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ON A DIFFERENTIAL INCLUSION GOVERNED BY TIME AND STATE-DEPENDENT MAXIMAL MONOTONE OPERATORS

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Abstract The current work focuses on the existence of absolutely continuous solutions to an evolution problem involving time and state-dependent maximal monotone operators. The well-posedness result is proved under an anti-monotone condition on the domain of the operators. Then, a minimization problem subject to this evolution problem is studied.

1 Introduction and preliminaries

We study, in a real separable Hilbert space H, the differential inclusion governed by time and state-dependent maximal monotone operators described by

$$\begin{cases} -\dot{x}(t) \in A_{(t,x(t))}x(t) + f(t,x(t)) \quad \text{a.e. } t \in I := [0,T], \\ x(0) = x_0 \in \mathrm{D}\left(A_{(0,x_0)}\right), \end{cases} \tag{1.1}$$

where $A_{(t,y)} : \mathrm{D}\left(A_{(t,y)}\right) \subset H \rightrightarrows H$, for each $(t,y) \in I \times H$, is a maximal monotone operator that varies in the sense of the pseudo-distance (see (1.3) and **Assumption 1** (1) below) and $f : I \times H \to H$ is a single-valued map.

The main concern of the present paper is to compensate for the lack of the ball-compactness assumption on the domain of the operators imposed in the recent contributions [3], [27] and [28]. This ball-compactness assumption has also occurred in the investigation of some second-order differential inclusions governed by such maximal monotone operators, see for instance [13], [25], [26]. The study in the aforementioned papers has been achieved under an absolutely continuous variation in the sense of the pseudo-distance (see **Assumption 1** (1) below).

As we will see below, invoking an anti-monotone condition (instead of the ball-compactness assumption) on the domain of the operators, that is, for each $(t_i, y_i) \in I \times H$, $x_i \in \mathrm{D}\left(A_{(t_i,y_i)}\right)$, $i = 1,2$, one has $\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0$, we succeed to establish the existence result related to (1.1). This assumption has been considered in [24], to study a class of second-order differential inclusions with maximal monotone operators. A suitable anti-monotone assumption on the sets $C(\cdot)$ (or $C(t,\cdot)$) can be found in [2], [8], [12], [22], dealing with second-order sweeping processes i.e., when $A_{(t,x)} = N_{C(t,x)}$ (or $A_x = N_{C(x)}$).

No need to use neither ball-compactness nor anti-monotone assumption when dealing with time-dependent maximal monotone operators A_t instead of $A_{(t,x)}$ in (1.1), we refer to e.g. [4], [5], [6], [15], [19], for some achievements on first-order evolution problems with A_t.

Differential inclusions with maximal monotone operators have many applications, we cite e.g., [1], [10], [18], [20] [23], [29], [30].

The authors in [28] have built a sequence of maps that converges to a solution of (1.1) by the Arzelà-Ascoli theorem, while in [3], the existence result is proved using Schauder's fixed point theorem (under a ball-compactness assumption in both papers). In our development, we proceed by a discretization approach to establish the main existence result. A sequence of solutions to the approximate problems is constructed. Then, the Cauchy criterion is proved so that the latter sequence converges uniformly to a solution of (1.1). Uniqueness is obtained under an hypo-monotone assumption on the operators and a Lipschitz property on the perturbation. The well-posedness result is therefore applied to optimal control theory.

The paper contains three sections. In Section 2, we prove the existence and uniqueness result to (1.1). In the last section, we provide an application to optimal control theory.

Now, we give notations and recall the background material needed later. Let $I := [0, T]$ ($T > 0$) be an interval of $\mathbb{R}$. Let H be a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$. Denote by $\overline{B}_H(x, r)$ the closed ball of center x and radius r on H, and by $\overline{B}_H$ the closed unit ball of H.
Let $\mathcal{C}_H(I)$ be the space of all continuous maps x from I to H endowed with the norm of uniform convergence, $\|x\|_\infty = \sup_{t \in I} \|x(t)\|$.
By $L^p_H(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\| \cdot \|_{L^\infty_H(I)}$).
By $W^{1,2}(I, H)$, we denote the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$.
Let us give some definitions and properties of maximal monotone operators, see [7], [9], [32].
Define the domain, range and graph of a set-valued operator $A : \mathrm{D}(A) \subset H \rightrightarrows H$ by

$$\begin{aligned} \mathrm{D}(A) &= \{x \in H : \ Ax \neq \emptyset\}, \\ R(A) &= \{y \in H : \ \exists x \in \mathrm{D}(A), \ y \in Ax\} = \cup\{Ax : \ x \in \mathrm{D}(A)\}, \\ Gr(A) &= \{(x, y) \in H \times H : \ x \in \mathrm{D}(A), \ y \in Ax\}. \end{aligned}$$

The operator $A : \mathrm{D}(A) \subset H \rightrightarrows H$ is said to be monotone, if for $(x_i, y_i) \in Gr(A)$, $i = 1, 2$ one has $\langle y_1 - y_2, x_1 - x_2 \rangle \ge 0$. It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H denotes the identity map of H.
If A is a maximal monotone operator, then, for every $x \in \mathrm{D}(A)$, Ax is non-empty, closed and convex. Then, the projection of the origin onto Ax, denoted $A^0 x$, exists and is unique.
Define for $\lambda > 0$, the resolvent of A by $J^A_\lambda = (I_H + \lambda A)^{-1}$ and the Yosida approximation of A by $A_\lambda = \frac{1}{\lambda}\left(I_H - J^A_\lambda\right)$. These operators are both single-valued and defined on the whole space H, and one has

$$J^A_\lambda x \in \mathrm{D}(A) \ \text{ and } \ A_\lambda x \in A(J^A_\lambda x) \text{ for every } \ x \in H, \tag{1.2}$$
$$\|A_\lambda x\| \le \|A^0 x\| \ \text{ for every } \ x \in \mathrm{D}(A).$$

Let S be a non-empty closed convex subset of H. Denote by $N_S(x)$ the normal cone to S at $x \in H$ given by

$$N_S(x) = \{y \in H : \ \langle y, z - x \rangle \le 0 \ \forall z \in S\}.$$

In such a case, $N_S(\cdot)$ is a maximal monotone operator.
Let $A : \mathrm{D}(A) \subset H \rightrightarrows H$ and $B : \mathrm{D}(B) \subset H \rightrightarrows H$ be two maximal monotone operators. Then, the pseudo-distance between A and B denoted by $\mathrm{dis}\,(A, B)$ (see [31]) is defined by

$$\mathrm{dis}\,(A, B) = \sup\left\{ \frac{\langle y_1 - y_2, x_2 - x_1 \rangle}{1 + \|y_1\| + \|y_2\|} : \ (x_1, y_1) \in Gr(A), \ (x_2, y_2) \in Gr(B) \right\}. \tag{1.3}$$

Clearly, $\mathrm{dis}\,(A, B) \in [0, +\infty]$, $\mathrm{dis}\,(A, B) = \mathrm{dis}\,(B, A)$ and $\mathrm{dis}\,(A, B) = 0$ iff $A = B$.

We need to recall the following lemmas (see [19]).

Lemma 1.1. *Let A be a maximal monotone operator of H. If $x \in \overline{\mathrm{D}(A)}$ and $y \in H$ are such that*

$$\langle A^0 z - y, z - x \rangle \ge 0 \ \ \forall z \in \mathrm{D}(A),$$

then, $x \in \mathrm{D}(A)$ and $y \in Ax$.

Lemma 1.2. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that* $\text{dis}\,(A_n, A) \to 0$. *Suppose also that $x_n \in \mathrm{D}\,(A_n)$ with $x_n \to x$ and $y_n \in A_n x_n$ with $y_n \to y$ weakly for some $x, y \in H$. Then, $x \in \mathrm{D}\,(A)$ and $y \in Ax$.*

Lemma 1.3. *Let A, B be maximal monotone operators of H. Then,*
(1) *for $\lambda > 0$ and $x \in \mathrm{D}\,(A)$*

$$\|x - J_\lambda^B(x)\| \leq \lambda \|A^0(x)\| + \text{dis}\,(A, B) + \sqrt{\lambda\big(1 + \|A^0 x\|\big)\text{dis}\,(A, B)};$$

(2) *for $\lambda > 0$ and $x, y \in H$*

$$\|J_\lambda^A(x) - J_\lambda^A(y)\| \leq \|x - y\|.$$

Lemma 1.4. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that* $\text{dis}\,(A_n, A) \to 0$ *and $\|A_n^0 x\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in \mathrm{D}\,(A_n)$. Then, for every $z \in \mathrm{D}\,(A)$ there exists a sequence (z_n) such that*

$$z_n \in \mathrm{D}\,(A_n), \quad z_n \to z \;\; \text{and} \;\; A_n^0 z_n \to A^0 z.$$

To close this section, recall a discrete version of Gronwall's lemma (see [16]).

Lemma 1.5. *Let $\alpha > 0$. Let (γ_i) and (η_i) be sequences of non-negative real numbers such that*

$$\eta_{i+1} \leq \alpha + (\sum_{k=0}^{i} \gamma_k \eta_k) \quad \text{for } i \in \mathbb{N}.$$

Then, one has

$$\eta_{i+1} \leq \alpha \exp(\sum_{k=0}^{i} \gamma_k) \quad \text{for } i \in \mathbb{N}.$$

2 Main result

In this section, we study the existence of absolutely continuous solutions to problem (1.1), under the following hypothesis:
Assumption 1: Let for $(t, x) \in I \times H$, $A_{(t,x)} : \mathrm{D}\,(A_{(t,x)}) \subset H \rightrightarrows H$ be a maximal monotone operator such that

(1) there exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $\alpha \in W^{1,2}(I, \mathbb{R})$ which is non-negative on $[0, T[$ and non-decreasing with $\alpha(T) < \infty$ and $\alpha(0) = 0$ such that

$$\text{dis}\,(A_{(t,y)}, A_{(s,z)}) \leq |\alpha(t) - \alpha(s)| + \lambda \|y - z\|, \; \forall t, s \in I, \; \forall y, z \in H;$$

(2) there exists a non-negative real constant c such that

$$\|A^0_{(t,y)} z\| \leq c(1 + \|y\| + \|z\|) \;\text{ for }\; t \in I, \; y \in H, \; z \in \mathrm{D}\,(A_{(t,y)});$$

(3) for any $(t_i, y_i) \in I \times H$, $x_i \in \mathrm{D}\,(A_{(t_i,y_i)})$, $i = 1, 2$, one has

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0.$$

Assumption 2: Let $f : I \times H \to H$ be a map such that

(1) for any fixed $x \in H$, $f(\cdot, x)$ is measurable on I and for any fixed $t \in I$, $f(t, \cdot)$ is continuous on H;

(2) there exists a non-negative real constant L such that

$$\|f(t, x)\| \leq L(1 + \|x\|) \text{ for all } (t, x) \in I \times H. \tag{2.1}$$

Now, we are able to state and prove our existence result regarding the evolution problem (1.1).

Theorem 2.1. *Suppose that* **Assumption 1** *and* **Assumption 2** *hold true. Then, for any* $x_0 \in \mathrm{D}\,(A_{(0,x_0)})$*, the evolution problem* (1.1) *has at least one absolutely continuous solution* $x(\cdot)$ *which satisfies*

$$\|\dot{x}(t)\| \leq \varrho(1+\dot{\alpha}(t)) \text{ a.e. } t \in I, \tag{2.2}$$

where ϱ *is a non-negative real constant depending on* λ*,* c*,* T*,* $\alpha(T)$ *and* $\|x_0\|$.

Proof. The proof is divided into three parts.
Part 1: Construction of the sequence (x_n).
Consider a subdivision of the interval I with

$$0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

For every $n \geq 1$ and $i = 0, 1, \cdots, n-1$, set

$$h_{i+1}^n = t_{i+1}^n - t_i^n, \quad \alpha_{i+1}^n = \alpha(t_{i+1}^n) - \alpha(t_i^n), \tag{2.3}$$

and suppose that

$$h_i^n \leq h_{i+1}^n, \quad \alpha_i^n \leq \alpha_{i+1}^n \text{ and } \quad \gamma_{i+1}^n = h_{i+1}^n + \alpha_{i+1}^n \leq \rho_n, \tag{2.4}$$

where $\rho_n = \frac{\gamma(T)}{n}$ and the map γ is defined by $\gamma(t) = t + \alpha(t)$, $t \in I$. It is clear that $\rho_n \to 0$ as $n \to +\infty$.
Put $x_0^n = x_0$ and set

$$x_{i+1}^n = J_{i+1}^n\Big(x_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\Big), \tag{2.5}$$

where

$$\begin{aligned} J_{i+1}^n &= J_{h_{i+1}^n}^{A_{(t_{i+1}^n, x_i^n)}} \\ &= \Big(I_H + h_{i+1}^n A_{(t_{i+1}^n, x_i^n)}\Big)^{-1}. \end{aligned}$$

Then, note by (1.2), that

$$x_{i+1}^n \in \mathrm{D}\,(A_{(t_{i+1}^n, x_i^n)}), \tag{2.6}$$

and, by (2.5), one writes

$$-\frac{1}{h_{i+1}^n}\Big(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\Big) \in A_{(t_{i+1}^n, x_i^n)} x_{i+1}^n. \tag{2.7}$$

Now, Lemma 1.3 yields

$$\begin{aligned} \|x_{i+1}^n - x_i^n\| &= \|J_{i+1}^n\Big(x_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\Big) - x_i^n\| \\ &\leq \|J_{i+1}^n\Big(x_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\Big) - J_{i+1}^n(x_i^n)\| + \|J_{i+1}^n(x_i^n) - x_i^n\| \\ &\leq \int_{t_i^n}^{t_{i+1}^n} \|f(s, x_i^n)\|ds + h_{i+1}^n\|A^0_{(t_i^n, x_{i-1}^n)} x_i^n\| + \mathrm{dis}\,(A_{(t_{i+1}^n, x_i^n)}, A_{(t_i^n, x_{i-1}^n)}) \\ &\quad + \Big(h_{i+1}^n(1 + \|A^0_{(t_i^n, x_{i-1}^n)} x_i^n\|)\mathrm{dis}\,(A_{(t_{i+1}^n, x_i^n)}, A_{(t_i^n, x_{i-1}^n)})\Big)^{\frac{1}{2}}. \end{aligned}$$

Using **Assumption 1** (1)-(2), **Assumption 2** (1)-(2) and the fact that $\sqrt{ab} \le \frac{1}{2}(a+b)$ for non-negative real constants a, b, gives

$$\begin{aligned}
\|x^n_{i+1} - x^n_i\| &\le h^n_{i+1} L(1+\|x^n_i\|) + h^n_{i+1} c(1+\|x^n_i\| + \|x^n_{i-1}\|) + \lambda \|x^n_i - x^n_{i-1}\| \\
&\quad + \frac{h^n_{i+1}}{2}(1 + c(1+\|x^n_i\| + \|x^n_{i-1}\|)) + \frac{3}{2}\alpha^n_{i+1} + \frac{\lambda}{2}\|x^n_i - x^n_{i-1}\| \\
&\le h^n_{i+1}(L + \frac{3c}{2})\|x^n_i\| + \frac{3c}{2} h^n_{i+1}\|x^n_{i-1}\| + \frac{3}{2}\alpha^n_{i+1} + \frac{3\lambda}{2}\|x^n_i - x^n_{i-1}\| \\
&\quad + h^n_{i+1}(L + \frac{3c}{2} + \frac{1}{2}).
\end{aligned}$$

This along with (2.4), one writes for any $i = 0, 1, \cdots, n-1$ with $t^n_{-1} = t^n_0$ and $x^n_{-1} = x^n_0$

$$\begin{aligned}
\|x^n_{i+1} - x^n_i\| &\le \gamma^n_{i+1}(L + \frac{3c}{2})\|x^n_i\| + \frac{3c}{2}\gamma^n_{i+1}\|x^n_{i-1}\| + \frac{3\lambda}{2}\|x^n_i - x^n_{i-1}\| \\
&\quad + \frac{3}{2}\gamma^n_{i+1} + \gamma^n_{i+1}(L + \frac{3c}{2} + \frac{1}{2}) \\
&\le \gamma^n_{i+1}(L + \frac{3c}{2})(\|x^n_i\| + \|x^n_{i-1}\|) + \frac{3\lambda}{2}\|x^n_i - x^n_{i-1}\| \\
&\quad + \gamma^n_{i+1}(L + \frac{3c}{2} + 2) \\
&\le \gamma^n_{i+1}(L + \frac{3c}{2} + 2)(1 + \|x^n_i\| + \|x^n_{i-1}\|) + \frac{3\lambda}{2}\|x^n_i - x^n_{i-1}\|.
\end{aligned}$$

Remember that $\lambda < \frac{2}{3}$, then, setting $\mu = \frac{3\lambda}{2}$ and $\eta = (L + \frac{3c}{2} + 2)$, it follows by iteration

$$\|x^n_{i+1} - x^n_i\| \le \eta \gamma^n_{i+1} \sum_{j=0}^{i} \mu^j (1 + \|x^n_{i-j}\| + \|x^n_{i-j-1}\|). \tag{2.8}$$

Thus, for any n and $i = 0, \cdots, n-1$ using (2.4), it results

$$\begin{aligned}
\|x^n_{i+1}\| &\le \|x^n_0\| + \sum_{j=0}^{i} \|x^n_{j+1} - x^n_j\| \\
&\le \|x^n_0\| + \eta \sum_{j=0}^{i} \gamma^n_{j+1} \sum_{k=0}^{j} \mu^k (1 + \|x^n_{j-k}\| + \|x^n_{j-k-1}\|) \\
&\le \|x^n_0\| + \eta \rho_n \left(\sum_{j=0}^{i} \left(\sum_{k=0}^{j} \mu^k \right) + \sum_{j=0}^{i} \sum_{k=0}^{j} \mu^k \left(\|x^n_{j-k}\| + \|x^n_{j-k-1}\| \right) \right) \\
&\le \|x^n_0\| + \eta \rho_n \left(\frac{i+1}{1-\mu} + 3 \sum_{j=0}^{i} \mu^j \left(\sum_{j=0}^{i} \|x^n_j\| \right) \right) \\
&\le \|x_0\| + \eta \frac{\gamma(T)}{1-\mu} + \frac{3\eta\rho_n}{1-\mu} \sum_{j=0}^{i} \|x^n_j\|.
\end{aligned}$$

Applying Lemma 1.5 gives for any $n \ge 1$ and $i = 1, \cdots, n-1$

$$\|x^n_{i+1}\| \le \left(\|x_0\| + \eta \frac{\gamma(T)}{1-\mu} \right) \exp\left(\frac{3\eta\gamma(T)}{1-\mu} \right) = \varrho_1.$$

This along with (2.8) yields

$$\|x^n_{i+1} - x^n_i\| \le \frac{\eta(1 + 2\varrho_1)}{1-\mu} \gamma^n_{i+1} = \varrho_2 \gamma^n_{i+1}.$$

Set $\varrho = \max(\varrho_1, \varrho_2)$, then, one writes

$$\|x_i^n\| \leq \varrho \quad \text{and} \quad \|x_{i+1}^n - x_i^n\| \leq \varrho \gamma_{i+1}^n. \tag{2.9}$$

For any $n \geq 1$, define the sequence $x_n : I \to H$ by

$$x_n(t) = x_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}\left(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\right) - \int_{t_i^n}^{t} f(s, x_i^n)ds, \tag{2.10}$$

for $t \in [t_i^n, t_{i+1}^n[,\ i = 0, 1, \cdots, n-1$ and $x_n(T) = x_n^n$. By derivation one gets

$$\dot{x}_n(t) = \frac{1}{t_{i+1}^n - t_i^n}\left(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, x_i^n)ds\right) - f(t, x_i^n). \tag{2.11}$$

Put for any $n \geq 1$

$$\theta_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_i^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}, \end{cases}$$

and

$$\phi_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_{i+1}^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}. \end{cases}$$

Hence, for each $n \in \mathbb{N}^*$, there is a null Lebesgue measure set $X_n \subset I$ such that

$$-\dot{x}_n(t) - f(t, x_n(\theta_n(t))) \in A_{(\phi_n(t), x_n(\theta_n(t)))} x_n(\phi_n(t)) \quad \forall t \in I \setminus X_n, \tag{2.12}$$

and

$$x_n(\phi_n(t)) \in \mathrm{D}\left(A_{(\phi_n(t), x_n(\theta_n(t)))}\right), \tag{2.13}$$

using (2.6) and (2.7).

Part 2: Convergence of the sequence (x_n).

Now, we show that the sequence $(x_n)_n$ is bounded in norm and variation.

From (2.1), (2.9) and (2.10), one has

$$\begin{aligned} \|x_n(t) - x_i^n\| &\leq \|x_{i+1}^n - x_i^n\| + 2Lh_{i+1}^n(1 + \|x_i^n\|) \\ &\leq \gamma_{i+1}^n(\varrho + 2L(1 + \varrho)). \end{aligned}$$

Then, using (2.4), one has for $t \in [t_i^n, t_{i+1}^n[, i \in \{0, 1, \cdots, n-1\}$

$$\|x_n(t) - x_i^n\| \leq (\varrho + 2L(1 + \varrho))\rho_n = \varrho_3 \rho_n, \tag{2.14}$$

along with (2.9) yields

$$\sup_n \|x_n(t)\| \leq (\varrho + 2L(1 + \varrho))\gamma(T) + \varrho = \varrho_0.$$

Thus, one deduces

$$\sup_n \|x_n(\cdot)\|_\infty \leq \varrho_0 \text{ and } \sup_n var(x_n(\cdot)) = \sup_n \left(\sum_{i=0}^{n-1} \|x_{i+1}^n - x_i^n\|\right) \leq \varrho\gamma(T).$$

Fix $s \in [t_i^n, t_{i+1}^n[$ and $t \in [t_j^n, t_{j+1}^n[$ with $i < j$. Then, by (2.4), (2.9), (2.14) one writes

$$\begin{aligned}
\|x_n(t) - x_n(s)\| &\leq \|x_n(t) - x_j^n\| + \|x_j^n - x_i^n\| + \|x_i^n - x_n(s)\| \\
&\leq \|x_j^n - x_i^n\| + 2\varrho_3\rho_n \\
&\leq \sum_{l=0}^{j-i-1} \|x_{i+l+1}^n - x_{i+l}^n\| + 2\varrho_3\rho_n \\
&\leq \varrho \sum_{l=0}^{j-i-1} \gamma_{i+l+1}^n + 2\varrho_3\rho_n \\
&= \varrho\Big(\gamma(t_j^n) - \gamma(t_i^n)\Big) + 2\varrho_3\rho_n \\
&\leq \varrho\Big(\gamma(t) - \gamma(t_i^n)\Big) + 2\varrho_3\rho_n \\
&= \varrho\Big(\gamma(t) - \gamma(s) + \gamma(s) - \gamma(t_i^n)\Big) + 2\varrho_3\rho_n \\
&\leq \varrho\Big(\gamma(t) - \gamma(s) + \gamma(t_{i+1}^n) - \gamma(t_i^n)\Big) + 2\varrho_3\rho_n \\
&\leq \varrho\Big(\gamma(t) - \gamma(s)\Big) + \varrho\gamma_{i+1}^n + 2\varrho_3\rho_n.
\end{aligned}$$

Once more by (2.4), it follows that for each $t, s \in I$ such that $s \leq t$

$$\|x_n(t) - x_n(s)\| \leq \varrho\Big(\gamma(t) - \gamma(s)\Big) + (\varrho + 2\varrho_3)\rho_n. \tag{2.15}$$

Now, since the sequence of bounded variation continuous functions (x_n) is uniformly bounded in norm and in variation, then, by Theorem 0.2.1 [22], we may assume that there is a bounded variation continuous map $x : I \to H$ such that $(x_n(t))$ converges weakly to $x(t)$ for all $t \in I$. Combining (2.1), (2.3), (2.4), (2.9), (2.11), one has for all $t \in [t_i^n, t_{i+1}^n[$

$$\begin{aligned}
\|\dot{x}_n(t)\| &\leq \frac{1}{h_{i+1}^n}\|x_{i+1}^n - x_i^n\| + 2L(1+\varrho) \qquad (2.16)\\
&\leq \varrho\frac{\gamma_{i+1}^n}{h_{i+1}^n} + 2L(1+\varrho) \\
&\leq \varrho\left(1 + \frac{\alpha(t_{i+1}^n) - \alpha(t_i^n)}{t_{i+1}^n - t_i^n}\right) + 2L(1+\varrho).
\end{aligned}$$

By the absolute continuity of $\alpha(\cdot)$, one has for a.e. $t \in]t_i^n, t_{i+1}^n[$, $\dot{\alpha}(t) = \lim_{n\to\infty} \frac{\alpha(t_{i+1}^n)-\alpha(t_i^n)}{t_{i+1}^n - t_i^n}$. Then, there is a Lebesgue measure null-set $Y \subset I$ such that for every $t \in I \setminus Y$, there exists $b_t < +\infty$

$$\|\dot{x}_n(t)\| \leq b_t. \tag{2.17}$$

Observe by (2.9) that

$$\|x_{i+1}^n - x_i^n\| \leq \int_{t_i^n}^{t_{i+1}^n} \delta(s)ds,$$

where the map δ is defined by $\delta(t) = \varrho(1 + \dot{\alpha}(t))$ for any $t \in I$.
Next, using the Cauchy-Schwarz inequality, one writes

$$\|x_{i+1}^n - x_i^n\| \leq (t_{i+1}^n - t_i^n)^{1/2}\left(\int_{t_i^n}^{t_{i+1}^n} \delta^2(s)ds\right)^{1/2}.$$

Combining the last inequality with (2.16), noting that $(x+y)^2 \le 2(x^2+y^2)$ for $x, y \in \mathbb{R}$, one gets

$$\begin{aligned}
\|\dot{x}_n\|^2_{L^2_H(I)} &= \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|\dot{x}_n(t)\|^2 dt \\
&\le \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \Big(\frac{1}{h_{i+1}^n} \|x_{i+1}^n - x_i^n\| + 2L(1+\varrho) \Big)^2 dt \\
&\le 2 \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \Big(\Big(\frac{\|x_{i+1}^n - x_i^n\|}{t_{i+1}^n - t_i^n} \Big)^2 + 4L^2(1+\varrho)^2 \Big) dt \\
&\le 2 \sum_{i=0}^{n-1} \Big(\frac{\|x_{i+1}^n - x_i^n\|^2}{t_{i+1}^n - t_i^n} + 4L^2(1+\varrho)^2 (t_{i+1}^n - t_i^n) \Big) \\
&\le 2 \Big(\sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \delta^2(s) ds + 4L^2(1+\varrho)^2 T \Big) \\
&= 2 \Big(\int_0^T \delta^2(s) ds + 4L^2(1+\varrho)^2 T \Big) < +\infty. \qquad (2.18)
\end{aligned}$$

Therefore, the sequence $(\dot{x}_n)$ is bounded in $L^2_H(I)$, extracting a subsequence that we do not relabel, $(\dot{x}_n)$ converges weakly to some map y. Recall that $(x_n(t))$ converges weakly to $x(t)$ for each $t \in I$. Let $z \in H$. One has for any $t \in I$

$$\begin{aligned}
\langle z, x(t) - x(0) \rangle &= \lim_{n\to\infty} \langle z, x_n(t) - x_n(0) \rangle \\
&= \lim_{n\to\infty} \Big\langle z, \int_0^t \dot{x}_n(s) ds \Big\rangle = \lim_{n\to\infty} \int_0^T \langle z 1_{]0,t]}(s), \dot{x}_n(s) \rangle ds \\
&= \int_0^T \langle z 1_{]0,t]}(s), y(s) \rangle ds = \Big\langle z, \int_0^t y(s) ds \Big\rangle,
\end{aligned}$$

where $1_{]0,t]}$ denotes the characteristic function of the interval $]0,t]$.
One deduces that $x(t) - x(0) = \int_0^t y(s) ds, \ t \in I$, that is, $\dot{x} = y$ a.e. on I and

$$(\dot{x}_n) \text{ converges weakly to } \dot{x} \text{ in } L^2_H(I). \qquad (2.19)$$

Now, let us prove that $(x_n)_n$ is a Cauchy sequence in $\mathcal{C}_H(I)$.
Let $n, m \in \mathbb{N}^*$ and let x_n, x_m be the absolutely continuous maps such that

$$-\dot{x}_n(t) - f(t, x_n(\theta_n(t))) \in A_{(\phi_n(t), x_n(\theta_n(t)))} x_n(\phi_n(t)), \qquad (2.20)$$

$$x_n(\phi_n(t)) \in \mathrm{D}\,(A_{(\phi_n(t), x_n(\theta_n(t)))}), \quad x_n(0) = x_0, \qquad (2.21)$$

and

$$-\dot{x}_m(t) - f(t, x_m(\theta_m(t))) \in A_{(\phi_m(t), x_m(\theta_m(t)))} x_m(\phi_m(t)), \qquad (2.22)$$

$$x_m(\phi_m(t)) \in \mathrm{D}\,(A_{(\phi_m(t), x_m(\theta_m(t)))}), \quad x_m(0) = x_0. \qquad (2.23)$$

Note that for each $t \in I$

$$\begin{aligned}
&\|x_n(\theta_n(t)) - x_m(\theta_m(t))\|^2 = \langle x_n(\theta_n(t)) - x_m(\theta_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t)) \rangle \\
&= \langle x_n(\theta_n(t)) - x_n(\phi_n(t)) + x_m(\phi_m(t)) - x_m(\theta_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t)) \rangle \\
&+ \langle x_n(\phi_n(t)) - x_m(\phi_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t)) \rangle \\
&\le (\|x_n(\theta_n(t)) - x_n(\phi_n(t))\| + \|x_m(\theta_m(t)) - x_m(\phi_m(t))\|) \|x_n(\theta_n(t)) - x_m(\theta_m(t))\| \\
&+ \langle x_n(\phi_n(t)) - x_m(\phi_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t)) \rangle,
\end{aligned}$$

simplifying using (2.9), one gets

$$\begin{aligned}&\|x_n(\theta_n(t)) - x_m(\theta_m(t))\|^2\\&\leq 2\varrho\Big(\|x_n(\theta_n(t)) - x_n(\phi_n(t))\| + \|x_m(\theta_m(t)) - x_m(\phi_m(t))\|\Big)\\&+ \langle x_n(\phi_n(t)) - x_m(\phi_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t))\rangle.\end{aligned}$$

In view of **Assumption 1** (3), (2.21) and (2.23), one remarks that

$$\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), x_n(\theta_n(t)) - x_m(\theta_m(t))\rangle \leq 0,$$

then, one writes

$$\|x_n(\theta_n(t)) - x_m(\theta_m(t))\| \leq (\Delta_{n,m}(t))^{\frac{1}{2}}, \tag{2.24}$$

where

$$\begin{aligned}\Delta_{n,m}(t) &= 2\varrho(\|x_n(\theta_n(t)) - x_n(\phi_n(t))\| + \|x_m(\theta_m(t)) - x_m(\phi_m(t))\|)\\&\leq 2\varrho(\|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x_n(\phi_n(t))\| + \|x_m(\theta_m(t)) - x_m(t)\|\\&+ \|x_m(t) - x_m(\phi_m(t))\|).\end{aligned}$$

Using (2.15) one has for all $t \in I$

$$\|x_n(\phi_n(t)) - x_n(t)\| \leq \varrho\Big(\gamma(\phi_n(t)) - \gamma(t)\Big) + (\varrho + 2\varrho_3)\rho_n,$$

and

$$\|x_n(\theta_n(t)) - x_n(t)\| \leq \varrho\Big(\gamma(t) - \gamma(\theta_n(t))\Big) + (\varrho + 2\varrho_3)\rho_n,$$

along with (2.4), it results that for any $t \in I$ and any $n \geq 1$

$$\|x_n(\phi_n(t)) - x_n(t)\| \leq \varrho_4\rho_n, \quad \|x_n(\theta_n(t)) - x_n(t)\| \leq \varrho_4\rho_n, \tag{2.25}$$

where $\varrho_4 = 2(\varrho + \varrho_3)$. Hence, it results that for any $t \in I$ and any $n, m \geq 1$

$$\Delta_{n,m}(t) \leq 4\varrho\varrho_4(\rho_n + \rho_m).$$

Coming back to (2.24), one deduces that

$$\|x_n(\theta_n(t)) - x_m(\theta_m(t))\| \leq 2\Big(\varrho\varrho_4(\rho_n + \rho_m)\Big)^{\frac{1}{2}}. \tag{2.26}$$

In the same way, one obtains

$$\|x_n(\phi_n(t)) - x_m(\phi_m(t))\| \leq 2\Big(\varrho\varrho_4(\rho_n + \rho_m)\Big)^{\frac{1}{2}}. \tag{2.27}$$

Set for any $n \geq 1$, $f_n(t) = f(t, x_n(\theta_n(t)))$, for all $t \in I$. In view of (2.1) and (2.9), one remarks that

$$\|f_n(t)\| \leq L(1 + \varrho) \text{ for all } t \in I \text{ and any } n \geq 1. \tag{2.28}$$

By the definition of the pseudo-distance in (1.3) and the differential inclusions (2.20) and (2.22), one writes

$$\begin{aligned}&\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), \dot{x}_n(t) + f_n(t) - \dot{x}_m(t) - f_m(t)\rangle\\&\leq \Big(1 + \|\dot{x}_n(t) + f_n(t)\| + \|\dot{x}_m(t) + f_m(t)\|\Big)\text{dis}\,(A_{(\phi_n(t), x_n(\theta_n(t)))}, A_{(\phi_m(t), x_m(\theta_m(t)))}).\end{aligned}$$

Combining **Assumption 1** (1), (2.3), (2.4), (2.26) and (2.28), one has

$$
\begin{aligned}
&\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), \dot{x}_n(t) + f_n(t) - \dot{x}_m(t) - f_m(t)\rangle \\
&\leq \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big) \operatorname{dis}\left(A_{(\phi_n(t), x_n(\theta_n(t)))}, A_{(\phi_m(t), x_m(\theta_m(t)))}\right) \\
&\leq \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(|\alpha(\phi_n(t)) - \alpha(\phi_m(t))| + \lambda \|x_n(\theta_n(t)) - x_m(\theta_m(t))\|\Big), \\
&\leq \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big).
\end{aligned}
$$

Next, using the latter inequality yields

$$
\begin{aligned}
&\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), \dot{x}_n(t) - \dot{x}_m(t)\rangle \\
&\leq \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big) \\
&+ \langle x_n(\phi_n(t)) - x_m(\phi_m(t)), f_m(t) - f_n(t)\rangle. \qquad (2.29)
\end{aligned}
$$

But from (2.27) and (2.28), it follows

$$
\begin{aligned}
\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), f_m(t) - f_n(t)\rangle &\leq \Big(\|f_n(t)\| + \|f_m(t)\|\Big)\|x_n(\phi_n(t)) - x_m(\phi_m(t))\| \\
&\leq 2L(1+\varrho)\|x_n(\phi_n(t)) - x_m(\phi_m(t))\| \\
&\leq 4L(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Combining back to (2.29) gives

$$
\begin{aligned}
&\langle x_n(\phi_n(t)) - x_m(\phi_m(t)), \dot{x}_n(t) - \dot{x}_m(t)\rangle \\
&\leq \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big) \\
&\quad + 4L(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}. \qquad (2.30)
\end{aligned}
$$

On the one hand, for all $t \in I$ and $n, m \in \mathbb{N}^*$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t) - x_m(t)\|^2 = \langle x_n(t) - x_m(t), \dot{x}_n(t) - \dot{x}_m(t)\rangle \\
&= \langle x_n(t) - x_n(\phi_n(t)), \dot{x}_n(t) - \dot{x}_m(t)\rangle - \langle x_m(t) - x_m(\phi_m(t)), \dot{x}_n(t) - \dot{x}_m(t)\rangle \\
&+ \langle x_n(\phi_n(t)) - x_m(\phi_m(t)), \dot{x}_n(t) - \dot{x}_m(t)\rangle. \qquad (2.31)
\end{aligned}
$$

On the other hand, from (2.30) and (2.31), one gets

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t) - x_m(t)\|^2 \\
&\leq \Big(\|\dot{x}_n(t)\| + \|\dot{x}_m(t)\|\Big)\Big(\|x_n(t) - x_n(\phi_n(t))\| + \|x_m(t) - x_m(\phi_m(t))\|\Big) \\
&+ \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big) \\
&+ 4L(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then, for a.e. $t \in I$,

$$\frac{1}{2}\frac{d}{dt}\|x_n(t) - x_m(t)\|^2 \le \varphi_{n,m}(t), \tag{2.32}$$

where for every $n, m \ge 1$ and every $t \in I$, the function $\varphi_{n,m}$ is defined by

$$\begin{aligned}\varphi_{n,m}(t) = &\Big(\|\dot{x}_n(t)\| + \|\dot{x}_m(t)\|\Big)\Big(\|x_n(t) - x_n(\phi_n(t))\| + \|x_m(t) - x_m(\phi_m(t))\|\Big)\\ &+ \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big)\\ &+ 4L(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}.\end{aligned}$$

Simplifying using (2.25) entails

$$\begin{aligned}\varphi_{n,m}(t) \le &\Big(\|\dot{x}_n(t)\| + \|\dot{x}_m(t)\|\Big)\Big(\varrho_4(\rho_n+\rho_m)\Big) + 4L(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\\ &+ \Big(1 + \|\dot{x}_n(t)\| + \|\dot{x}_m(t)\| + 2L(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big).\end{aligned}$$

Next, deduce from (2.18) that $(\dot{x}_n)$ is bounded in $L^1_H(I)$, then, one obtains

$$\begin{aligned}&\int_0^T \varphi_{n,m}(s)ds \le \varrho_4\Big(\|\dot{x}_n\|_{L^1_H(I)} + \|\dot{x}_m\|_{L^1_H(I)}\Big)(\rho_n+\rho_m)\\ &+ \Big(T + \|\dot{x}_n\|_{L^1_H(I)} + \|\dot{x}_m\|_{L^1_H(I)} + 2TL(1+\varrho)\Big)\Big(\rho_n + \rho_m + 2\lambda\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}\Big)\\ &+ 4TL(1+\varrho)\Big(\varrho\varrho_4(\rho_n+\rho_m)\Big)^{\frac{1}{2}}.\end{aligned}$$

Thus, it follows that

$$\lim_{n,m\to+\infty}\int_0^T \varphi_{n,m}(s)ds = 0. \tag{2.33}$$

Integrating (2.32) over $[0, t]$, yields

$$\|x_n(t) - x_m(t)\|^2 \le 2\int_0^T \varphi_{n,m}(s)ds,$$

noting that $\|x_n(0) - x_m(0)\| = 0$. This along with (2.33), one concludes that $(x_n(\cdot))$ is a Cauchy sequence in $\mathcal{C}_H(I)$. So by the uniform Cauchy's criterion, the sequence $(x_n(\cdot))$ converges uniformly in $\mathcal{C}_H(I)$ to $x(\cdot)$ (since the weak pointwise convergence of $(x_n(\cdot))$ to $x(\cdot)$ is proved above).
Observe moreover that

$$\|x_n(\theta_n(t)) - x(t)\| \quad \le \quad \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\|.$$

This along with the uniform convergence above and (2.25) gives

$$\|x_n(\theta_n(t)) - x(t)\| \to 0, \quad as\ n \to +\infty. \tag{2.34}$$

In the same vein, one obtains

$$\|x_n(\phi_n(t)) - x(t)\| \to 0, \quad as\ n \to +\infty. \tag{2.35}$$

Part 3: Statement of the following differential inclusions

$$-\dot{x}(t) \in A_{(t,x(t))}x(t) + f(t, x(t)) \quad \text{a.e. } t \in I, \tag{2.36}$$

$$x(t) \in \mathrm{D}\left(A_{(t,x(t))}\right) \quad t \in I. \tag{2.37}$$

First, we prove (2.37).
Recall by (2.13) that $x_n(\phi_n(t)) \in \mathrm{D}\left(A_{(\phi_n(t),x_n(\theta_n(t)))}\right)$ for all $t \in I$. In view of **Assumption 1** (1), (2.4) and (2.34), it follows that

$$\begin{aligned} \mathrm{dis}\left(A_{(\phi_n(t),x_n(\theta_n(t)))}, A_{(t,x(t))}\right) &\leq \alpha(\phi_n(t)) - \alpha(t) + \lambda\|x_n(\theta_n(t)) - x(t)\| \\ &\leq \rho_n + \lambda\|x_n(\theta_n(t)) - x(t)\| \to 0, \quad as \quad n \to \infty. \end{aligned} \tag{2.38}$$

Remark that $(u_n) = (A^0_{(\phi_n(t),x_n(\theta_n(t)))}x_n(\phi_n(t)))$ is bounded by **Assumption 1** (2) and (2.9). Then, we may extract from (u_n) a subsequence that converges weakly to some $u \in H$. Since the sequence $(x_n(\phi_n(t)))$ converges to $x(t)$ in H (see (2.35)), applying Lemma 1.2, one concludes that $x(t) \in \mathrm{D}\left(A_{(t,x(t))}\right)$, $t \in I$.
Now, let us show (2.36).
Combining (2.34) with **Assumption 2** (1) then, one gets $f(t, x_n(\theta_n(t))) \to f(t, x(t))$ a.e, along with (2.28), then, applying the Lebesgue dominated convergence theorem yields

$$f_n(\cdot) \to f(\cdot, x(\cdot)) \text{ in } L^2_H(I). \tag{2.39}$$

From (2.19) and (2.39), one deduces that $(\dot{x}_n + f_n)$ converges weakly to $\dot{x}(\cdot) + f(\cdot, x(\cdot))$ in $L^2_H(I)$. Then, there exists a sequence (v_j) such that for each $j \in \mathbb{N}$, $v_j \in co\{\dot{x}_l + f_l,\ l \geq j\}$ and (v_j) converges strongly to $\dot{x}(\cdot) + f(\cdot, x(\cdot))$ in $L^2_H(I)$. Then, we may extract from (v_j) a subsequence that converges a.e. to $\dot{x}(\cdot) + f(\cdot, x(\cdot))$. Hence, there exists a subset Y_n of I with null Lebesgue measure and a subsequence (j_p) of $\mathbb{N}$ such that for all $t \in I \backslash Y_n$, $(v_{j_p}(t))$ converges to $\dot{x}(t) + f(t, x(t))$. Hence, for $t \in I \setminus Y_n$

$$\dot{x}(t) + f(t, x(t)) \in \bigcap_{p \in \mathbb{N}} \overline{co}\{\dot{x}_l(t) + f_l(t),\ l \geq j_p\},$$

which means that for $t \in I \setminus Y_n$ and any $\zeta \in H$

$$\langle \dot{x}(t) + f(t, x(t)), \zeta \rangle \leq \limsup_{n \to \infty} \langle \dot{x}_n(t) + f_n(t), \zeta \rangle. \tag{2.40}$$

Recall that $x(t) \in \mathrm{D}\left(A_{(t,x(t))}\right)$, $t \in I$. By Lemma 1.1, it remains to prove that

$$\langle \dot{x}(t) + f(t, x(t)), x(t) - y \rangle \leq \langle A^0_{(t,x(t))}y, y - x(t) \rangle \quad \text{a.e. } t \in I,$$

for all $y \in \mathrm{D}\left(A_{(t,x(t))}\right)$.
Let $y \in \mathrm{D}\left(A_{(t,x(t))}\right)$. Now, applying Lemma 1.4 to the maximal monotone operators $A_{(t,x(t))}$ and $A_{(\phi_n(t),x_n(\theta_n(t)))}$ that satisfy (2.38), then, there exists $y_n \in \mathrm{D}\left(A_{(\phi_n(t),x_n(\theta_n(t)))}\right)$ such that

$$y_n \to y \text{ and } A^0_{(\phi_n(t),x_n(\theta_n(t)))}y_n \to A^0_{(t,x(t))}y. \tag{2.41}$$

For every $t \in I \setminus X_n$, from (2.12) and the monotone property of $A_{(\phi_n(t),x_n(\theta_n(t)))}$ it follows that

$$\langle \dot{x}_n(t) + f_n(t), x_n(\phi_n(t)) - y_n \rangle \leq \langle A^0_{(\phi_n(t),x_n(\theta_n(t)))}y_n, y_n - x_n(\phi_n(t)) \rangle. \tag{2.42}$$

Combining (2.17), (2.28) with (2.42) , it results for $t \in I \setminus (\bigcup_{n \in \mathbb{N}} X_n \cup Y_n \cup Y)$

$$\begin{aligned} \langle \dot{x}_n(t) + f_n(t), x(t) - y \rangle &= \langle \dot{x}_n(t) + f_n(t), x_n(\phi_n(t)) - y_n \rangle \\ &\quad + \langle \dot{x}_n(t) + f_n(t), (x(t) - x_n(\phi_n(t))) - (y - y_n) \rangle \\ &\leq \langle A^0_{(\phi_n(t),x_n(\theta_n(t)))}y_n, y_n - x_n(\phi_n(t)) \rangle \\ &\quad + \Big(b_t + L(1 + \varrho)\Big)\Big(\|x_n(\phi_n(t)) - x(t)\| + \|y_n - y\|\Big). \end{aligned}$$

Taking (2.35), (2.40) and (2.41) into account entail that

$$\langle \dot{x}(t) + f(t, x(t)), x(t) - y\rangle \leq \limsup_{n\to\infty} \langle \dot{x}_n(t) + f_n(t), x(t) - y\rangle \leq \langle A^0_{(t,x(t))} y, y - x(t)\rangle.$$

Thus, the differential inclusion (2.36) holds true. In other words, the evolution problem (1.1) has at least one absolutely continuous solution $x(\cdot) : I \to H$.
In view of (2.15) and the convergence above, it follows

$$\|x(t) - x(s)\| \leq \varrho\Big(\gamma(t) - \gamma(s)\Big), \text{ for } 0 \leq s \leq t \leq T,$$

that is,

$$\|x(t) - x(s)\| \leq \varrho(t - s + \alpha(t) - \alpha(s)), \text{ for } 0 \leq s \leq t \leq T,$$

then, the estimate (2.2) is fulfilled.
The proof of the theorem is therefore finished. □

To end this section, we impose extra assumptions to obtain the uniqueness of the solution to problem (1.1).

Theorem 2.2. *Suppose that assumptions of Theorem 2.1 hold true. Let for any $(t, y) \in I \times H$ the operator $A_{(t,y)}$ be hypo-monotone in the sense that for every $\eta > 0$, there exists a non-negative real function $\beta_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for any $t \in I$, for any $x_i \in \overline{B}_H(0, \eta)$ and for $z_i \in A_{(t,x_i)} x_i$, $i = 1, 2$, one has*

$$\langle z_1 - z_2, x_1 - x_2\rangle \geq -\beta_\eta(t)\|x_1 - x_2\|^2.$$

Moreover, suppose that for every $\rho > 0$, there exists a non-negative real function $\kappa_\rho(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for all $t \in I$ and for $x, y \in \overline{B}_H(0, \rho)$

$$\|f(t, x) - f(t, y)\| \leq \kappa_\rho(t)\|x - y\|. \tag{2.43}$$

Then, for any $x_0 \in \mathrm{D}\left(A_{(0,x_0)}\right)$, the evolution problem (1.1) has one and only one absolutely continuous solution which satisfies (2.2).

Proof. Existence of the solution follows from Theorem 2.1.
Let us study the uniqueness of the solution. Suppose that $x_1(\cdot)$ and $x_2(\cdot)$ are two solutions to problem (1.1). Since each solution satisfies (2.2), then, there exists $\eta > 0$ such that

$$\|x_i(t)\| \leq \eta, \quad i = 1, 2, \quad t \in I,$$

and

$$-\dot{x}_1(t) - f(t, x_1(t)) \in A_{(t,x_1(t))} x_1(t), \quad x_1(0) = x_0 \in \mathrm{D}\left(A_{(0,x_0)}\right) \quad \text{a.e. } t \in I,$$
$$-\dot{x}_2(t) - f(t, x_2(t)) \in A_{(t,x_2(t))} x_2(t), \quad x_2(0) = x_0 \in \mathrm{D}\left(A_{(0,x_0)}\right) \quad \text{a.e. } t \in I.$$

By the hypo-monotone property above, one gets

$$\langle -\dot{x}_1(t) - f(t, x_1(t)) + \dot{x}_2(t) + f(t, x_2(t)), x_1(t) - x_2(t)\rangle \geq -\beta_\eta(t)\|x_1(t) - x_2(t)\|^2,$$

then,

$$\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle + \langle f(t, x_1(t)) - f(t, x_2(t)), x_1(t) - x_2(t)\rangle \leq \beta_\eta(t)\|x_1(t) - x_2(t)\|^2.$$

Using the Lipschitz assumption on $f(t, \cdot)$ on bounded sets in (2.43), there exists $\kappa_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that

$$\frac{1}{2}\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 = \langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle$$
$$\leq \beta_\eta(t)\|x_1(t) - x_2(t)\|^2 + \kappa_\eta(t)\|x_1(t) - x_2(t)\|^2.$$

Thus, integrating over $[0, t]$ yields

$$\|x_1(t) - x_2(t)\|^2 \leq 2\int_0^t (\beta_\eta(s) + \kappa_\eta(s))\|x_1(s) - x_2(s)\|^2 ds,$$

and Gronwall's lemma allows to conclude that $x_1 \equiv x_2$. Consequently, the solution of (1.1) is unique. □

We close this section by the following particular case of the sweeping process.

Corollary 2.3. *Let $C : I \times H \rightrightarrows H$ be a set-valued map such that*
for each $(t, y) \in I \times H$, $C(t, y)$ is a non-empty closed convex subset of H;
there exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $\alpha \in W^{1,2}(I, \mathbb{R})$ which is non-negative on $[0, T[$ and non-decreasing with $\alpha(T) < \infty$ and $\alpha(0) = 0$ such that

$$d(x, C(t, u)) - d(x, C(s, v)) \leq |\alpha(t) - \alpha(s)| + \lambda\|v - u\|, \ \forall t, s \in I \text{ and } \forall x, v, u \in H;$$

for any $(t_i, y_i) \in I \times H,\ x_i \in C(t_i, y_i), i = 1, 2$, one has

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0.$$

Let for any $(t, y) \in I \times H$, the operator $N_{C(t,y)}$ be hypo-monotone in the sense that for every $\eta > 0$, there exists a non-negative real function $\beta_\eta(\cdot) \in L^1_{\mathbb{R}}(I)$ such that for any $t \in I$, for any $x_i \in \overline{B}_H(0, \eta)$ and for $z_i \in N_{C(t,x_i)} x_i,\ i = 1, 2$, one has

$$\langle z_1 - z_2, x_1 - x_2 \rangle \geq -\beta_\eta(t)\|x_1 - x_2\|^2.$$

Let $f : I \times H \to H$ be a map satisfying **Assumption 2** (1)-(2) *which is Lipschitz in the sense of* (2.43)*. Then, for any $x_0 \in C(0, x_0)$, the following perturbed sweeping process*

$$\begin{cases} -\dot{x}(t) \in N_{C(t,x(t))} x(t) + f(t, x(t)) & \text{a.e. } t \in I, \\ x(0) = x_0, \end{cases}$$

has a unique solution $x(\cdot) \in W^{1,2}(I, H)$.

Proof. Set for each $(t, y) \in I \times H$, $A_{(t,y)} = N_{C(t,y)}$. Then, this operator is maximal monotone and satisfies **Assumption 1** (1)-(2)-(3). Hence, all assumptions of Theorem 2.2 are satisfied. Theorem 2.1 yields the desired existence result, while Theorem 2.2 ensures the uniqueness of the solution. □

3 Application to optimal control theory

In this section we are interested in the existence of an optimal solution to a minimization problem subject to the differential inclusion studied in Section 2.

Theorem 3.1. *Let for any $(t, y) \in I \times H$, $A_{(t,y)} : \mathrm{D}\,(A_{(t,y)}) \subset H \rightrightarrows H$ be a maximal monotone operator satisfying assumptions of Theorem 2.2. Let $f : I \times H \to H$ be measurable on I such that for a non-negative real constant $l > 0$, one has*

$$\|f(t, x)\| \leq l, \quad \textit{and} \quad \|f(t, x) - f(t, y)\| \leq l\|x - y\|, \tag{3.1}$$

for all $t \in I$ and for all $x, y \in H$.
The cost functional $J : I \times H \times H \to [0, +\infty[$ is measurable, such that $J(t, \cdot, \cdot)$ is lower semi-continuous on $H \times H$ for every $t \in I$, and $J(t, x, \cdot)$ is convex on H for every $(t, x) \in I \times H$.
Define the set $\mathcal{V}$ by

$$\mathcal{V} := \{\ v \in L^\infty_{\mathbb{R}}(I)\ :\ |v(t)| \leq 1 \text{ a.e.}\}.$$

Then, the minimization problem

$$\min_{v \in \mathcal{V}} \int_0^T J(t, x_v(t), \dot{x}_v(t))dt, \tag{3.2}$$

has an optimal solution, where $x_v(\cdot)$ denotes the unique absolutely continuous solution associated with the control $v(\cdot) \in \mathcal{V}$, to the control problem

$$\begin{cases} -\dot{x}_v(t) \in A_{(t,x_v(t))} x_v(t) + v(t) f(t, x_v(t)) & \text{a.e. } t \in I, \\ x_v(0) = x_0 \in \mathrm{D}\,(A_{(0,x_0)}). \end{cases} \tag{3.3}$$

Proof. Note that the set $\mathcal{V}$ is $\sigma(L^\infty_{\mathbb{R}}(I), L^1_{\mathbb{R}}(I))$-compact and also $\sigma(L^1_{\mathbb{R}}(I), L^\infty_{\mathbb{R}}(I))$-compact. For each $v \in \mathcal{V}$, define the map $f_v(t,x) = v(t)f(t,x)$ for any $(t,x) \in I \times H$. Then, f_v satisfies the assumptions of Theorem 2.2. Consequently, the problem (3.3) has a unique absolutely continuous solution x_v, by Theorem 2.2.
Let $v_n(\cdot)$ be a minimizing sequence of problem (3.2) i.e.,

$$\lim_{n\to+\infty} \int_0^T J(t, x_{v_n}(t), \dot{x}_{v_n}(t))dt = \min_{w\in\mathcal{V}} \int_0^T J(t, x_w(t), \dot{x}_w(t))dt,$$

where $x_{v_n}(\cdot)$ denotes the unique absolutely continuous solution to

$$\begin{cases} -\dot{x}_{v_n}(t) \in A_{(t,x_{v_n}(t))}x_{v_n}(t) + v_n(t)f(t, x_{v_n}(t)) & \text{a.e. } t \in I, \\ x_{v_n}(0) = x_0 \in \mathrm{D}\left(A_{(0,x_0)}\right). \end{cases}$$

Since the set $\mathcal{V}$ is $\sigma(L^\infty_{\mathbb{R}}(I), L^1_{\mathbb{R}}(I))$-compact. Then, suppose that $(v_n(\cdot))$ $\sigma(L^\infty_{\mathbb{R}}(I), L^1_{\mathbb{R}}(I))$-converges in $L^\infty_{\mathbb{R}}(I)$ to $v \in \mathcal{V}$. Hence $(v_n(\cdot))$ $\sigma(L^1_{\mathbb{R}}(I), L^\infty_{\mathbb{R}}(I))$-converges to v.
From (2.2), there exists $\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that for any $n \geq 1$

$$\|\dot{x}_{v_n}(t)\| \leq \eta(t) \quad \text{a.e} \quad t \in I,$$

and

$$\sup_n \int_0^T \|\dot{x}_{v_n}(t)\|^2 dt < constant < +\infty.$$

Remark by (3.1) and the construction of the set $\mathcal{V}$ that

$$\sup_n \|v_n(t)f(t, x_{v_n}(t))\| \leq l \quad \text{a.e } \ t \in I. \tag{3.4}$$

Thus, there is $x \in W^{1,2}(I, H)$ such that

$$(x_{v_n}) \text{ converges pointwisely to } x, \tag{3.5}$$

$$(\dot{x}_{v_n}) \text{ converges weakly to } \dot{x} \text{ in } L^1_H(I). \tag{3.6}$$

The Lipschitz property of $f(t,\cdot)$ in (3.1) along with (3.5), yields

$$f(t, x_{v_n}(t)) \to f(t, x(t)) \ \text{ a.e } t \in I. \tag{3.7}$$

Moreover, since f is measurable on I along with (3.1) and (3.7), the Lebesgue dominated convergence theorem entails that

$$(f(\cdot, x_{v_n}(\cdot))) \text{ converges to } f(\cdot, x(\cdot)) \text{ in } L^1_H(I).$$

By (3.4), there exists $z(\cdot) \in L^1_H(I)$ such that

$$(v_n(\cdot)f(\cdot, x_{v_n}(\cdot))) \text{ converges weakly in } L^1_H(I) \text{ to } z(\cdot)\ .$$

It remains to prove that

$$(v_n(\cdot)f(\cdot, x_{v_n}(\cdot))) \text{ converges weakly in } L^1_H(I) \text{ to } v(\cdot)f(\cdot, x(\cdot)).$$

Let $h \in L^\infty_H(I)$, then, one has for each $t \in I$

$$\langle h(t), v_n(t)f(t, x_{v_n}(t))\rangle = \langle v_n(t)h(t), f(t, x_{v_n}(t))\rangle,$$

and

$$\sup_n \|v_n(t)h(t)\| \leq \|h\|_{L^\infty_H(I)} \text{ and } \|v(t)h(t)\| \leq \|h\|_{L^\infty_H(I)}.$$

Observe that

$$\begin{aligned}
& \Big|\int_0^T \langle h(t), v_n(t) f(t, x_{v_n}(t))\rangle dt - \int_0^T \langle h(t), v(t) f(t, x(t))\rangle dt\Big| \\
= & \Big|\int_0^T \langle v_n(t) h(t), f(t, x_{v_n}(t))\rangle dt - \int_0^T \langle v(t) h(t), f(t, x(t))\rangle dt\Big| \\
= & \Big|\int_0^T \langle v_n(t) h(t), f(t, x_{v_n}(t)) - f(t, x(t))\rangle dt + \int_0^T \langle (v_n(t) - v(t)) h(t), f(t, x(t))\rangle dt\Big| \\
\leq & \Big|\int_0^T \langle v_n(t) h(t), f(t, x_{v_n}(t)) - f(t, x(t))\rangle dt\Big| + \Big|\int_0^T \langle (v_n(t) - v(t)) h(t), f(t, x(t))\rangle dt\Big|.
\end{aligned}$$

Since $(v_n(\cdot)h(\cdot))$ converges weakly to $v(\cdot)h(\cdot)$ in $L^1_H(I)$ and $f(\cdot, x(\cdot)) \in L^\infty_H(I)$, then, one concludes that

$$\lim_{n\to\infty} \int_0^T \langle v_n(t)h(t), f(t, x(t))\rangle dt = \int_0^T \langle v(t)h(t), f(t, x(t))\rangle dt.$$

Since $(f(\cdot, x_{v_n}(\cdot)))$ is uniformly bounded, and $(f(\cdot, x_{v_n}(\cdot)))$ converges pointwisely to $f(\cdot, x(\cdot))$, along with the fact that $(v_n(\cdot)h(\cdot))$ is bounded in $L^\infty_H(I)$ and $(v_n(\cdot)h(\cdot))$ $\sigma(L^\infty_H(I), L^1_H(I))$-converges to $v(\cdot)h(\cdot)$, it results that $(f(\cdot, x_{v_n}(\cdot)))$ converges to $f(\cdot, x(\cdot))$ with respect to the Mackey topology $\tau(L^\infty_H(I), L^1_H(I))$ (see [11]), that is,

$$\lim_{n\to\infty} \int_0^T \langle v_n(t)h(t), f(t, x_{v_n}(t))\rangle dt = \int_0^T \langle v_n(t)h(t), f(t, x(t))\rangle dt.$$

Combining the two last equalities gives

$$\lim_{n\to+\infty} \int_0^T \langle h(t), v_n(t) f(t, x_{v_n}(t))\rangle dt = \int_0^T \langle h(t), v(t) f(t, x(t))\rangle dt,$$

i.e.,

$$(v_n(\cdot) f(\cdot, x_{v_n}(\cdot))) \text{ converges weakly in } L^1_H(I) \text{ to } v(\cdot) f(\cdot, x(\cdot)). \tag{3.8}$$

Now, an application of the lower semi-continuity for integral functionals (see Theorem 8.1.6 [14]) yields

$$\liminf_{n\to\infty} \int_0^T J(t, x_{v_n}(t), \dot{x}_{v_n}(t)) dt \geq \int_0^T J(t, x(t), \dot{x}(t)) dt.$$

Thus, it follows that

$$\inf_{w\in\mathcal{V}} \int_0^T J(t, x_w(t), \dot{x}_w(t)) dt = \int_0^T J(t, x(t), \dot{x}(t)) dt.$$

Let us verify that

$$-\dot{x}(t) - v(t) f(t, x(t)) \in A_{(t,x(t))} x(t) \quad \text{a.e. } t \in I, \tag{3.9}$$

$$x(t) \in \mathrm{D}\,(A_{(t,x(t))}). \tag{3.10}$$

Recall that for each n, one has

$$-\dot{x}_{v_n}(t) - v_n(t) f(t, x_{v_n}(t)) \in A_{(t,x_{v_n}(t))} x_{v_n}(t) \quad \text{a.e. } t \in I.$$

From the proceeding convergence modes (see (3.5)-(3.6) and (3.8)) then, arguing as in **Part 3** of the proof of Theorem 2.1, the inclusions (3.9)-(3.10) hold true. We omit the details for shortness. Since the solution of (3.9) is unique (see Theorem 2.2), one concludes that $x(\cdot) = x_v(\cdot)$ where $x_v(\cdot)$ is the unique absolutely continuous solution associated with the control v to problem (3.3). This completes the proof of the theorem. □

We close this section by the following corollary.

Corollary 3.2. *Let $C : I \times H \rightrightarrows H$ be a set-valued map such that assumptions of Corollary 2.3 hold true. Let the maps $f : I \times H \to H$, $J : I \times H \times H \to [0, +\infty[$, and the set $\mathcal{V}$ satisfy assumptions of Theorem 3.1. Then, the minimization problem*

$$\min_{v \in \mathcal{V}} \int_0^T J(t, x_v(t), \dot{x}_v(t))dt,$$

has an optimal solution, where $x_v(\cdot)$ denotes the unique absolutely continuous solution associated with the control $v(\cdot) \in \mathcal{V}$, to the control sweeping process

$$\begin{cases} -\dot{x}_v(t) \in N_{C(t,x_v(t))}x_v(t) + v(t)f(t, x_v(t)) & \text{a.e. } t \in I, \\ x_v(0) = x_0 \in C(0, x_0). \end{cases}$$

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