the maximum degree Δ and minimum degree δ and let $|\zeta_1|$ be the largest absolute ${}^{\alpha}DS(G)$-eigenvalue. Then*

$$|\zeta_1| \le n\left|\Delta^{\alpha} - \delta^{\alpha}\right| . \tag{3.8}$$

Equality holds if G is a regular graph.

Remark 3.5. Since $\sqrt{\frac{n(n-1)}{2}} < n$, the upper bound Theorem (3.2) is more sharper than the upper bound corollary (3.4).

Remark 3.6. Since $|\zeta_1| = |\zeta_2|$. Then $|\zeta_1|$ and $|\zeta_2|$ have the same upper bound.

4 Energy of ${}^{\alpha}DS(G)$ and its Bounds

Theorem 4.1. *Let G be a graph of order n and d_j the degree of v_j of G. Let α be an arbitrary real number, not 0 and 1. Then the general degree subtraction energy of G is*

$$ {}^{\alpha}DSE(G) = 2\sqrt{n\sum_{j=1}^{n} d_j^{2\alpha} - (M_1^{\alpha}(G))^2}. $$

Proof. From Theorem (2.1) we get the required result. □

Theorem 4.2. *Let G be a graph. Then*

$$ {}^{\alpha}DSE(G) = 2\sqrt{Z}. \tag{4.1} $$

Proof. From (1.1)

$$ {}^{\alpha}DSE(G) = \sum_{j=1}^{n} |\zeta_j| $$

$$ {}^{\alpha}DSE(G) = 2|\zeta_1| \tag{4.2} $$

$$ {}^{\alpha}DSE(G) = 2\sqrt{Z}. $$

□

Theorem 4.3. *Let G be a bi-regular graph with n vertices. Then*

$$ {}^{\alpha}DSE(G) = 2((d_1^{\alpha} - d_2^{\alpha})\sqrt{(n-k)k}. $$

Proof. Consider a graph G with k vertices having a degree of d_1, and $n-k$ vertices with a degree of d_2 and from (2.7) we get

$$ {}^{\alpha}DSE(G) = 2\sqrt{n\sum_{j=1}^{n} d_j^{2\alpha} - \left(\sum_{j=1}^{n} d_j^{\alpha}\right)^2}. \tag{4.3} $$

$$ \begin{aligned} {}^{\alpha}DSE(G) &= 2\sqrt{n[kd_1^{2\alpha} + (n-k)d_2^{2\alpha}] - [kd_1^{\alpha} + (n-k)d_2^{\alpha}]^2} \\ &= 2\sqrt{n[kd_1^{2\alpha} + (n-k)d_2^{2\alpha}] - [kd_1^{\alpha} + (n-k)d_2^{\alpha}]^2} \\ &= 2\sqrt{(n-k)kd_2^{2\alpha} + k(n-k)d_1^{2\alpha} - k(n-k)d_1^{\alpha}d_2^{\alpha}} \\ &= 2((d_1^{\alpha} - d_2^{\alpha})\sqrt{(n-k)k}. \end{aligned} $$

□

Corollary 4.4. *Let $K_{a,b}$ be a complete bipartite graph with $n = a + b$ vertices. Then from Theorem (4.3)*

$$ {}^{\alpha}DSE(K_{a,b}) = 2(a^{\alpha} - b^{\alpha})\sqrt{ab}. $$

Corollary 4.5. *Let P_n be a path with $n \geq 2$ vertices. Then from Theorem (4.3)*

$$ {}^{\alpha}DSE(P_n) = 2(2^{\alpha} - 1)\sqrt{2n-4}. $$

Corollary 4.6. *Let S_n be a star graph with $n \geq 1$ vertices. Then from Theorem (4.3)*

$$ {}^{\alpha}DSE(S_n) = 2\sqrt{n-1}[(n-1)^{\alpha} - 1]. $$

Corollary 4.7. *If P_n and S_n are path and star graph with $n > 1$ vertices respectively.*

(i) If $\alpha < 0$, then

$$^{\alpha}DSE(S_n) \leq {}^{\alpha}DSE(P_n).$$

(ii) If $\alpha > 0$, then

$$^{\alpha}DSE(P_n) \leq {}^{\alpha}DSE(S_n).$$

Equality holds if $n = 2, 3$ or $\alpha = 0$.

Proof. If P_n and S_n are path and star graph respectively then from corollaries (4.5) and (4.6), we get the required result. □

Theorem 4.8. *Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then*

$$^{\alpha}DSE(G) \quad \leq |\Delta^{\alpha} - \delta^{\alpha}| \sqrt{2n(n-1)}.$$

Equality holds if G is a regular graph or $\alpha = 0$.

Proof. From (4.2) and Theorem (3.2) we get the required result. □

Theorem 4.9. *Let G be a graph of order n with maximum degree Δ and minimum degree δ and ζ_j where $j = 1, 2, \cdots, n$ are $^{\alpha}DS(G)$-eigenvalues and let α be an arbitrary real number, not 0 and 1.*

(i) If $\alpha < 0$, then

$$^{\alpha}DSE(G) \leq 2\left(n\delta^{\alpha} - M_1^{\alpha}(G)\right).$$

(ii) If $\alpha > 0$, then

$$^{\alpha}DSE(G) \leq 2\left(n\Delta^{\alpha} - M_1^{\alpha}(G)\right).$$

Equality holds if G is a regular graph.

Proof. From (4.2) and Theorem (3.3) we get the required result. □

Theorem 4.10. *Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then*

$$^{\alpha}DSE(G) \quad \leq 2n\left|\Delta^{\alpha} - \delta^{\alpha}\right|.$$

Equality holds if G is a regular graph.

Proof. From the corollary (3.4) we get the required result. □

Remark 4.11. Since $\sqrt{2n(n-1)} < 2n$, the upper bound in theorem (4.8) is more sharper than the upper bound in theorem (4.10).

Theorem 4.12. *Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then*

$$^{\alpha}DSE(G) \quad \leq 2n\sqrt{|\Delta^{2\alpha} - \delta^{2\alpha}|}.$$

Equality holds if G is a regular graph or $\alpha = 0$.

Proof. From (4.3) we get the required result. □

Remark 4.13. Since $|\Delta^{\alpha} - \delta^{\alpha}| < \sqrt{|\Delta^{2\alpha} - \delta^{2\alpha}|}$, the upper bound in theorem (4.10) is more sharper than the upper bound in theorem (4.12).

Lemma 4.14. *Let G be a graph of order n and let α be an arbitrary real number, not 0 and 1. Then*

$$^{\alpha}DSE(G) \quad \leq 2M_1^{\alpha}(G)\sqrt{n-1}.$$

Equality holds for null graphs.

Proof. If d_j is a degree of the vertex v_j, where $j = 1, 2, \cdots, n$, then

$$\sum_{j=1}^{n} d_j^{2\alpha} \leq \left(\sum_{j=1}^{n} d_j^{\alpha}\right)^2 . \tag{4.4}$$

Thus from (4.3) we get

$$^{\alpha}DSE(G) \quad \leq 2\left(\sum_{j=1}^{n} d_j^{\alpha}\right)\sqrt{n-1}. \tag{4.5}$$

If α is an arbitrary real number, not 0 and 1, then from Theorems (4.1) and (4.5) we get the required result. □

Theorem 4.15. *Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then*

$$^{\alpha}DSE(G) \leq \begin{cases} 2n\delta^{\alpha}\sqrt{n-1} & if\ \alpha < 0 \\ 2n\Delta^{\alpha}\sqrt{n-1} & if\ \alpha > 0 \end{cases}$$

Equality holds for null graphs.

Proof. From (4.5) we get the required result. □

Lemma 4.16. *Cauchy-Schwarz inequality[29] states that if $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are n real vector, then*

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \quad \leq \quad \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

Lemma 4.17. *Let G be a graph of order n and size m_1 with real adjacency eigenvalues $\lambda_j,\ j = 1, 2, \cdots, n$. Let H be another graph of order n and size m_2 with vertex degree $d_j, j = 1, 2, \cdots, n$. Let $|\zeta_j|$, where $j = 1, 2, \cdots, n$ are the non increasing absolute $^{\alpha}DS(H)$-eigenvalues and $|\zeta_1|$ be the largest $^{\alpha}DS(H)$-eigenvalue.*

(i) Consider the adjacency eigenvalues of G such as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$^{\alpha}DSE(G) \quad \leq \quad \frac{4\sqrt{m_1 Z}}{\lambda_1 + \lambda_2}. \tag{4.6}$$

Equality holds if G is a regular graph or $\alpha = 0$ or $\lambda_1 = \lambda_2 = \sqrt{m_1}$.

(ii) Consider the absolute adjacency eigenvalues of G such as $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Then

$$^{\alpha}DSE(G) \quad \leq \quad \frac{4\sqrt{m_1 Z}}{|\lambda_1| + |\lambda_2|}. \tag{4.7}$$

Equality holds if G is a regular graph or $\alpha = 0$ or $|\lambda_1| = |\lambda_2| = \sqrt{m_1}$.

Proof. (i) If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are adjacency eigenvalues of G, then by Cauchy-Schwarz inequality[29], we get

$$\left(\sum_{j=1}^{n}(\lambda_j|\zeta_j|)\right)^2 \quad \leq \quad \left(\sum_{j=1}^{n} {\lambda_j}^2\right)\left(\sum_{j=1}^{n} |\zeta_j|^2\right). \tag{4.8}$$

Since $\sum_{j=1}^{n} |\zeta_j|^2 = 2Z$ and $\sum_{j=1}^{n} {\lambda_j}^2 = 2m_1$.
On substituting and simplifying (4.8), we get

$$|\zeta_1| \quad \leq \quad \frac{\sqrt{4m_1 Z}}{\lambda_1 + \lambda_2}. \tag{4.9}$$

From (4.9) and (4.2) we get the required result.

(ii) If $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ are absolute adjacency eigenvalues of G, then by Cauchy-Schwarz inequality[29], we get

$$\left(\sum_{j=1}^{n}(|\lambda_j||\zeta_j|)\right)^2 \leq \left(\sum_{j=1}^{n}\lambda_j{}^2\right)\left(\sum_{j=1}^{n}|\zeta_j|^2\right). \tag{4.10}$$

Since $\sum_{j=1}^{n}|\lambda_j|^2 = \sum_{j=1}^{n}\lambda_j{}^2 = 2m_1$.
On substituting and simplifying (4.10), we get

$$|\zeta_1| \leq \frac{\sqrt{4m_1 Z}}{|\lambda_1| + |\lambda_2|}. \tag{4.11}$$

From (4.11) and (4.2) we get the required result.

□

Theorem 4.18. *Let G be a graph of order n and size m with vertex degree $d_j, j = 1, 2, \cdots, n$. Let $|\zeta_1|$ be the nonzero largest absolute ${}^{\alpha}DS(G)$-eigenvalue. Let $H = K_a \cup \overline{K_{n-a}}$ be another graph.*

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq \begin{cases} \sqrt{\frac{8aZ}{a-1}} & 2 \leq a < n. \\ \frac{\sqrt{8n(n-1)Z}}{n-2} & a = n. \end{cases}$$

Equality holds if G is a regular graph or $\alpha = 0$.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq \sqrt{\frac{8(a-1)Z}{a}} \quad 2 \leq a \leq n.$$

Equality holds if G is a regular graph or $\alpha = 0$ or $a = 2$.

Proof. Let $|\zeta_1|, |\zeta_2|, \cdots, |\zeta_{n-a+1}|, |\zeta_{n-a+2}|, \cdots, |\zeta_n|$ be the non increasing absolute ${}^{\alpha}DS(G)$-eigenvalues of G. The adjacency eigenvalues of H are $a-1$, 0 $(n-a\ times)$, and -1 $(a-1\ times)$ and the size of H, $m_1 = \frac{a(a-1)}{2}$.

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then from (4.6), we get

$$|\zeta_1| \leq \begin{cases} \sqrt{\frac{2a}{a-1}Z} & 2 \leq a < n. \\ \frac{\sqrt{2n(n-1)Z}}{n-2} & a = n. \end{cases} \tag{4.12}$$

From (4.12) and (4.2) we get the required result.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then from (4.7), we get

$$|\zeta_1| \leq \sqrt{\frac{2(a-1)}{a}Z} \quad 2 \leq a \leq n. \tag{4.13}$$

From (4.13) and (4.2) we get the required result. □

Theorem 4.19. *Let G be a graph with n vertices, where the degrees of the vertices are denoted as d_j for $j = 1, 2, \cdots, n$ and let $|\zeta_1|$ be the nonzero largest absolute ${}^{\alpha}DS(G)$-eigenvalue. Let H be the union of k copies of complete graph K_a, $H = \cup_k K_a$ with $n = ka$ vertices.*

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq \begin{cases} \sqrt{\frac{2kaZ}{a-1}} & 1 < k \leq n \text{ and } 2 \leq a \leq n. \\ \frac{\sqrt{8a(a-1)Z}}{a-2} & k = 1 \text{ and } 2 < a \leq n. \end{cases}$$

Equality holds if G is a regular graph or $\alpha = 0$.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq \begin{cases} \sqrt{\frac{2kaZ}{a-1}} & 1 < k \leq n \text{ and } 2 < a \leq n. \\ \sqrt{\frac{8(a-1)Z}{a}} & k = 1 \text{ and } 2 \leq a \leq n. \end{cases}$$

Equality holds if G is a regular graph or $\alpha = 0$ or $a = 2$.

Proof. Let $|\zeta_1|, |\zeta_2|, \cdots, |\zeta_k|, |\zeta_{k+1}|, \cdots, |\zeta_n|$ be the non increasing absolute $^{\alpha}DS(G)$-eigenvalues of G. The H-adjacency eigenvalues are $a-1$ $(k\ times)$, -1 $(n-k\ times)$ and the order and size of H are $n = ak$ and $\frac{ka(a-1)}{2}$ respectively.

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then from (4.6), we get

$$|\zeta_1| \leq \begin{cases} \sqrt{\frac{kaZ}{2(a-1)}} & 1 < k \leq n \text{ and } 2 \leq a \leq n. \\ \frac{\sqrt{2a(a-1)Z}}{a-2} & k = 1 \text{ and } 2 < a \leq n. \end{cases} \tag{4.14}$$

From (4.14) and (4.2) we get the required result.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then from (4.7), we get

$$|\zeta_1| \leq \begin{cases} \sqrt{\frac{kaZ}{2(a-1)}} & 1 < k \leq n \text{ and } 2 < a \leq n. \\ \sqrt{\frac{2(a-1)Z}{a}} & k = 1 \text{ and } 2 \leq a \leq n. \end{cases} \tag{4.15}$$

From (4.15) and (4.2) we get the required result.

□

Theorem 4.20. *Let G be a graph of order n with m edges. Let $d_j, j = 1, 2, \cdots, n$ are degree vertices of G. Let $|\zeta_1|$ be the nonzero largest absolute $^{\alpha}DS(G)$-eigenvalue. Let H be the union of k copies of complete bipartite graph $K_{a,b}$, $H = \cup_k K_{a,b}$, where $n = k(a+b)$.*

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq \begin{cases} 2\sqrt{kZ} & k \geq 2. \\ 4\sqrt{Z} & k = 1. \end{cases}$$

Equality holds if G is a regular graph, $\alpha = 0$.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then

$$^{\alpha}DSE(G) \leq 2\sqrt{kZ} \quad k \geq 1.$$

Equality holds if G is a regular graph, $\alpha = 0$ or $k = 1$.

Proof. Let $|\zeta_1|, |\zeta_2|, \cdots, |\zeta_k|, |\zeta_{k+1}|, \cdots, |\zeta_n|$ be the non increasing absolute $^{\alpha}DS(G)$-eigenvalues of G. The adjacency eigenvalues of H are $\sqrt{ab}$ of multiplicity k, $zero$ of multiplicity $n-2k$ and $-\sqrt{ab}$ of multiplicity k and the order and size of H are $n = k(a+b)$ and kab respectively.

(i) If the adjacency eigenvalues of H are arranged in non increasing order, then from (4.6), we get

$$|\zeta_1| \leq \begin{cases} \sqrt{kZ} & k \geq 2. \\ \sqrt{4Z} & k = 1. \end{cases} \tag{4.16}$$

From (4.16) and (4.2) we get the required result.

(ii) If the absolute adjacency eigenvalues of H are arranged in non increasing order, then from (4.7), we get

$$|\zeta_1| \leq \sqrt{kZ} \quad k \geq 1. \tag{4.17}$$

From (4.17) and (4.2) we get the required result.

□

Definition 4.21. [13] Suppose $(r) = (r_1, r_2, \dots, r_n)$ and $(t) = (t_1, t_2, \dots, t_n)$ are two non-increasing sequences of real numbers. Then we say (r) is majorized by (t), denoted by $(r) \trianglelefteq (t)$, if and only if $\sum_{k=1}^{n} r_k = \sum_{k=1}^{n} t_k$ and $\sum_{k=1}^{j} r_k \leq \sum_{k=1}^{j} t_k$ for all $j = 1, 2, \dots, n$.

Lemma 4.22. *[13] Suppose $(r) = (r_1, r_2, \dots, _n)$ and $(t) = (t_1, t_2, \dots, t_n)$ are two non-increasing sequences of real numbers. If $(r) \trianglelefteq (t)$, then for any convex function Φ, $\sum_{k=1}^{n} \Phi(r_k) \leq \sum_{k=1}^{n} \Phi(t_k)$.*

Theorem 4.23. *Let G_1 be a connected graph with graphic sequence $(\pi_1) = (d_j)$, $j = 1, 2, \dots, n$ and G_2 be a connected graph with graphic sequence $(\pi_2) = (d'_j)$, $j = 1, 2, \dots, n$.*

(i) If $(\pi_1) \trianglelefteq (\pi_2)$, $\alpha < 0$ or $\alpha > 1$, then ${}^{\alpha}DSE(G_1) \leq {}^{\alpha}DSE(G_2)$.

(ii) If $(\pi_1) \trianglelefteq (\pi_2)$, $0 < \alpha < 1$, then ${}^{\alpha}DSE(G_1) \geq {}^{\alpha}DSE(G_2)$.

The equality holds if and only if $(\pi_1) = (\pi_2)$.

Proof. (i) For the condition $\alpha < 0$ or $\alpha > 1$, in [16] authors have stated that

$$M_1^{\alpha}(G_1) \quad \leq \quad M_1^{\alpha}(G_2), \tag{4.18}$$

the equality holds if and only if $(\pi_1) = (\pi_2)$. From (4.18) and Lemma (4.14) we get the required result.

(ii) For the condition $0 < \alpha < 1$, in [16] authors have stated that

$$M_1^{\alpha}(G_1) \quad \geq \quad M_1^{\alpha}(G_2), \tag{4.19}$$

the equality holds if and only if $(\pi_1) = (\pi_2)$. From (4.19) and Lemma (4.14) we get the required result.

□

Consider T_n is a class of trees of order n. The tree $S(n, i)$ on n vertices is called a double star graph, which is obtained by joining the center of $K_{1,i-1}$ to that of $K_{1,n-1-i}$ by an edge, where $i \geq \lceil \frac{n}{2} \rceil$. Particularly, $S(n, n-1) = K_{1,n-1}$. Let $T_n^s = \{T \in T_n | \Delta(T) = s\}$.

Corollary 4.24. *Let T be a tree in T_n^s, where $s \geq \lceil \frac{n}{2} \rceil$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then ${}^{\alpha}DSE(T) \leq {}^{\alpha}DSE(S(n, s))$.

(ii) If $0 < \alpha < 1$, then ${}^{\alpha}DSE(T) \geq {}^{\alpha}DSE(S(n, s))$.

The equality holds if and only if $\pi(T) = \pi(S(n, s))$.

Proof. (i) If $\alpha < 0$ or $\alpha > 1$ and $T \in T_n^s$, then in [16] authors have stated that

$$M_1^\alpha(T) \quad \leq \quad M_1^\alpha(S(n,s)). \tag{4.20}$$

From (4.20) and Lemma (4.14) we get the required result.

(ii) If $0 < \alpha < 1$ and $T \in T_n^s$, then in [16] authors have stated that

$$M_1^\alpha(T) \quad \leq \quad M_1^\alpha(S(n,s)). \tag{4.21}$$

From (4.21) and Lemma (4.14) we get the require result.

□

Theorem 4.25. *Let G_1 be a connected graph with a cut edge $e = uv$, where $d_v \geq 2$ and $d_u \geq 2$. Let G_2 be the graph obtained from G_1 by contracting the edge e into a new vertex u_e, which becomes adjacent to all the former neighbors of u and of v, and adding a new pendent edge $u_e v_e$, where v_e is a new pendent vertex.*

(i) If $\alpha < 0$ or $\alpha > 1$, then ${}^\alpha DSE(G_1) < {}^\alpha DSE(G_2)$.

(ii) If $0 < \alpha < 1$, then ${}^\alpha DSE(G_1) > {}^\alpha DSE(G_2)$.

Proof. (i) For the condition $\alpha < 0$ or $\alpha > 1$, in [16] authors have stated that

$$M_1^\alpha(G_1) \quad < \quad M_1^\alpha(G_2). \tag{4.22}$$

From (4.22) and Lemma (4.14) we get the required result.

(ii) For the condition $0 < \alpha < 1$, in [16] authors have stated that

$$M_1^\alpha(G_1) \quad > \quad M_1^\alpha(G_2). \tag{4.23}$$

From (4.23) and Lemma (4.14) we get the required result.

□

Corollary 4.26. *Let T be a tree in $T_n \backslash K_{1,n-1}$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then ${}^\alpha DSE(T) < {}^\alpha DSE(K_{1,n-1})$.

(ii) If $0 < \alpha < 1$, then ${}^\alpha DSE(T) > {}^\alpha DSE(K_{1,n-1})$.

Proof. (i) If $\alpha < 0$ or $\alpha > 1$ and $T \in T_n \backslash K_{1,n-1}$, then in [16] authors have stated that

$$M_1^\alpha(T) \quad < \quad M_1^\alpha(K_{1,n-1}). \tag{4.24}$$

From (4.24) and Lemma (4.14) we get the required result.

(ii) If $0 < \alpha < 1$ and $T \in T_n^s$, then in [16] authors have stated that

$$M_1^\alpha(T) \quad > \quad M_1^\alpha(K_{1,n-1}). \tag{4.25}$$

From (4.25) and Lemma (4.14) we get the required result.

□

Theorem 4.27. *Let G_1 be a connected graph with two vertices u, v such that $d_{G_1}(v) \geq d_{G_1}(u)$. Consider $v_1, \ldots, v_s (1 \leq s \leq d_{G_1}(u) - 1)$ are some vertices of $N_{G_1}(u) \backslash \{N_{G_1}(v) \bigcup v\}$. Let G_2 be the graph obtained from G_1 by deleting the edges uv_j and adding edges vv_j for $j = 1, \ldots, s$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then ${}^\alpha DSE(G_1) < {}^\alpha DSE(G_2)$.

(ii) If $0 < \alpha < 1$*, then* ${}^{\alpha}DSE(G_1) > {}^{\alpha}DSE(G_2)$.

Proof. (i) For the condition $\alpha < 0$ or $\alpha > 1$, in [16] authors have stated that

$$M_1^{\alpha}(G_1) \quad < \quad M_1^{\alpha}(G_2). \tag{4.26}$$

From (4.26) and Lemma (4.14) we get the required result.

(ii) For the condition $0 < \alpha < 1$, in [16] authors have stated that

$$M_1^{\alpha}(G_1) \quad > \quad M_1^{\alpha}(G_2). \tag{4.27}$$

From (4.27) and Lemma (4.14) we get the required result.

□

Suppose v is a vertex of graph G. Let $G_{p,q}(q \geq p \geq 1)$ be the graph obtained from G by attaching two new paths $P : v(= v_0)v_1v_2 \ldots v_p$ and $Q : v(= u_0)u_1u_2 \ldots u_q$ of length p and q, respectively, at v, where $v_1, v_2, \ldots, v_p$ and $u_1, u_2, \ldots, u_q$ are distinct new vertices. Let $G_{p-1,q+1} = G_{p,q} - v_{p-1}v_p + u_qv_p$.

Theorem 4.28. *Let G be a connected graph with $n \geq 2$ and $q \geq p \geq 1$. Then* ${}^{\alpha}DSE(G_{p,q}) = {}^{\alpha}DSE(G_{p-1,q+1})$.

Proof. Since $G_{p,q}$ and $G_{p-1,q+1}$ has same graphic sequence so from (4.3) we get the required result. □

Corollary 4.29. *Let T be a tree in $T_n \backslash P_n$.*

(i) If $\alpha < 0$ *or* $\alpha > 1$*, then* ${}^{\alpha}DSE(P_n) < {}^{\alpha}DSE(T)$.

(ii) If $0 < \alpha < 1$*, then* ${}^{\alpha}DSE(P_n) > {}^{\alpha}DSE(T)$.

Proof. (i) If $\alpha < 0$ or $\alpha > 1$ and $T \in T_n \backslash P_n$, then in [16] authors have stated that

$$M_1^{\alpha}(P_n) \quad < \quad M_1^{\alpha}(T). \tag{4.28}$$

From (4.28) and Lemma (4.14) we get the required result.

(ii) If $0 < \alpha < 1$ and $T \in T_n \backslash P_n$, then in [16] authors have stated that

$$M_1^{\alpha}(P_n) \quad > \quad M_1^{\alpha}(T). \tag{4.29}$$

From (4.29) and Lemma (4.14) we get the require result.

□

Theorem 4.30. *Let T be a tree with $n \geq 2$.*

(i) If $\alpha < 0$ *or* $\alpha > 1$*, then* ${}^{\alpha}DSE(T) < 2\sqrt{n[(n-1)^{\alpha} + n - 1]^2 - [2^{\alpha}(n-2) + 2]^2}$.

(ii) If $0 < \alpha < 1$*, then* ${}^{\alpha}DSE(T) < 2\sqrt{n[2^{\alpha}(n-2) + 2]^2 - [(n-1)^{\alpha} + n - 1]^2}$.

Proof. (i) For $\alpha < 0$ or $\alpha > 1$ in [20] the author has stated that

$$(n-2)2^{\alpha} + 2 \leq M_1^{\alpha}(T) \leq (n-1)^{\alpha} + n - 1. \tag{4.30}$$

From (4.3), (4.4) and (4.30) we get the required result.

(ii) For $0 < \alpha < 1$ in [20] the author has stated that

$$(n-1)^{\alpha} + n - 1 \leq M_1^{\alpha}(T) \leq (n-2)2^{\alpha} + 2. \tag{4.31}$$

From (4.3), (4.4) and (4.31) we get the required result.

□

Definition 4.31. [19] Unicyclic graph is a graph containing exactly one cycle.

Theorem 4.32. *Let G be a unicyclic graph with $n \geq 4$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then ${}^{\alpha}DSE(G) < 2\sqrt{n[(n-1)^{\alpha} + 2^{\alpha+1} + n - 3]^2 - n^2 4^{\alpha}}$.

(ii) If $0 < \alpha < 1$, then ${}^{\alpha}DSE(G) < 2\sqrt{n^3 4^{\alpha} - [(n-1)^{\alpha} + 2^{\alpha+1} + n - 3]^2}$.

Proof. (i) For $\alpha < 0$ or $\alpha > 1$, in [19] the author has stated that

$$n2^{\alpha} \leq M_1^{\alpha}(G) \leq (n-1)^{\alpha} + 2^{\alpha+1} + n - 3. \tag{4.32}$$

From (4.3), (4.4) and (4.32) we get the required result.

(ii) For $0 < \alpha < 1$, in [19] the author has stated that

$$(n-1)^{\alpha} + 2^{\alpha+1} + n - 3 \leq M_1^{\alpha}(G) \leq n2^{\alpha}. \tag{4.33}$$

From (4.3), (4.4) and (4.33) we get the required result.

□

5 Algorithm to Compute the general degree subtraction energy (${}^{\alpha}DSE(G)$)

In this section, we obtain an algorithm to compute the general degree subtraction energy of (${}^{\alpha}DSE(G)$).

Algorithm 1 Computation of general degree subtraction energy of a graph (${}^{\alpha}DSE(G)$)

1: **Start**
2: **input**: A simple connected graph G with n vertices
3: **Declare**: Adjacency Array list $A[n]$, j, $N1 = 0$, $N2 = 0$ as integers and α which is a nonzero real number
4: **Declare**: Result as floating point
5: **Compute**:
6: **for** $j = 0$ *to* $j = n - 1$ **do**
7: $N1 = N1 + pow(A[j].size, 2\alpha)$ ▷ $pow(A[j].size, 2\alpha)$ means $A[j].size$ to the power 2α)
8: $N2 = N2 + pow(A[j].size, \alpha)$
9: **Compute the Result**:
10: Result = $2 * \sqrt{n * N1 - pow(N2, 2)}$
11: **Display**: The Result
12: **Stop**

In this Algorithm (1), we try to achieve the better with the time complexity $O(n)$. From which we can find general degree subtraction energy (${}^{\alpha}DSE(G)$), similarly we can easily calculate characteristic polynomial of ${}^{\alpha}DS(G)$ and ${}^{\alpha}DS(G)$-eigenvalues.

6 Conclusion

Here we have introduced a new matrix associated with simple connected graph G called general degree subtraction. We have studied characteristic polynomial, spectra, and energy of ${}^{\alpha}DS(G)$. We have also explored its bounds for spectral radius. Moreover we have observed the effects on the general degree subtraction energy when some operations are applied. We have also given an algorithm with time complexity $O(n)$ to find the general degree subtraction energy of a simple connected graph G.

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