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Stochastic Analysis on Interaction between Palm Leaf and Caterpillar Life-Cycle

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Abstract *In this paper, stochastic analysis of interaction between palm leaf and caterpillar life-cycle is presented analytically and numerically. Existence, stabilities and extinction are analysed theoretically. Further, the results explained theoretically are simulated by using numerical representation.*

1 Introduction

In Indonesia, the food and energy industries have mostly benefited from palm oil during the past ten years, and it has also made a big impact on global trade (cites [4] and [6]). Additionally, palm oil has grown to be a significant domestic and global commodity as well as an important Indonesian plantation product. Large-scale palm oil production necessitates substantial land modification, which contributes to the loss of primary forests. The usage of pesticides that have an adverse effect on the environment is one of the issues in the sector. The biggest challenges that might drastically lower production are pests and illnesses. There are many causes for the emergence of insect attacks on plants. Therefore, insect infestations must be controlled with minimal impact on the environment [7]. Insect predators, parasitoids, and pollinators are only a few of the non-targeted pests that are negatively impacted by the extensive use of insecticides [8]. The latter situation was noticed in 2012 when a new caterpillar, Pseudoresia desmierdechenoni, began to appear. In Batubara, North Sumatra, this outbreak has resulted in a substantial loss of oil palm [5]. The persistent emergence of nettle caterpillars in Indonesian plantations is a sign that helpful insects that were effective in keeping the pest under control economically have disappeared. These caterpillars become oil palm defoliators and drastically reduce productivity by eating the leaves of either young or elderly palms [7]. To preserve sustainable crop production, the use of biological management by natural predators is being preferred [1]. For construction of ordinary differential equation system model, Syukriyah et al. (2019) [7] assumed that the entire life cycle of a caterpillar is influenced by food availability (leaf surface area) and its interaction with the predator as follows:

$$\begin{cases} dM = (\alpha P - \beta M)dt \\ dE = (\sigma M - \delta E)dt \\ dL = (\delta E - \theta L - aL(1-S) - cLR)dt \\ dS = (\eta(1-S) - dL)dt \\ dP = (\theta L - \alpha P)dt \\ dR = (\tau R(1-bR) + eLR - \mu R)dt \end{cases} \tag{1.1}$$

where parameters mean:

(i) $\alpha \equiv$ *Transition rate from pupa to moth,*

(ii) $\beta \equiv$ *Moth natural death rate,*

(iii) $\sigma \equiv$ *Egg production rate by a moth,*

(iv) $\delta \equiv$ *Transition rate from egg to larva,*

(v) $\theta \equiv$ *Transition rate from larva to pupa,*

(vi) $\eta \equiv$ *Growth rate of leaf surface area,*

(vii) $\tau \equiv$ *Predator natural growth rate,*

(viii) $\mu \equiv$ *Predator natural death rate,*

(ix) $a \equiv$ *Interaction coefficient between* L *and* S*,*

(x) $b \equiv$ *Increase of natural carrying capacity of* R*,*

(xi) $c \equiv$ *Interaction coefficient between* L *and* R*,*

(xii) $e \equiv$ *Interaction coefficient between* L *and* R*.*

As was mentioned in [7], there have been many random situations that have affected palm tree environment. For that reason, it is necessary tu study model (1.1) under random conditions, in this case, under environmental variability. In order to define the stochastic model, we shall consider that moth natural death rate (β)*, transition rate from egg to larva* (δ)*, transition rate from larva to pupa* (θ)*, growth rate of leaf surface area* (η)*, transition rate from pupa to moth* (α) *and predator natural death rate* (μ)*; since those rate can be susceptible of a white noise (environmental variability) [3]. Therefore, stochastic predator-prey model is given as follows:*

$$\begin{cases} dM = (\alpha P - \beta M)dt - \sigma_1 M dB_1(t) \\ dE = (\sigma M - \delta E)dt - \sigma_2 E dB_2(t) \\ dL = (\delta E - \theta L - aL(1-S) - cLR)dt - \sigma_3 L dB_3(t) \\ dS = (\eta(1-S) - \alpha L)dt - \sigma_6 S dB_6(t) \\ dP = (\theta L - \alpha P)dt - \sigma_4 P dB_4(t) \\ dR = (\tau R(1-bR) + eLR - \mu R)dt - \sigma_5 R dB_5(t), \end{cases} \tag{1.2}$$

where $\sigma_i B_i(t)$ *for* $i = 1, 2, 3, 4, 5, 6$ *are independent each other.*

2 Main Results

The coefficients of model (1.2) are continuos and locally Lipschitz. For instance, we can show that in a finite time, the solution does not diverge, thus it has a positive $\mathbb{R}^6_+ = \{(a, b, c, d, e, f) \in \mathbb{R}^6 : a > 0, b > 0, c > 0, d > 0, e > 0, f > 0\}$ *as an invariant set. Therefore, the following theorem come up with:*

Theorem 2.1. *For initial values* $(M(0), E(0), L(0), S(0), P(0), R(0)) \in \mathbb{R}^6_+$*, the system (1.2) has an unique solution* $(M(t), E(t), L(t), S(t), P(t), R(t))$ *for all* $t \geq 0$ *and the solution remains in* $\mathbb{R}^6_+$ *with probability one.*

Proof. Let $q = (q_1, q_2, q_3, q_4, q_5, q_6)$ y $u = (u_1, u_2, u_3, u_4, u_5, u_6)$, where

$$
\begin{aligned}
q_1 &= \alpha P - \beta M \\
q_2 &= \sigma M - \delta E \\
q_3 &= \delta E - \theta L - aL(1-S) - cLR \\
q_4 &= \eta(1-S) - dL \\
q_5 &= \theta L - \alpha P \\
q_6 &= \tau R(1-bR) + eLR - \mu R \\
u_1 &= -\sigma_1 M \\
u_2 &= -\sigma_2 E \\
u_3 &= -\sigma_3 L \\
u_4 &= -\sigma_6 S \\
u_5 &= -\sigma_4 P \\
u_6 &= -\sigma_5 R
\end{aligned}
$$

Lyapunov's operator associated (1.1) is given by

$$
\begin{aligned}
L =& q_1 \frac{\partial}{\partial M} + q_2 \frac{\partial}{\partial E} + q_3 \frac{\partial}{\partial L} + q_4 \frac{\partial}{\partial S} + q_5 \frac{\partial}{\partial P} + q_1 \frac{\partial}{\partial R} \\
&+ \frac{1}{2} u_1^2 \frac{\partial^2}{\partial M^2} + \frac{1}{2} u_2^2 \frac{\partial^2}{\partial E^2} + \frac{1}{2} u_3^2 \frac{\partial^2}{\partial L^2} + \frac{1}{2} u_4^2 \frac{\partial^2}{\partial S^2} \\
&+ \frac{1}{2} u_5^2 \frac{\partial^2}{\partial P^2} + \frac{1}{2} u_6^2 \frac{\partial^2}{\partial R^2}.
\end{aligned}
$$

Now, let's define $V : \mathbb{R}_+^6 \times [0, \infty) \to [0, \infty)$ by

$$
\begin{aligned}
V(M, E, L, S, P, R) =& M - 1 - ln(M) + E - 1 - ln(E) + L - 1 - ln(L) \\
&+ S - 1 - ln(S) + P - 1 - ln(P) + R - 1 - ln(R).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&q_1\frac{\partial V}{\partial M}=\alpha P-\beta M-\frac{\alpha P}{M}+\beta,\\
&q_2\frac{\partial V}{\partial E}=\sigma M-\delta E-\frac{\sigma M}{E}+\delta,\\
&q_3\frac{\partial V}{\partial L}=\delta E-\theta L-aL+aLS-cLR-\frac{\delta E}{L}+\theta+a+aS+cR,\\
&q_4\frac{\partial V}{\partial S}=\eta-\eta S-\alpha P-\frac{\eta}{S}+\eta+\frac{\alpha P}{S},\\
&q_5\frac{\partial V}{\partial P}=\theta L-\alpha P-\frac{\theta L}{P}+\alpha,\\
&q_6\frac{\partial V}{\partial R}=\tau R-\tau bR^2+eLR-\mu R-\tau+\tau bR-eL+\mu,\\
&\frac{u_1^2}{2}\frac{\partial^2 V}{\partial M^2}=\sigma_1^2\frac{1}{2},\\
&\frac{u_2^2}{2}\frac{\partial^2 V}{\partial E^2}=\sigma_2^2\frac{1}{2},\\
&\frac{u_3^2}{2}\frac{\partial^2 V}{\partial L^2}=\sigma_3^2\frac{1}{2},\\
&\frac{u_4^2}{2}\frac{\partial^2 V}{\partial S^2}=\sigma_6^2\frac{1}{2},\\
&\frac{u_5^2}{2}\frac{\partial^2 V}{\partial P^2}=\sigma_4^2\frac{1}{2},\\
&\frac{u_6^2}{2}\frac{\partial^2 V}{\partial R^2}=\sigma_5^2\frac{1}{2},
\end{aligned}
$$

Hence, $LV(M,E,L,S,P,R)$ is given by

$$
\begin{aligned}
LV(M,E,L,S,P,R) &\leq \alpha P+\sigma M+\delta E+\theta L+R(a+\tau+\tau b)\\
&+(\sigma_1^2/2+\sigma_2^2/2+\sigma_3^2/2+\sigma_4^2/2+\sigma_5^2/2+\sigma_6^2/2\\
&+2\eta+\beta+\delta+\theta+a+\alpha+\mu)\\
&\leq \max\{\alpha,\sigma,\delta,\theta,K_1,1\}(P+M+E+L+R+S)+K_2
\end{aligned}
$$

where $K_1=\tau+\tau b$ y $K_2=\sigma_1^2/2+\sigma_2^2/2+\sigma_3^2/2+\sigma_4^2/2+\sigma_5^2/2+\sigma_6^2/2+2\eta+\beta+\delta+\theta+a+\alpha+\mu$

$$LV(M,E,L,S,P,R)\leq 2K_3(P+M+E+L+R+S)+K_2$$

where $K_3=\max\{\alpha,\sigma,\delta,\theta,K_1,1\}$. Thus,

$$
\begin{aligned}
LV(M,E,L,S,P,R) &\leq 2K_3(V(M,E,L,S,P,R)+3)+K_2\\
&=K_4V(M,E,L,S,P,R)+K_5.
\end{aligned}
$$

where $K_4=2K_3$ y $K_5=6K_3+K_2$.

The arguments below are standard for the theory of stochastic differential equations. Now, let $W(0)\equiv(M(0),E(0),L(0),S(0),P(0),R(0))\in\mathbb{R}_+^6$ fixed and let $m_0\in\mathbb{N}$ sufficiently large, thus $\min\{|M(0)|,|E(0)|,|L(0)|,|S(0)|,|P(0)|,|R(0)|\}>1/m$ for all $m>m_0$. Let's define the following stopping time $\tau_m=\inf\{t>0: W(t)\notin[1/n,n]^6\}$ where $W(t)=(M(t),E(t),L(t),S(t),P(t),R(t))$, for all $m\geq m_0$. It can be seen that τ_m is increasing in m if $\tau_\infty \lim_{m\to\infty}\tau_m$, then $\tau_\infty\leq\tau_e$ with probability one, where τ_e is the explosion time defined as

$\tau_e = \inf\{t \geq 0 : |W(t)| = \infty\}$. We shall show that $\mathbb{P}[\tau_\infty = \infty] = 1$, which implies that, with probability one, the process remains in $\mathbb{R}^6_+$ besides, it does not diverges in a finite time, because $\tau_e = \infty$ with probability one. Let's consider the opposite, that is $\mathbb{P}[\tau_\infty = \infty] < 1$. Therefore, there exists $T > 0$ and $\eta > 0$ such that $\mathbb{P}[\tau_\infty < T] > \eta$. Then, there is $m_1 \geq m_0$ such that $\mathbb{P}[\tau_m < T] > \eta$, for all $m \geq m_1$. Applying the Itô formula,

$$dV(W(t),t) = LV(W(t),t)dt + \frac{\partial V}{\partial M}dB_1(t) + \frac{\partial V}{\partial E}dB_2(t) + \frac{\partial V}{\partial L}dB_3(t) + \frac{\partial V}{\partial S}dB_4(t) + \frac{\partial V}{\partial P}dB_5(t) + \frac{\partial V}{\partial R}dB_6(t).$$

Thus, $V(W(\tau_n \wedge T), \tau_n T)$ is equal to

$$V(W(0),0) + \int_0^{\tau_n \wedge T} LV(X(\tau_n \wedge T), \tau_n \wedge T)dt + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial M}dB_1(t) + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial E}dB_2(t) + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial L}dB_3(t) + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial S}dB_4(t) + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial P}dB_5(t) + \int_0^{\tau_n \wedge T} \frac{\partial V}{\partial R}dB_6(t)$$

By getting the expected value and by using $LV(M,E,L,S,P,R,t) \leq K_4V(M,E,L,S,P,R,t) + K_5$ and the assumption that $\mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial M}dB_1(t)\right] = \mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial E}dB_2(t)\right] = \mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial L}dB_3(t)\right] = \mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial S}dB_4(t)\right] = \mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial P}dB_5(t)\right] = \mathbb{E}\left[\int_0^{\tau_n \wedge T} \frac{\partial V}{\partial R}dB_6(t)\right] = 0$. Therefore, we get

$$\begin{aligned}\mathbb{E}V(W(\tau_n \wedge T), \tau_n \wedge T) &\leq V(W(0),0) + \mathbb{E}\int_0^{\tau_n \wedge T} (K_4V(M,E,L,S,P,R,t) + K_5)dt \\ &\leq V(W(0),0) + \mathbb{E}\int_0^{T} (K_4V(W(\tau_n \wedge T)), \tau_n \wedge T) + K_5)dt \\ &\leq V(W(0),0) + K_5T + K_4\mathbb{E}\int_0^{T} V(W(\tau_n \wedge T)), \tau_n \wedge T)dt.\end{aligned}$$

Now, by Fubini's Theorem,

$$\mathbb{E}\int_0^{T} V(W(\tau_n \wedge T)), \tau_n \wedge T)dt = \int_0^{T} \mathbb{E}V(W(\tau_n \wedge T)), \tau_n \wedge T)dt$$

and from Gronwall's inequality, we have

$$\mathbb{E}V(W(\tau_n \wedge T)), \tau_n \wedge T) \leq [V(W(0),0) + K_5T]e^{K_4T}.$$

Furthermore, if $k(n) = \min\{1/n - 1 - log(1/n), n - 1 - log(n)\}$, therefore

$$\mathbb{E}V(W(\tau_n \wedge T)), \tau_n \wedge T)1_{\{\tau_n < T} \geq k(n)\mathbb{P}[\tau_n < T] \geq k(n)\eta.$$

Nonetheless, the previous inequality contradicts the fact that $\mathbb{E}V(W(\tau_n \wedge T)), \tau_n \wedge T) \leq [V(W(0),0) + K_5T]e^{K_4T}$. Hence, $\mathbb{P}[\tau_\infty = \infty] = 1$. □

Next, we show some stability results.

Definition 2.2. [3] The trivial solution of a stochastic differential equation is called stable in probability if for all $t \geq 0$ and for any $\varepsilon \in (0,1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that

$$\mathbb{P}(\|x(t)\| < r \text{ for all } t \geq 0) \geq \varepsilon,$$

for any $\|x(0)\| < \delta$.

Theorem 2.3. *(See Theorem 2.2 in [3]) If there exists a function $V(x,t) \in C^{2,1} \in (\mathbb{R}^d_+ \times \mathbb{R}_+, \mathbb{R}_+)$ positive-definitive such that $LV(x,t) \leq 0$ for all $(x,t) \in \mathbb{R}^d_+ \times \mathbb{R}_+$, then the trivial solution of the stochastic differential equation is stable in probability.*

Theorem 2.4. *Let's consider the stochastic model (1.2). Then, the trivial solution of (1.2) is said to be stable in probability, if the following conditions hold:*

(i) $\sigma \leq \beta$,

(ii) $\tau \leq \mu$.

Proof. Let's define $V(M,E,L,S,P,R,t) = M+E+L+S+P+R \in C^{2,1} \in (\mathbb{R}^6_+ \times \mathbb{R}_+, \mathbb{R}_+)$; we can see that $V(M,E,L,S,P,R,t)$ is positive-definitive. Therefore, Lyapunov's operator associated to (1.2) over $C^{2,1}(M,E,L,S,P,R,t)$, is given by

$$\begin{aligned} L = & q_1 \frac{\partial}{\partial M} + q_2 \frac{\partial}{\partial E} + q_3 \frac{\partial}{\partial L} + q_4 \frac{\partial}{\partial S} + q_5 \frac{\partial}{\partial P} + q_1 \frac{\partial}{\partial R} \\ & + \frac{1}{2} u_1^2 \frac{\partial^2}{\partial M^2} + \frac{1}{2} u_2^2 \frac{\partial^2}{\partial E^2} + \frac{1}{2} u_3^2 \frac{\partial^2}{\partial L^2} + \frac{1}{2} u_4^2 \frac{\partial^2}{\partial S^2} \\ & + \frac{1}{2} u_5^2 \frac{\partial^2}{\partial P^2} + \frac{1}{2} u_6^2 \frac{\partial^2}{\partial R^2}. \end{aligned}$$

where,

$$\begin{aligned} q_1 &= \alpha P - \beta M \\ q_2 &= \sigma M - \delta E \\ q_3 &= \delta E - \theta L - aL \\ q_4 &= \eta - \eta S - \alpha L \\ q_5 &= \theta L - \alpha P \\ q_6 &= \tau R - \mu R \\ u_1 &= -\sigma_1 M \\ u_2 &= -\sigma_2 E \\ u_3 &= -\sigma_3 L \\ u_4 &= -\sigma_6 S \\ u_5 &= -\sigma_4 P \\ u_6 &= -\sigma_5 R \end{aligned}$$

By mentioned above, we can see that

$$LV = -\beta M + \sigma M - aL + \eta - \eta S - \alpha L + \tau R - \mu R$$

If $\sigma \leq \beta$ and $\tau \leq \mu$, then $LV \leq 0$.
Therefore, by theorem (2.3), trivial solution of (1.2) is stable in probability. □

Definition 2.5. [3] The trivial solution of a stochastic differential equation is said to be stochastically asymptotically stable in probability if for every $\varepsilon \in (0,1)$ there exists $\delta = \delta(\varepsilon)$ such that

$$\mathbb{P}\left(\lim_{t \to \infty} x(t) = 0\right) \geq 1 - \varepsilon,$$

for any $\|x(0)\| < \delta$.

Theorem 2.6. *(See Theorem 2.4 in [3]) If there exists a positive-define decreasing unbounded fucntion $V(x,t) \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R}_+)$ such that $Lv(x,t)$ is negatove-definitive, the the trivial solution of the stochastic differential equation is stochastically stable in the large.*

Theorem 2.7. *(See Theorem 7.1 in [2]) If the trivial solution of the linear system associated to a stochastic differential equation is stochastically asymptotically stable, then the trivial solution with respect to the stochastic differential equation is stochastically stable.*

Theorem 2.8. *Consider the stochastic model (1.2), then the trivial solution of (1.2) is stochastically asymptotically stable in probability if the following conditions holds:*

(i) $(\sigma_1^2 - \beta)(\sigma_2^2 - \delta) < \frac{\alpha\sigma}{4}$,

(ii) $\eta < 2\sigma_6^2$

(iii) $\sigma_4^2 < \alpha$ *or* $\tau + \sigma_5^2 < \mu$

(iv) $\sigma_3^2 > \theta + a$

(v) $(\sigma_2^2 - \delta)(\sigma_3^2 - \theta - a) < \frac{\sigma\delta}{4}$

Proof. According to Theorems (2.6) and (2.7) , the model (1.2) should be linearised around the trivial solution, thus

$$\begin{cases} dM = (\alpha P - \beta M)dt - \sigma_1 M dB_1(t) \\ dE = (\sigma M - \delta E)dt - \sigma_2 E dB_2(t) \\ dL = (\delta E - \theta L - aL)dt - \sigma_3 L dB_3(t) \\ dS = (\eta - \eta S - \alpha L)dt - \sigma_6 S dB_6(t) \\ dP = (\theta L - \alpha P)dt - \sigma_4 P dB_4(t) \\ dR = (\tau R - \mu R)dt - \sigma_5 R dB_5(t), \end{cases} \tag{2.1}$$

Now, let

$$\begin{aligned} q_1 &= \alpha P - \beta M \\ q_2 &= \sigma M - \delta E \\ q_3 &= \delta E - \theta L - aL \\ q_4 &= \eta - \eta S - \alpha L \\ q_5 &= \theta L - \alpha P \\ q_6 &= \tau R - \mu R \\ u_1 &= -\sigma_1 M \\ u_2 &= -\sigma_2 E \\ u_3 &= -\sigma_3 L \\ u_4 &= -\sigma_6 S \\ u_5 &= -\sigma_4 P \\ u_6 &= -\sigma_5 R \end{aligned}$$

By (2.6) and (2.7), it is enough to define a function $V(M,E,L,S,P,R,t) \in C^{2,1}(\mathbb{R}_+^6 \times \mathbb{R}_+;\mathbb{R}_+)$ positive-definite. Now, we define Lyapunov's operator associated to (1.2) is given by:

$$\begin{aligned} L =& q_1 \frac{\partial}{\partial M} + q_2 \frac{\partial}{\partial E} + q_3 \frac{\partial}{\partial L} + q_4 \frac{\partial}{\partial S} + q_5 \frac{\partial}{\partial P} + q_1 \frac{\partial}{\partial R} \\ &+ \frac{1}{2} u_1^2 \frac{\partial^2}{\partial M^2} + \frac{1}{2} u_2^2 \frac{\partial^2}{\partial E^2} + \frac{1}{2} u_3^2 \frac{\partial^2}{\partial L^2} + \frac{1}{2} u_4^2 \frac{\partial^2}{\partial S^2} \\ &+ \frac{1}{2} u_5^2 \frac{\partial^2}{\partial P^2} + \frac{1}{2} u_6^2 \frac{\partial^2}{\partial R^2}. \end{aligned}$$

Let's define

$$V(M,E,L,S,P,R,t) = c_1M^2 + c_2E^2 + c_3L^2 + c_6S^2 + c_4P^2 + c_5R^2$$

where $c_i, i = 1,2,3,4,5,6$, are non-negative constants. Applying L in V, we have

$$\begin{aligned} LV =& 2c_1\alpha MP - 2c_1\beta M^2 + 2c_1\sigma_1^2 M^2 + 2c_2\sigma ME - 2c_2\delta E^2 + 2c_2\sigma_2^2 E^2 \\ &+ 2c_3\delta EL - 2c_3(\theta + a)L^2 + 2c_3\sigma_3^2 L^2 + 2c_6\eta S - 2c_6\eta S^2 \\ &- 2c_6\alpha PS + 2c_6\sigma_6^2 S^2 + 2c_4\theta LP - 2c_4\alpha P^2 + 2c_4\sigma_4^2 P^2 \\ &+ 2c_5(\tau - \mu)R^2 + 2c_5\sigma_5^2 R^2 \end{aligned}$$

$$\begin{aligned} LV =& M^2(2c_1\sigma_1^2 - 2c_1\beta) + E^2(2c_2\sigma_2^2 - 2c_2\delta) + L^2(2c_3\sigma_3^2 - 2c_3(\theta + a)) \\ & S^2(2c_6\sigma_6^2 - 2c_6\eta) + 2c_6\eta S + P^2(2c_4\sigma_4^2 - 2c_4\alpha) + R^2(2c_5(\tau - \mu) + 2c_5\sigma_5^2) \\ & MP(2c_1\alpha) + ME(2c_2\sigma) + EL(2c_3\delta) + LP(2c_4\theta) - PS(2c_6\alpha). \end{aligned}$$

Taking $c_1 = \dfrac{1}{2\alpha}$, $c_2 = \dfrac{1}{2\sigma}$, $c_3 = \dfrac{1}{2\delta}$, $c_4 = \dfrac{1}{2\theta}$, $c_5 = \dfrac{1}{2}$ y $c_6 = \dfrac{1}{2}$, we get

$$\begin{aligned} LV =& M^2(\sigma_1^2/\alpha - \beta\alpha) + E^2(\sigma_2^2/\sigma - \delta/\sigma) + L^2(\sigma_3^2/\delta - (\theta + a)/\delta) \\ & S^2(\sigma_6^2 - \eta) + \eta S + P^2(\sigma_4^2/\theta - \alpha/\theta) + R^2(\tau - \mu + \sigma_5^2) \\ & MP + ME + EL + LP - PS. \end{aligned}$$

Therefore, LV can be represented by $LV = AM_1^2 + BE^2 + CL^2 + DS^2 + EP^2 + FR^2\ GMP + HME + IEL + JLP - KPS + NS$, where $A = -\dfrac{\sigma_1^2 - \beta}{\alpha}$, $B = \dfrac{\sigma_2^2 - \delta}{\sigma}$, $C = \dfrac{\sigma_3^2 - \theta - a}{\delta}$, $D = \sigma_6^2 - \eta$, $E = \dfrac{\sigma_4^2 - \alpha}{\theta}$, $F = \tau - \mu + \sigma_5^2$, $N = \eta$ and $G = H = I = J = K = 1$. Now, consider the vector $y := (M,E,L,S,P,R)$ and coefficient matrix Q

$$\begin{pmatrix} 2A & H & 0 & 0 & G & 0 \\ H & 2B & I & 0 & 0 & 0 \\ 0 & I & 2C & 0 & J & 0 \\ 0 & 0 & 0 & 2D + N & -K & 0 \\ G & 0 & J & -K & 2E & 0 \\ 0 & 0 & 0 & 0 & 0 & 2F \end{pmatrix}$$

we can rewrite $LV = AM_1^2 + BE^2 + CL^2 + DS^2 + EP^2 + FR^2\ GMP + HME + IEL + JLP - KPS + NS$ in its quadratic form $LV = \dfrac{1}{2}y^TQy$.

Now, let's define the following sub-matrices on Q,

$$Q_1 = \begin{pmatrix} 2A & H \\ H & 2B \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 2D + N \\ J & -K \end{pmatrix}, Q_3 = \begin{pmatrix} 2E & 0 \\ 0 & 2F \end{pmatrix},$$

$$Q_4 = 2C, Q_5 = \begin{pmatrix} 2B & I \\ I & 2C \end{pmatrix}, Q_6 = 2F.$$

Thus, LV is negative-definitive since $det(Q_1) < 0$ if $(\sigma_1^2 - \beta)(\sigma_2^2 - \delta) < \frac{\alpha\sigma}{4}$; $det(Q_2) < 0$ if $\eta < 2\sigma_6^2$; $det(Q_3) < 0$ if $\sigma_4^2 < \alpha$ or $\tau + \sigma_5^2 < \mu$; $det(Q_4) > 0$ if $\sigma_3^2 > \theta + a$; $det(Q_5) < 0$ if $(\sigma_2^2 - \delta)(\sigma_3^2 - \theta - a) < \frac{\sigma\delta}{4}$ and $det(Q_6) > 0$. Therefore, LV is negative-definitive for the trajectories in $\mathbb{R}_+^6$, except in the point $(0,0,0,0,0,0)$. □

Definition 2.9. [3] The trivial solution of a stochastic differential equation is said to be stochastically asymptomatically stable in the large if it is stochastically stable and moreover for all $x(0) \in \mathbb{R}^d$,

$$\mathbb{P}\left(\lim_{t\to\infty} x(t) = 0\right) = 1$$

Remark 2.10. In the stochastic differential equation (1.2), we can see if number of month get extinction; number of egg, number of larva and number of pupa get extinction too. Thus, the following theorem show the conditions under which number of moth is getting extinction.

Theorem 2.11. *Let $(M(t), E(t), L(t), S(t), P(t), R(t))$ be the solution of of system (1.2) for any initial value $(M(0), E(0), L(0), S(0), P(0), R(0)) \in \mathbb{R}^6_+$. Then, the number of month get extinction exponentially with probability one, i.e.,*

$$\mathbf{P}\left(\lim_{t\to\infty} M(t) = 0\right) = 1$$

if $\alpha < \beta + \sigma_1^2$.

Proof. Let

$$\begin{aligned} dM &= (\alpha P - \beta M)dt - \sigma_1 M dB_1(t) \\ &\leq [\alpha(P+M) - \beta(M+P)]dt - \sigma_1(M+P)dB_1(t) \\ &= (\alpha-\beta)(M+P)dt - \sigma_1(M+P)dB_1(t). \end{aligned}$$

Let's define $M + P \equiv X$, then $dM \leq (\alpha-\beta)Xdt - \sigma_1 X dB_1(t)$. Then, let $X(t)$ be the unique solution of the equation:

$$\begin{aligned} dX(t) &= (\alpha-\beta)X(t)dt - \sigma_1 X dB_1(t) \\ X(0) &= M(0) + P(0). \end{aligned}$$

Now, let $W_1(0) = \dfrac{1}{M(0)+P(0)}$. Now, by applying Itô formula, we obtain

$$\begin{aligned} dW_1 &= \left(-(\alpha-\beta)W_1 + W_1\sigma_1\right)dt + \sigma_1 W_1 dB_1(t) \\ &\leq \left(\beta - \alpha + \sigma_1\right)W_1 dt + \sigma_1 W_1 dB_1(t), \end{aligned}$$

with $W_1(0) = \dfrac{1}{M(0)+P(0)}$. Now, by the numerical solution of stochastic differential equation, we get

$$W_1(t) \leq \frac{1}{M(0)+P(0)} e^{\beta-\alpha+\sigma_1^2)t+\sigma_1 B_1}$$

Therefore, we have

$$X(t) \leq (M(0)+P(0))e^{(\alpha-\beta-\sigma_1^2)t-\sigma_1 B_1} \tag{2.2}$$

where $M(t) \leq X(t)$. Now, form (2.2), we obtain

$$M(t) \leq (M(0)+P(0))e^{(\alpha-\beta-\sigma_1^2)t-\sigma_1 B_1},$$

for every $t \geq 0$. Therefore,

$$ln(M(t)) \leq ln((M(0)+P(0)) + (\alpha-\beta-\sigma_1^2)t - \sigma_1 B_1,$$

i.e.,

$$\frac{ln(M(t))}{t} \leq \frac{ln((M(0)+P(0))}{t} + \frac{(\alpha-\beta-\sigma_1^2)t}{t} - \frac{\sigma_1 B_1}{t}.$$

Now, by applying the strong law of large number for local martingale, we get

$$\begin{aligned}\limsup_{t\to\infty}\frac{ln(M(t))}{t} &\leq \limsup_{t\to\infty}\frac{(M(0)+P(0))}{t} + \frac{(\alpha-\beta-\sigma_1^2)t}{t} - \frac{\sigma_1 B_1}{t}\\ &= \alpha-\beta-\sigma_1^2\end{aligned}$$

and $\alpha-\beta-\sigma_1^2 < 0$ if $\alpha < \beta+\sigma_1^2$ which leads to

$$\mathbf{P}\left(\lim_{t\to\infty} M(t) = 0\right) = 1.$$

□

3 Numerical Simulations

Simulation 1*: This simulation shows the behaviour of the system under any circumstance at* $t = 20$ *with initial values* $(M, E, L, S, P, R) \equiv (10, 10, 9, 20, 20, 5)$ *and with parameters* $\alpha = 0.3$*,* $\beta = 0.2$*,* $\sigma = 0.2$*,* $\delta = 0.2$*,* $\theta = 0.3$*,* $\eta = 0.4$*,* $\tau = 0.3$*,* $\mu = 0.15$*,* $a = 0.1$*,* $b = 0.2$*,* $c = 0.3$*,* $e = 0.2$*,* $\sigma_1 = 0.03$*,* $\sigma_2 = 0.02$*,* $\sigma_3 = 0.25$*,* $\sigma_4 = 0.032$*,* $\sigma_5 = 0.045$ *and* $\sigma_6 = 0.05$*. We have* $(M, E, L, S, P, R) \equiv (2, 4, 0.5, 0.3, 0.7, 5)$.

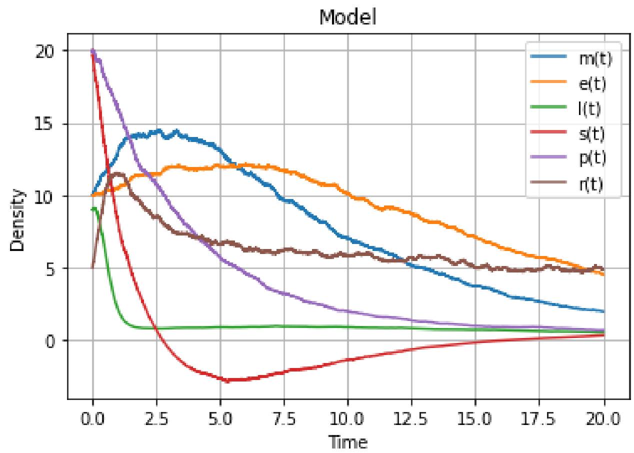

Figure 1*: Stochastic trajectories with white noise on the system.*

Simulation 2*: Taking values showed in Theorem (2.4), at* $t = 20$ *with initial values* $(M, E, L, S, P, R) \equiv (10, 10, 9, 20, 20, 5)$ *and with parameters* $\alpha = 0.01$*,* $\beta = 0.2$*,* $\sigma = 0.01$*,* $\delta = 0.2$*,* $\theta = 0.2$*,* $\eta = 0.1$*,* $\tau = 0.1$*,* $\mu = 0.3$*,* $a = 0.3$*,* $b = 0.01$*,* $c = 0.3$*,* $e = 0.03$*,* $\sigma_1 = 0.6$*,* $\sigma_2 = 0.01$*,* $\sigma_3 = 0.1$*,* $\sigma_4 = 0.5$*,* $\sigma_5 = 0.01$ *and* $\sigma_6 = 0.01$*. We have* $(M, E, L, S, P, R) \equiv (0.4, 0.2, 0.1, 0.22, 0.1, 0.9)$.

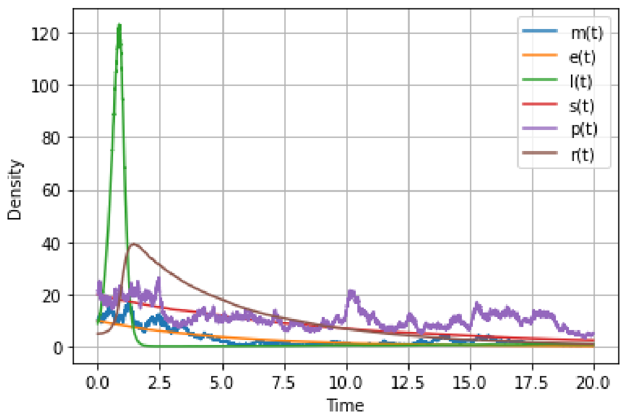

***Figure 2**: Stochastic trajectories with white noise on the system are described by (1.2) with parameters presented in (2.4).*

***Simulation 3**: Taking values showed in Theorem (2.8), at* $t = 20$ *with initial values* $(M, E, L, S, P, R) \equiv (10, 10, 9, 20, 20, 5)$ *and with parameters* $\alpha = 0.3$, $\beta = 0.2$, $\sigma = 0.01$, $\delta = 0.2$, $\theta = 0.01$, $\eta = 0.1$, $\tau = 0.1$, $\mu = 0.3$, $a = 0.03$, $b = 0.01$, $c = 0.3$, $e = 0.03$, $\sigma_1 = 0.6$, $\sigma_2 = 0.01$, $\sigma_3 = 0.4$, $\sigma_4 = 0.1$, $\sigma_5 = 0.01$ *and* $\sigma_6 = 0.4$. *We have* $(M, E, L, S, P, R) \equiv (0.4, 0.2, 0.3, 0.2, 0.06, 0.17)$.

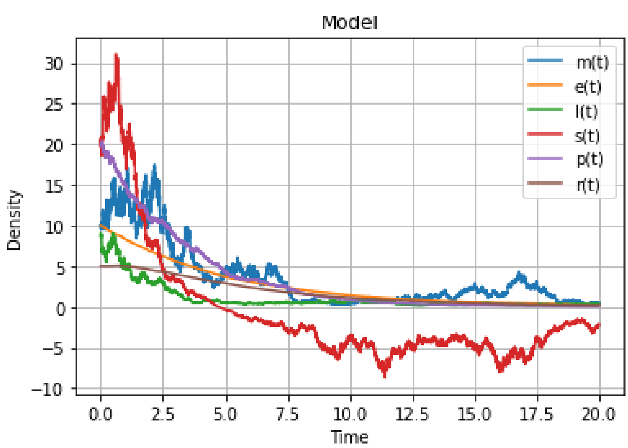

***Figure 3**: Stochastic trajectories with white noise on the system are described by (1.2) with parameters presented in (2.8).*

***Simulation 4**: The following simulation shows condition under which number of month, number of egg, number of larva and number of pupa get extinction. Besides, leaf surface area and number of larva's predator is maintained persistent. At* $t = 100$ *with initial values* $(M, E, L, S, P, R) \equiv (10, 10, 9, 20, 20, 5)$ *and with parameters* $\alpha = 0.3$, $\beta = 0.4$, $\sigma = 0.2$, $\delta = 0.2$, $\theta = 0.3$, $\eta = 0.4$, $\tau = 0.3$, $\mu = 0.15$, $a = 0.1$, $b = 0.2$, $c = 0.3$, $e = 0.2$, $\sigma_1 = 0.2$, $\sigma_2 = 0.02$, $\sigma_3 = 0.025$, $\sigma_4 = 0.032$, $\sigma_5 = 0.045$ *and* $\sigma_6 = 0.05$. *We have* $(M, E, L, S, P, R) \equiv (0, 0, 0, 1, 0, 3)$.

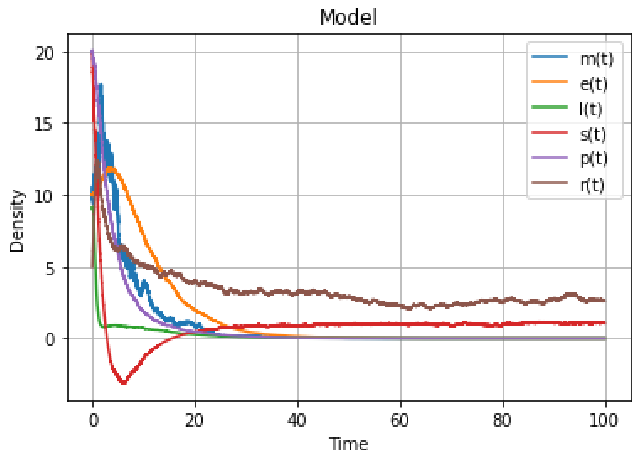

Figure 4: *Stochastic trajectories with white noise on the system are described by (1.2) with parameters presented in (2.11).*

4 Conclusion

A stochastic predator-prey interaction with two levels of predation between palm leaf, nettle caterpillar, and predator is represented in a system of stochastic differential equations as was shown in (1.2). The existence, probability stability and stochastically asymptotically stable in probability have been gotten in Theorems 2.1, 2.4 and 2.8, respectively. Further, some extinction conditions were establish (see Theorem 2.11). These results were supported with numerical simulations. For future works, we suggest conditions under which predators extinction can be determinated as well as ergodicity stationary distribution for persistent of the stochastic system (1.2).

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Conflicts of Interest

The author declares no conflict of interest.

Data availability

The data that support the findings of this study are available within the paper.

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