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μ-SYMMETRY ANALYSIS, μ-CONSERVATION LAWS, and EXACT SOLUTIONS OF THE MODIFIED EQUAL WIDTH EQUATION

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Abstract This study discusses the μ-symmetries, μ-conservation laws, and exact solutions of the modified equal width equation (MEWE). MEWE is used as a model in partial differential equations (PDE) to simulate one-dimensional wave transmission in nonlinear media with dispersion processes. First and foremost, we present some essential pieces of information about the offered techniques. In light of such information, we discover μ-symmetries. The essential idea behind the μ-symmetry approach is that it reduces one-independent variables in a system of PDEs by employing μ-symmetries and invariance surface conditions. The μ-symmetry method has been applied to MEWE and transformed into an ordinary differential equation (ODE). Then, we employ a modified version of the generalized exponential rational function method (mGERFM) to this reduced ODE to obtain soliton solutions. Thanks to the mGERFM, we discover unique wave solutions in the forms of exponential function solutions, combined periodic soliton solution, singular periodic wave solution, shock wave solutions, trigonometric function solutions, mixed-form soliton solution, hyperbolic solution in mixed form, and periodic soliton solution. Furthermore, by employing the variational problem procedure, we get the Lagrangian and the μ-conservation laws. The mGERFM, μ-symmetry analysis, and μ-conservation laws have not been discussed in previous investigations for the MEWE. We also demonstrate the properties with figures for these solutions. Here, we use Maple software to validate the complete outcomes of the study.

1 Introduction

Real-world phenomenons can be converted into mathematical language by employing NLPDEs. The solution of nonlinear partial differential equations (NLPDEs) plays an essential role in understanding the behavior of a complex system. The effort to find exact solutions to nonlinear equations is paramount for comprehending most nonlinear physical phenomena. Nonlinear wave phenomena occur in various scientific and engineering disciplines, such as solid-state physics, chemical physics, and geometry.

Recently, influential and efficient approaches to finding approximate and analytic solutions to nonlinear equations have attracted substantial interest from different groups of scientists, such as the extended (G'/G)-expansion method [1], Bifurcation analysis [2], Stability analysis [3], Functional variable method (FVM) [4], Residual power series method [5], Hirota bilinear method [6], Lie symmetry analysis [7], μ-symmetry analysis [8, 9, 10, 11], Differential transform method (DTM) [12], F-Expansion Procedure [13], Homotopy Perturbation Method [14], $S(\xi)$-expansion method [15], Convergence analysis [16], Exact and numerical solutions [17], and so on.

Lie symmetry procedure, first examined by S. Lie, is one of the most general and effective procedures for obtaining exact solutions for NLPDEs. A symmetry group of a differential equation means a transformation that maps (smooth) solutions to solutions. Lie employed a continuous group of transformations to develop solution methods for ODEs. ODEs with trivial Lie or no symmetries but possess λ-symmetries can be integrated using the λ-symmetry procedure. λ-symmetry was introduced by Muriel and Romero as a new kind of symmetry [18]. Morando and Gaeta viewed the case of PDEs and extended the λ-symmetries to the μ-symmetries [19, 20, 21]. In the case of the μ-symmetries of the Lagrangian, the conservation law is referred to as the μ-conservation law.

The principal purpose of the current study is to scrutinize the μ-symmetries, reductions, invariant solutions, and conservation laws for the MEWE.

The paper is assembled as follows. Section 2 offers the main concepts of the $\mu-$symmetry, conservation law of μ and the mGERFM. In Section 3, firstly, we introduce the MEWE, then we yield the μ-symmetries of the MEWE and construct the invariant solutions of the model by employing the accepted μ-symmetries. Also, Section 3 is devoted to reducing the MEWE into an ODE using similarity variables. Then, the exact solutions for the MEWE are obtained by using mGERFM. In Section 4, we obtain Lagrangian in potential form using the variational problem method and the Frechet derivative. The conservation law of μ is investigated in Section 5 for the MEWE. Lastly, in Section 6, conclusions are given.

2 Portrayal of offered techniques

2.1 μ-symmetry analysis

Surmise that $\mu = \lambda_i dx_i$ be a semi basic one-form on first order jet space $(J^{(1)}\aleph, \pi, \aleph)$, which is compatible, namely, $\wp_j \lambda_i = \wp_i \lambda_j$ [8, 9, 10, 11, 21]. Here, $\wp_i$ and $\wp_j$ are total derivative with respect to x_i, and λ_i defines from $J^{(1)}\aleph$ to $\mathbb{R}$.

Think that Δ be the sth-order partial differential equation (PDE) as follows

$$\Delta : Z(x, w^{(s)}) = 0. \tag{2.1}$$

Here, $w = w(x) = w(x_1, x_2, ..., x_p)$ and $w^{(s)}$ symbolizes all sth order derivatives of w as to x.

Let Ω be a vector field on $J^{(s)}\aleph$. Then, we describe the Ω as

$$\Omega = \Upsilon + \sum_{|J|=1}^{s} \psi_J \partial w_J, \tag{2.2}$$

in which Υ is a vector field on $\aleph$ and defines as

$$\Upsilon = \xi^i(x, w)\frac{\partial}{\partial x^i} + \varphi(x, w)\frac{\partial}{\partial w}. \tag{2.3}$$

Eq.(2.2) is the prolongation of μ of Eq.(2.3) if its coefficient provides the prolongation formula of μ

$$\psi_{J,i} = (\wp_i + \lambda_i)\psi_J - w_{J,m}(\wp_i + \lambda_i)\xi^m, \tag{2.4}$$

in which $\psi_0 = \varphi$. Let $R \subset J^{(s)}\aleph$ be the solution manifold for Δ. If $\Omega : R \to TR$, it is said that, for Eq.(2.1), Eq.(2.3) is a μ-symmetry. To get μ-symmetry of Eq.(2.1), then applies Eq.(2.2) to Eq.(2.1), and restrain the got outcomes to the solution manifold $R_\Delta \subset \aleph^{(s)}$ that will be up to ξ, φ, λ_i. If we deem the λ as functions on $\aleph^{(s)}$ and compatibility conditions between the λ_i, a system of all the dependence on w_J form the determining equations [21]. $V = \exp(\int \mu)\Upsilon$ is an exponential vector field if Eq.(2.3) is a vector field on $\aleph$.

Theorem 2.1. *Let sth-order PDE defines as $\Delta(x, w^s)$, Eq.(2.3) be a vector field on $\aleph$, with invariant surface condition $Q = \varphi - w_i\xi^i$, and Ω be the μ−prolong of order s of Υ. In this case, for $\triangle$, Eq.(2.3) is a μ-symmetry, then $\Omega : R_\Upsilon \to TR_\Upsilon$, in which $R_\Upsilon \subset J^{(s)}\aleph$ is the solution manifold for Δ_Υ made of Δ and $\grave{E}_J := \wp_J Q = 0, \forall J$ with $|J| = 0, 1, ..., s-1$ [8, 9, 10, 11, 21].*

2.2 μ−conservation law

Surmise that $\mu = \lambda_i dx_i$ be a semi-basic one-form and with the compability condition $\wp_j\lambda_i = \wp_i\lambda_j$.

A conservation law of μ is

$$(\wp_i + \lambda_i)P^i = 0. \tag{2.5}$$

Here, P^i is a conserved vector of μ and this vector is a matrix-valued $\aleph$-vector.

Think that $\mathcal{L} = \mathcal{L}(x, w^{(s)})$ depicts the sth order Lagrangian. For $\mathcal{L}$, Eq.(2.3) is a μ-symmetry, namely, $\exists\aleph$-vector P^i such that $(\wp_i + \lambda_i)P^i = 0$ where the necessary and sufficient condition is $\Omega[\mathcal{L}] = 0$ [19].
Let second-order Lagrangian defines as $\mathcal{L} = \mathcal{L}(x, t, w, w_x, ..., w_{tt})$ and for $\mathcal{L}$, $\Upsilon = \varphi(\frac{\partial}{\partial w})$ be a μ-symmetry. $\aleph$-vector P^i is got as [19]

$$P^i := \varphi\frac{\partial \mathcal{L}}{\partial w_i} + [(\wp_j + \lambda_j)\varphi]\frac{\partial \mathcal{L}}{\partial w_{ij}} - \varphi\wp_j(\frac{\partial \mathcal{L}}{\partial w_{ij}}). \tag{2.6}$$

Here, $\wp_j$ is the total derivative.

The Frechet derivative $\wp_\Delta$ is self adjoint, namely, $\wp^*_\Delta = \wp_\Delta$ is necessary and sufficient condition in which a system admits a variational formulation [8, 9, 10, 11, 22].

Theorem 2.2. *Let $\Delta = 0$ be a system of differential equations. For some variational problem $\pounds = \int Ldx$, Δ is the Euler-Lagrange expression, i.e., $\wp_\Delta = \wp^*_\Delta$ if and only if $\Delta = \grave{E}(L)$. Then, by employing the homotopy formula $L[u] = \int_0^1 u\Delta[\lambda u]d\lambda$, a Lagrangian can be found for Δ.*

2.3 The mGERFM

In this part, we think mGERFM, scrutinized in [23]. Utilizing the approach is mainly based on the subsequent framework [24].

Let us consider the following NLPDE as follows:

$$Q_1[q, q_t, q_x, q_{tt}, q_{xx}, ...] = 0. \tag{2.7}$$

Using $q = q(x, t) = v(\Phi)$, and $\Phi = kx - wt$, Eq.(2.7) is transferred to

$$Q_2[v(\Phi), -w\frac{\partial v(\Phi)}{\partial \Phi}, k\frac{\partial v(\Phi)}{\partial \Phi}, w^2\frac{\partial^2 v(\Phi)}{\partial \Phi^2}, k^2\frac{\partial^2 v(\Phi)}{\partial \Phi^2}, ...] = 0. \tag{2.8}$$

This approach includes a symbolic configuration for the solution that can be characterized as follows:

$$v(\Phi) = \gamma_0 + \sum_{n=1}^{n_0}\gamma_n\left(\frac{\Lambda'(\Phi)}{\Lambda(\Phi)}\right)^n + \sum_{n=1}^{n_0}\delta_n\left(\frac{\Lambda'(\Phi)}{\Lambda(\Phi)}\right)^{-n}, \tag{2.9}$$

in which

$$\Lambda(\Phi) = \frac{\varsigma_1\exp(\epsilon_1\Phi) + \varsigma_2\exp(\epsilon_2\Phi)}{\varsigma_3\exp(\epsilon_3\Phi) + \varsigma_4\exp(\epsilon_4\Phi)}. \tag{2.10}$$

Unknown coefficients $\gamma_0, \gamma_n, \delta_n$ $(1 \le n \le n_0)$ and ς_i, ϵ_i $(1 \le i \le 4)$ are real (or complex) constants to be evaluated, such that Eq.(2.9) satisfies the Eq.(2.8). Besides, the positive integer

n_0 is calculated by the principles of balancing. Substituting Eq.(2.9) together with Eq.(2.10) into Eq.(2.8) and gathering all terms, the left-hand side of the resultant the equation is converted into a polynomial equation $\Psi(T_1, T_2, T_3, T_4) = 0$ as to $T_r = \exp(\epsilon_r \Phi)$ for $r = 1, 2, 3, 4$. Taking each coefficient of Ψ to zero, we reach a set of algebraic equations. Solving these algebraic equations with the aid of a symbolic computation package and then inserting non-trivial solutions in Eq.(2.9), the explicit shape of the solutions of Eq.(2.7) will be extracted.

3 Mathematical discussion for the nonlinear model

One particularly well-known NLPDE is the KdV equation derived by Korteweg and de Vries [25]. The KdV equation can be offered as

$$\Theta_t + \Theta\Theta_x + \Theta_{xxx} = 0. \tag{3.1}$$

Eq.(3.1) is a NLPDE in one dimension and defines the time-dependent motion of shallow water waves. Another equation is the Regularised Long-wave equation (RLWE) [26]. This equation can be written as

$$\Theta_t + \Theta_x + \varepsilon\Theta\Theta_x - \kappa\Theta_{xxt} = 0.$$

RLWE is more common than the KdV equation to describe the behavior of nonlinear dispersive waves.

Morrison et al. [27] presented an equal width equation (EWE) derived by utilizing both KdV and RLWE. The EWE is also known as one-dimensional NLEE in the offered form

$$\Theta_t + \Theta\Theta_x - \kappa\Theta_{xxt} = 0.$$

Because of the soliton solution with permanent speed and form, the wave has EW for all amplitude. That is why it is named the EW wave equation. Here, $\Theta = \Theta(x, t)$ represents wave amplitude with boundary condition $\Theta \to 0$ as $x \to \pm\infty$. Also, x represents the space coordinate, t denotes the time coordinate, and κ is a positive parameter.

The MEWE is derived from the EWE, and it has cubic nonlinearity with dispersive waveform

$$\Delta : \Theta_t + \Theta^2\Theta_x - \Theta_{xxt} = 0. \tag{3.2}$$

Different analytical and numerical approaches have been used to discover the solution to MEWE, for example, Collocation Method [28], Multigrid Method [29], Sine-cosine method [30], Classical Lie symmetry analysis [31].

3.1 Primary outcomes of solving model (3.2) using the μ-symmetry analysis

Suppose that, we have a semi-basic one-form $\mu = \lambda_1 dx + \lambda_2 dt$ such that $\wp_t\lambda_1 = \wp_x\lambda_2$ when $\Theta_t + \Theta^2\Theta_x - \Theta_{xxt} = 0$.

Let

$$\Upsilon = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial \Theta} \tag{3.3}$$

be a vector field on $\aleph$, and ξ, τ, φ based on x, t, Θ. The third prolongation is given as

$$\Omega = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial \Theta} + \psi^x\frac{\partial}{\partial \Theta_x} + \psi^t\frac{\partial}{\partial \Theta_t} + \psi^{xxt}\frac{\partial}{\partial \Theta_{xxt}}. \tag{3.4}$$

Ω satisfies the following μ-symmetry condition:

$$\psi^t + \psi^x\Theta^2 + 2\varphi\Theta\Theta_x - \psi^{xxt} \mid_{\Delta_\Theta = 0} = 0, \tag{3.5}$$

where

$$\begin{aligned}
\psi^{x} &= (\wp_x + \lambda_1)\varphi - \Theta_x(\wp_x + \lambda_1)\xi - \Theta_t(\wp_x + \lambda_1)\tau,\\
\psi^{t} &= (\wp_t + \lambda_2)\varphi - \Theta_x(\wp_t + \lambda_2)\xi - \Theta_t(\wp_t + \lambda_2)\tau,\\
\psi^{xx} &= (\wp_x + \lambda_1)\psi^{x} - \Theta_{xx}(\wp_x + \lambda_1)\xi - \Theta_{xt}(\wp_x + \lambda_1)\tau,\\
\psi^{xxt} &= (\wp_t + \lambda_2)\psi^{xx} - \Theta_{xxx}(\wp_t + \lambda_2)\xi - \Theta_{xxt}(\wp_t + \lambda_2)\tau, \qquad (3.6)
\end{aligned}$$

and $\wp_t$ and $\wp_x$ denote the total differentiations with respect to t and x:

$$\begin{aligned}
\wp_t &= \frac{\partial}{\partial t} + \Theta_t\frac{\partial}{\partial\Theta} + \Theta_{xt}\frac{\partial}{\partial\Theta_x} + \Theta_{tt}\frac{\partial}{\partial\Theta_t} + ..., \qquad (3.7)\\
\wp_x &= \frac{\partial}{\partial x} + \Theta_x\frac{\partial}{\partial\Theta} + \Theta_{xt}\frac{\partial}{\partial\Theta_t} + \Theta_{xx}\frac{\partial}{\partial\Theta_x} + ...
\end{aligned}$$

To begin with, we substitute Eq.(3.6) into Eq.(3.5) together with (3.7). Then, we write $\Theta_t + \Theta^2\Theta_x$ instead of Θ_{xxt}, and expanding them, we obtain an over-determined system for λ_1, λ_2, ξ, τ, φ :

$$3\xi_{\Theta\Theta} = 0, \quad 2\tau_{\Theta} = 0,$$

$$3\lambda_2\xi_{\Theta} + 3\xi_{t\Theta} = 0,$$

$$4\tau_{x\Theta} + 4\tau_{\Theta}\lambda_1 + 3\tau\lambda_{1\Theta} = 0,$$

$$2\tau_{x\Theta} + 2\tau_{\Theta}\lambda_1 + \tau\lambda_{1\Theta} = 0,$$

$$\vdots$$

$$2\tau_{\Theta\Theta x} + 2\tau_{\Theta\Theta}\lambda_1 + 3\tau_{\Theta}\lambda_{1\Theta} + \tau\lambda_{1\Theta\Theta} = 0,$$

$$-\varphi_{\Theta\Theta} + 2\xi_{x\Theta} + \tau_{t\Theta} + 2\xi_{\Theta}\lambda_1 + 2\xi\lambda_{1\Theta} + \lambda_2\tau_{\Theta} = 0. \qquad (3.8)$$

Surmise that λ_1 and λ_2 are any choices of the type

$$\lambda_1 = \wp_x[H] + y, \quad \lambda_2 = \wp_t[H] + z, \qquad (3.9)$$

where $H = H(x,t)$, $y = y(x)$ and $z = z(t)$ are arbitrary functions, and λ_1, λ_2 satisfy to $\wp_x\lambda_2 = \wp_t\lambda_1$ on solutions to Eq.(3.2).

Case-1: When $y = 0$, $z = 0$, and $H = -\ln(\varrho)$ in the functions of λ_1 and λ_2, then by substituting the functions

$$\lambda_1 = -\frac{\varrho_x}{\varrho}, \quad \lambda_2 = -\frac{\varrho_t}{\varrho} \qquad (3.10)$$

into the system of Eq.(3.8) and solving them, we get

$$\xi = 0, \quad \tau = \varrho, \quad \varphi = 0. \qquad (3.11)$$

Then, by substituting the ξ, τ, and φ into Eq.(3.3), we obtain

$$\Upsilon_1 = \varrho \frac{\partial}{\partial t}. \tag{3.12}$$

Eq.(3.12) is μ-symmetry of Eq.(3.2). Also,

$$\begin{aligned} V &= \exp\left(\int \lambda_1 dx + \lambda_2 dt\right) \Upsilon \\ &= \exp\left(\int (-\frac{\varrho_x}{\varrho}) dx + (-\frac{\varrho_t}{\varrho}) dt\right) \Upsilon_1. \end{aligned} \tag{3.13}$$

Thanks to the **Theorem 2.1**, the order reduction of Eq.(3.2) is

$$\begin{aligned} Q_1 &= \varphi - \xi \Theta_x - \tau \Theta_t \\ &= -\varrho \Theta_t. \end{aligned} \tag{3.14}$$

Case-2: When $y = 0$, $z = \frac{1}{t-d_1}$, and $H = -\ln(\varrho)$ in the functions of λ_1 and λ_2, then by placing the functions

$$\lambda_1 = -\frac{\varrho_x}{\varrho}, \quad \lambda_2 = -\frac{\varrho_t}{\varrho} + \frac{1}{t-d_1} \tag{3.15}$$

into the system of Eq.(3.8) and solving them, we attain

$$\xi = 0, \quad \tau = \varrho, \quad \varphi = \frac{\Theta}{2(d_1 - t)} \varrho. \tag{3.16}$$

Here, d_1 is an integration constant. Then, by inserting the ξ, τ, and φ into Eq.(3.3), we obtain

$$\Upsilon_2 = \varrho \left(\frac{\partial}{\partial t} + \frac{\Theta}{2(d_1 - t)} \frac{\partial}{\partial \Theta} \right). \tag{3.17}$$

Eq.(3.17) is $\mu-$symmetry of Eq.(3.2). Also,

$$V = \exp\left(\int (-\frac{\varrho_x}{\varrho}) dx + (-\frac{\varrho_t}{\varrho} + \frac{1}{t-d_1}) dt\right) \Upsilon_2. \tag{3.18}$$

By using the **Theorem 2.1**, the order reduction of Eq.(3.2) is

$$\begin{aligned} Q_2 &= \varphi - \xi \Theta_x - \tau \Theta_t \\ &= \varrho \left(\frac{\Theta}{2(d_1 - t)} - \Theta_t \right). \end{aligned} \tag{3.19}$$

Here, d_1 is an integration constant.

Case-3: When $y = 0$, $z = 0$, and $H = -\ln(\varrho)$ in the functions of λ_1 and λ_2, then, by substituting the functions

$$\lambda_1 = -\frac{\varrho_x}{\varrho}, \quad \lambda_2 = -\frac{\varrho_t}{\varrho} \tag{3.20}$$

into the system of Eq.(3.8) and solving them, we reach

$$\xi = \varrho, \quad \tau = (d_1 + t d_2) \varrho, \quad \varphi = -\frac{\Theta d_2}{2} \varrho. \tag{3.21}$$

Then, by inserting the ξ, τ, and φ into the vector field, we obtain

$$\Upsilon_3 = \varrho \left(\frac{\partial}{\partial x} + (d_1 + t d_2) \frac{\partial}{\partial t} - \frac{\Theta d_2}{2} \frac{\partial}{\partial \Theta} \right). \tag{3.22}$$

Eq.(3.22) is $\mu-$symmetry of Eq.(3.2). Also,

$$V = \exp\left(\int\left(-\frac{\varrho_x}{\varrho}\right)dx + \left(-\frac{\varrho_t}{\varrho}\right)dt\right)\Upsilon_3. \tag{3.23}$$

By using the **Theorem 2.1**, the order reduction of Eq.(3.2) is

$$\begin{aligned} Q_3 &= \varphi - \xi\Theta_x - \tau\Theta_t \\ &= -\varrho\left(\frac{\Theta d_2}{2} + \Theta_x + (d_1 + td_2)\Theta_t\right). \end{aligned} \tag{3.24}$$

Note that $\varrho = \varrho(x,t)$ is an arbitrary function, and d_1, d_2 are integration constants.

3.2 μ-invariant solutions for the MEWE

The characteristic equation forms are constructed by using the invariant surface condition. By solving the characteristic equation form, similarity variables are obtained. Then, thanks to the similarity variables and the original equation, a PDE can be converted to an ODE. Then, by solving the ODE, the invariant solution is obtained.

The characteristic equation corresponding to Eq.(3.14) is written as

$$\frac{dx}{0} = \frac{dt}{-\varrho} = \frac{d\Theta}{0}. \tag{3.25}$$

By solving Eq.(3.25), we get similarity variables as indicated below

$$\rho = x, \quad \Theta = \varrho_1(\rho).$$

After placing Θ into Eq.(3.2), Eq.(3.2) can be reduced to the ODE

$$\begin{aligned} \varrho_1(\rho)^2\left(\frac{d}{d\rho}\varrho_1(\rho)\right) &= 0, \\ \varrho_1(\rho) &= C. \end{aligned}$$

Therefore, we have an invariant solution

$$\Theta = C.$$

For Eq.(3.24), let us consider $-\varrho \neq 0$. Then, we have $\frac{\Theta d_2}{2} + \Theta_x + (d_1 + td_2)\Theta_t = 0$. The characteristic equation corresponding to Eq.(3.22) is written as

$$\frac{dx}{1} = \frac{dt}{(d_1 + td_2)} = \frac{d\Theta}{\frac{d_2}{2}}. \tag{3.26}$$

Specially, if we choose $d_1 = 1$, $d_2 = 0$ in Eq.(3.26), we have

$$\frac{dx}{1} = \frac{dt}{1} = \frac{d\Theta}{0}. \tag{3.27}$$

By solving Eq.(3.27), we get similarity variables as indicated below

$$\varpi_1 = t - x, \quad \Theta = \varrho_2(\varpi_1). \tag{3.28}$$

After placing Θ into Eq.(3.2), Eq.(3.2) can be reduced to the ODE as

$$\frac{d}{d\varpi_1}\varrho_2(\varpi_1) - \varrho_2(\varpi_1)^2\left(\frac{d}{d\varpi_1}\varrho_2(\varpi_1)\right) - \left(\frac{d^3}{d\varpi_1^3}\varrho_2(\varpi_1)\right) = 0. \tag{3.29}$$

If we solve Eq.(3.29) directly, we get an integral form as follows:

$$\int^{\varrho_2(\varpi_1)} \frac{6}{\sqrt{-6x^4 - 72C_1x + 36x^2 + 72C_2}} dx - \varpi_1 - C_3 = 0. \tag{3.30}$$

Thanks to the Eq.(3.30), we have:
Set-1:
letting $C_1 = C_3 = 0$, $C_2 = 1$, and solving them, we obtain

$$\Theta(x,t) = -\frac{6\ JacobiSN\left(\frac{1}{6}\sqrt{-18+6\sqrt{21}}(t-x), \frac{i}{2}\left(\sqrt{3}+\sqrt{7}\right)\right)}{\sqrt{-9+3\sqrt{21}}}. \tag{3.31}$$

Set-2:
Let $C_2 = C_3 = 1$, $C_1 = 0$, we get

$$\Theta(x,t) = -\frac{6\ JacobiSN\left(\frac{1}{6}\sqrt{-18+6\sqrt{21}}(t-x+1), \frac{i}{2}\left(\sqrt{3}+\sqrt{7}\right)\right)}{\sqrt{-9+3\sqrt{21}}}. \tag{3.32}$$

Set-3:
If we choose $C_2 = -1$, $C_1 = C_3 = 0$, we reach

$$\Theta(x,t) = -\frac{6\ JacobiSN\left(\frac{1}{6}\sqrt{-18+6i\sqrt{3}}(t-x), \frac{1}{2}\left(\sqrt{2+2i\sqrt{3}}\right)\right)}{\sqrt{9-3i\sqrt{3}}}. \tag{3.33}$$

Set-4:
If we take $C_1 = 0$, $C_2 = C_3 = -1$, we attain

$$\Theta(x,t) = -\frac{6\ JacobiSN\left(\frac{1}{6}\sqrt{-18+6i\sqrt{3}}(t-x-1), \frac{1}{2}\left(\sqrt{2+2i\sqrt{3}}\right)\right)}{\sqrt{9-3i\sqrt{3}}}. \tag{3.34}$$

Particular Case
We know that the combination of a vector field (infinitesimal generator) is also a vector field. Then, let us consider the following linear combination of the μ-symmetry generators:

$$\Upsilon_{1,3} = k\Upsilon_1 + w\Upsilon_3. \tag{3.35}$$

Especially, if we take $d_1 = d_2 = 0$ in Eq.(3.22), then, Υ_3 becomes

$$\Upsilon_3 = \varrho\left(\frac{\partial}{\partial x}\right).$$

Thus, from the Eq.(3.35) we attain

$$\Upsilon_{1,3} = \varrho\left(k\frac{\partial}{\partial t} + w\frac{\partial}{\partial x}\right). \tag{3.36}$$

By using the **Theorem 2.1**, we have

$$\begin{aligned} Q &= \varphi - \xi\Theta_x - \tau\Theta_t. \\ &= -\varrho\left[w\Theta_x + k\Theta_t\right]. \end{aligned} \tag{3.37}$$

Here, $\varrho \neq 0$, thus, we say that $w\Theta_x + k\Theta_t = 0$.
The characteristic equation corresponding to Eq.(3.37) is written as

$$\frac{dx}{w} = \frac{dt}{k} = \frac{d\Theta}{0}. \tag{3.38}$$

By solving Eq.(3.38), we get similarity variables as indicated below

$$\Phi = kx - wt, \quad \Theta = u(\Phi).$$

So, the group-invariant solution is

$$\Theta(x,t) = u(kx - wt). \tag{3.39}$$

Substitution of Θ into Eq.(3.2) yields a third order ODE

$$-wu' + u^2u'k + k^2wu''' = 0. \tag{3.40}$$

Integration of Eq.(3.40) with respect to u yields

$$k^2wu'' - wu + \frac{k}{3}u^3 = 0, \tag{3.41}$$

in which we take 0 as a constant of integration.

Remark 3.1. In Eq.(3.26), particularly, we choose $d_1 = 1$, $d_2 = 0$, we reach Eq.(3.27). By solving Eq.(3.27), we obtain similarity variables as Eq.(3.28). We get reduced ODE as Eq.(3.29) thanks to similarity variables. We obtain an integral form if we solve directly Eq.(3.29). By solving this integral form, particularly, we take $(C_1, C_2, C_3) = (0, 1, 0)$, $(C_1, C_2, C_3) = (0, 1, 1)$, $(C_1, C_2, C_3) = (0, -1, 0)$, $(C_1, C_2, C_3) = (0, -1, -1)$, we obtain Jacobi elliptic function solutions. On the other hand, as a particular case, we deal with Eq.(3.35). Let us focus on Eq.(3.38). If we choose $k = w = 1$, then, Eq.(3.29) is the same as Eq.(3.40). However, we do not solve Eq.(3.40) directly. First, we integrate Eq.(3.40) concerning u, then we obtain Eq.(3.41).

3.3 Main results of solving Eq.(3.41) employing the mGERFM

Balancing u'' with u^3 in Eq.(3.41) gives $n_0 + 2 = 3n_0$ and $n_0 = 1$. From Eq.(2.9), we have

$$u(\Phi) = \gamma_0 + \gamma_1\left(\frac{\Lambda'(\Phi)}{\Lambda(\Phi)}\right) + \delta_1\left(\frac{\Lambda'(\Phi)}{\Lambda(\Phi)}\right)^{-1}, \tag{3.42}$$

in which $\Lambda(\Phi)$ is defined by Eq.(2.10) as

$$\Lambda(\Phi) = \frac{\varsigma_1 \exp(\epsilon_1\Phi) + \varsigma_2 \exp(\epsilon_2\Phi)}{\varsigma_3 \exp(\epsilon_3\Phi) + \varsigma_4 \exp(\epsilon_4\Phi)}. \tag{3.43}$$

Category 1

Taking $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, 1, 1, 0]$ and $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4] = [0, 1, -2, 0]$ in Eq.(3.43) yields

$$\Lambda_1(\Phi) = \frac{1 + \exp(\Phi)}{\exp(-2\Phi)}. \tag{3.44}$$

To get the values of parameters, we need to solve algebraic equations with the aid of Maple and the pursuing set of solutions can be delivered as

Sub-category 1.1

$$\gamma_0 = \pm 5\left(-\frac{9w^2}{2}\right)^{\frac{1}{4}}, \quad \pm i \times 5\left(-\frac{9w^2}{2}\right)^{\frac{1}{4}}, \quad \gamma_1 = 0,$$

$$\delta_1 = \pm 12\left(-\frac{9w^2}{2}\right)^{\frac{1}{4}}, \quad \pm i \times 12\left(-\frac{9w^2}{2}\right)^{\frac{1}{4}},$$

$$k = \pm\frac{2}{3w}\sqrt{-\frac{9w^2}{2}}.$$

Inserting these above values of γ_0, γ_1, δ_2 into Eq.(3.42), we have

$$u_{1,1}(\Phi) = \frac{2^{\frac{3}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(-2 + 3\exp(\Phi))}{6\exp(\Phi) + 4}. \tag{3.45}$$

By using the Eq.(3.45) together with Eq.(3.39), then, the exponential function can be expressed as

$$\Theta_{1,1}(x,t) = \frac{2^{\frac{3}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(3\exp(kx-wt)-2)}{2(3\exp(kx-wt)+2)}. \tag{3.46}$$

Sub-category 1.2

$$\gamma_0 = \pm\frac{5}{2}\left(-72w^2\right)^{\frac{1}{4}}, \quad \pm\frac{5i}{2}\left(-72w^2\right)^{\frac{1}{4}}, \quad \delta_1 = 0,$$

$$\gamma_1 = \pm\left(-72w^2\right)^{\frac{1}{4}}, \quad \pm i\left(-72w^2\right)^{\frac{1}{4}},$$

$$k = \pm\frac{\sqrt{-72w^2}}{6w}.$$

Substituting the values of γ_0, γ_1, δ_2 into Eq.(3.42), we have

$$u_{1,2}(\Phi) = \frac{2^{\frac{3}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(\exp(\Phi)-1)}{2(\exp(\Phi)+1)}. \tag{3.47}$$

Using the Eq.(3.47) together with Eq.(3.39), the exponential function solution is obtained as

$$\Theta_{1,2}(x,t) = \frac{2^{\frac{3}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(\exp(kx-wt)-1)}{2(\exp(kx-wt)+1)}. \tag{3.48}$$

Category 2

When we choose $[\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4] = [1, -1, 2i, 0]$ and $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4] = [i, -i, 1, 0]$ in Eq.(3.43) gives

$$\Lambda_2(\Phi) = \frac{\sin(\Phi)}{\exp(\Phi)}. \tag{3.49}$$

The next **Sub-category** are scheduled:

Sub-category 2.1

$$\gamma_0 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \gamma_1 = 0,$$

$$\delta_1 = \pm 2 \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm 2i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}},$$

$$k = \pm\frac{\sqrt{18}\sqrt{w^2}}{6w}.$$

By considering these values in Eq.(3.42), we have

$$u_{2,1}(\Phi) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}(\cos(\Phi)+\sin(\Phi))}{\cos(\Phi)-\sin(\Phi)}. \tag{3.50}$$

By using the Eq.(3.50) together with Eq.(3.39), then the combined periodic solution can be written as

$$\Theta_{2,1}(x,t) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}(\cos(kx-wt)+\sin(kx-wt))}{\cos(kx-wt)-\sin(kx-wt)}. \tag{3.51}$$

Sub-category 2.2

$$\gamma_0 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \delta_1 = 0,$$

$$\gamma_2 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}},$$

$$k = \pm\frac{\sqrt{18}\sqrt{w^2}}{6w}.$$

Substituting the values of γ_0, γ_1, γ_2 into (3.42), we have

$$u_{2,2}(\Phi) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}\cos(\Phi)}{\sin(\Phi)}. \tag{3.52}$$

By using the Eq.(3.52) together with Eq.(3.39), we find singular periodic soliton solution as

$$\Theta_{2,2}(x,t) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}\cos(kx-wt)}{\sin(kx-wt)}. \tag{3.53}$$

Category 3
For $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4] = [1,-1,2,0]$ and $[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4] = [1,-1,-1,0]$ in Eq.(3.43) offers

$$\Lambda_3(\Phi) = \frac{\sinh(\Phi)}{\exp(-\Phi)}. \tag{3.54}$$

Proceeding as the outline of mGERFM, we attain

$$\gamma_0 = \gamma_1 = \pm\left(-18w^2\right)^{\frac{1}{4}}, \quad \pm i\times\left(-18w^2\right)^{\frac{1}{4}},$$

$$\delta_1 = 0, \quad k = \pm\frac{\sqrt{-18w^2}}{6w}.$$

By regarding these values in Eq.(3.42), one receives

$$u_3(\Phi) = -\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}\sinh(\Phi)}{\cosh(\Phi)}. \tag{3.55}$$

By using the Eq.(3.55) together with Eq.(3.39), then, we obtain the shock wave solution as

$$\Theta_3(x,t) = -\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}\sinh(kx-wt)}{\cosh(kx-wt)}. \tag{3.56}$$

Category 4
On selecting $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4] = [2,0,1,1]$ and $[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4] = [1,0,i,-i]$ in Eq.(3.43) yields

$$\Lambda_4(\Phi) = \frac{\exp(\Phi)}{\cos(\Phi)}. \tag{3.57}$$

The subsequent **Sub-category** are planned:
Sub-category 4.1

$$\gamma_0 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i\times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \gamma_1 = 0,$$

$$\delta_1 = \pm 2\times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm 2i\times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}},$$

$$k = \pm\frac{\sqrt{18}\sqrt{w^2}}{6w}.$$

Combining these outcomes with Eq.(3.42) yields

$$u_{4,1}(\Phi) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}(2\cos(\Phi)\sin(\Phi)-1)}{2\cos^2(\Phi)-1}. \tag{3.58}$$

Using the Eq.(3.58) together with Eq.(3.39), in this way, we attain the following trigonometric solution as

$$\Theta_{4,1}(x,t) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}(2\cos(kx-wt)\sin(kx-wt)-1)}{2\cos^2(kx-wt)-1}. \tag{3.59}$$

Sub-category 4.2

$$\gamma_0 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \delta_1 = 0,$$

$$\gamma_1 = \pm 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}}, \quad \pm i \times 18^{\frac{1}{4}}\left(w^2\right)^{\frac{1}{4}},$$

$$k = \pm\frac{\sqrt{18}\sqrt{w^2}}{6w}.$$

Inserting these values in Eq.(3.42), offers

$$u_{4,2}(\Phi) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}\sin(\Phi)}{\cos(\Phi)}. \tag{3.60}$$

Consequently, we discover the singular periodic solution can be expressed as

$$\Theta_{4,2}(x,t) = \frac{2^{\frac{1}{4}}\sqrt{3}(w^2)^{\frac{1}{4}}\sin(kx-wt)}{\cos(kx-wt)}. \tag{3.61}$$

Category 5

Considering $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4] = [2,0,1,1]$ and $[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4] = [0,0,1,-1]$ in Eq.(3.43) yields

$$\Lambda_5(\Phi) = \frac{1}{\cosh(\Phi)}. \tag{3.62}$$

Also, we reach

$$\gamma_0 = 0, \quad \gamma_1 = \pm\sqrt{\pm\frac{3i}{2}\sqrt{2}w}, \quad \delta_1 = \frac{\frac{3i}{2}w\sqrt{2}}{\sqrt{\pm\frac{3i}{2}\sqrt{2}w}},$$

$$k = \pm\frac{i\sqrt{2}}{4}.$$

Consequently, regarding these solutions and Eq.(3.42), it is likely to reach the subsequent outcome

$$u_5(\Phi) = -\frac{2^{\frac{3}{4}}\sqrt{3}(\sqrt{-iw})(2\cosh^2(\Phi)-1)}{2\sinh(\Phi)\cosh(\Phi)}. \tag{3.63}$$

Hence, we discover the mixed-form soliton solution can be written as

$$\Theta_5(x,t) = -\frac{2^{\frac{3}{4}}\sqrt{3}(\sqrt{-iw})(2\cosh^2(kx-wt)-1)}{2\cosh(kx-wt)\sinh(kx-wt)}. \tag{3.64}$$

Category 6

As long as, if it is allocated $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4] = [2,0,1,1]$ and $[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4] = [-2,0,1,-1]$ into account in Eq.(3.43) produces

$$\Lambda_6(\Phi) = \frac{\exp(-2\Phi)}{\cosh(\Phi)}. \tag{3.65}$$

We obtain,

Sub-category 6.1

$$\gamma_0 = \pm 2\left(-18w^2\right)^{\frac{1}{4}}, \quad \pm 2i \times \left(-18w^2\right)^{\frac{1}{4}}, \quad \gamma_1 = 0,$$

$$\delta_1 = \pm 3 \times \left(-18w^2\right)^{\frac{1}{4}}, \quad \pm 3i \times \left(-18w^2\right)^{\frac{1}{4}},$$

$$k=\pm\frac{\sqrt{-18w^2}}{6w}.$$

Plugging the values of γ_0, γ_1, δ_2 into Eq.(3.42), we have

$$u_{6,1}(\Phi)=\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(2\sinh(\Phi)+\cosh(\Phi))}{\sinh(\Phi)+2\cosh(\Phi)}. \tag{3.66}$$

By use of Eq.(3.66) together with Eq.(3.39), then the hyperbolic solution in mixed form can be formulated as

$$\Theta_{6,1}(x,t)=\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}(2\sinh(kx-wt)+\cosh(kx-wt))}{\sinh(kx-wt)+2\cosh(kx-wt)}. \tag{3.67}$$

Sub-category 6.2

$$\gamma_0=\pm 2\left(-18w^2\right)^{\frac{1}{4}}, \quad \pm 2i\times\left(-18w^2\right)^{\frac{1}{4}}, \quad \delta_1=0,$$

$$\gamma_1=\pm\left(-18w^2\right)^{\frac{1}{4}}, \quad \pm i\times\left(-18w^2\right)^{\frac{1}{4}},$$

$$k=\pm\frac{\sqrt{-18w^2}}{6w}.$$

Plugging the values of γ_0, γ_1, δ_2 into Eq.(3.42), one obtains

$$u_{6,2}(\Phi)=-\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}\sinh(\Phi)}{\cosh(\Phi)}. \tag{3.68}$$

By using the Eq.(3.55) together with Eq.(3.39), the next shape is derived as the shock wave solution

$$\Theta_{6,2}(x,t)=-\frac{2^{\frac{1}{4}}\sqrt{3}(-w^2)^{\frac{1}{4}}\sinh(kx-wt)}{\cosh(kx-wt)}. \tag{3.69}$$

Category 7

If we take $[\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4]=[1,1,2,0]$ and $[\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4]=[i,-i,0,0]$ in Eq.(3.43) offers

$$\Lambda_7(\Phi)=\cos(\Phi). \tag{3.70}$$

We get

Sub-category 7.1

$$\gamma_0=0, \quad \gamma_1=\pm\sqrt{-\frac{3w\sqrt{2}}{2}}, \quad \pm\frac{\sqrt{3}\sqrt{2}\sqrt{w\sqrt{2}}}{2},$$

$$\delta_1=\pm\frac{3w\sqrt{2}}{2\sqrt{-\frac{3w\sqrt{2}}{2}}}, \quad \pm\frac{w\sqrt{3}}{\sqrt{w\sqrt{2}}},$$

$$k=\pm\frac{\sqrt{2}}{4}.$$

These outcomes along with Eq.(3.42) lead to the next construction

$$u_{7,1}(\Phi)=\frac{2^{\frac{3}{4}}\sqrt{3}\sqrt{-w}(2\cos^2(\Phi)-1)}{2\cos(\Phi)\sin(\Phi)}. \tag{3.71}$$

Hence, we reach the following trigonometric solution as

$$\Theta_{7,1}(x,t)=\frac{2^{\frac{3}{4}}\sqrt{3}\sqrt{-w}(2\cos^2(kx-wt)-1)}{2\cos(kx-wt)\sin(kx-wt)}. \tag{3.72}$$

Sub-category 7.2

$$\gamma_0 = 0, \quad \gamma_1 = \pm\sqrt{\pm 3iw},$$

$$\delta_1 = \pm\frac{3iw}{\sqrt{\pm 3iw}},$$

$$k = \pm\frac{i}{2}.$$

If these outcomes are regarded in conjunction with Eq.(3.42), the next product is got

$$u_{7,2}(\Phi) = -\frac{\sqrt{3}\sqrt{-iw}}{\cos(\Phi)\sin(\Phi)}. \tag{3.73}$$

By using the Eq.(3.73) together with Eq.(3.39), hence, we get following periodic solution

$$\Theta_{7,2}(x,t) = -\frac{\sqrt{3}\sqrt{-iw}}{\cos(kx-wt)\sin(kx-wt)}. \tag{3.74}$$

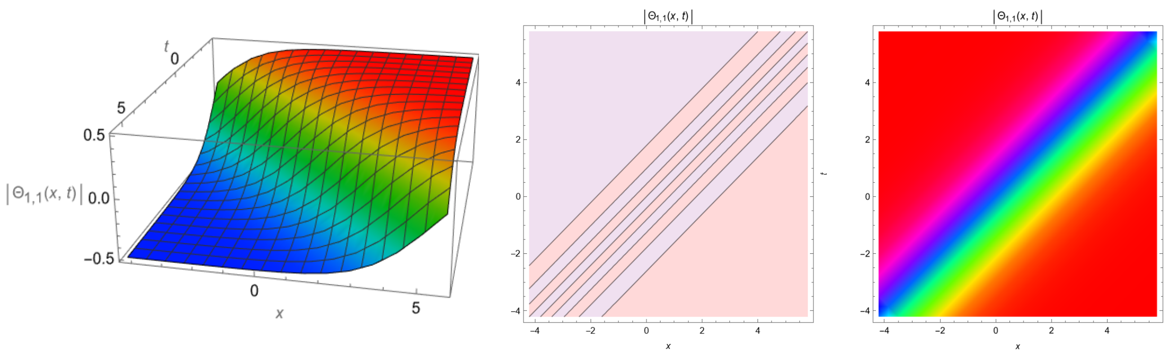

Figure 1. The 3-dimensional, contour and density figures of $|\Theta_{1,1}(x,t)|$ in (3.46), when $k = 1,\ w = 1$.

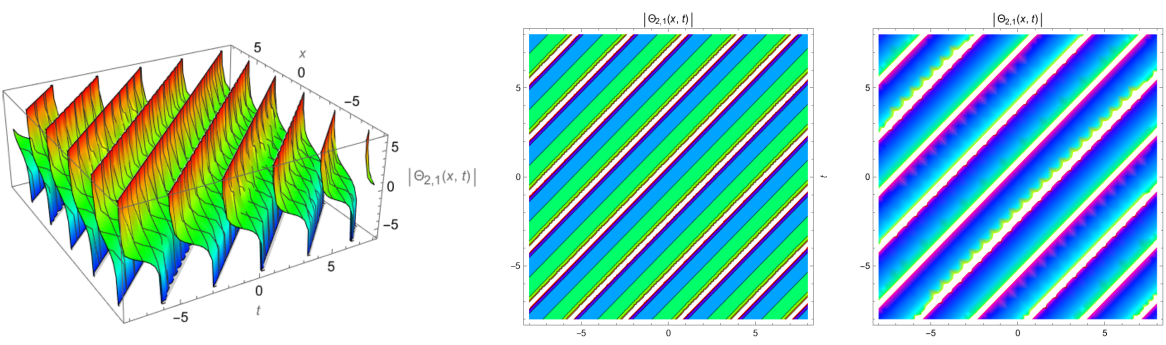

Figure 2. The 3-dimensional, contour and density figures of $|\Theta_{2,1}(x,t)|$ in (3.51), when $k = 1,\ w = 1$.

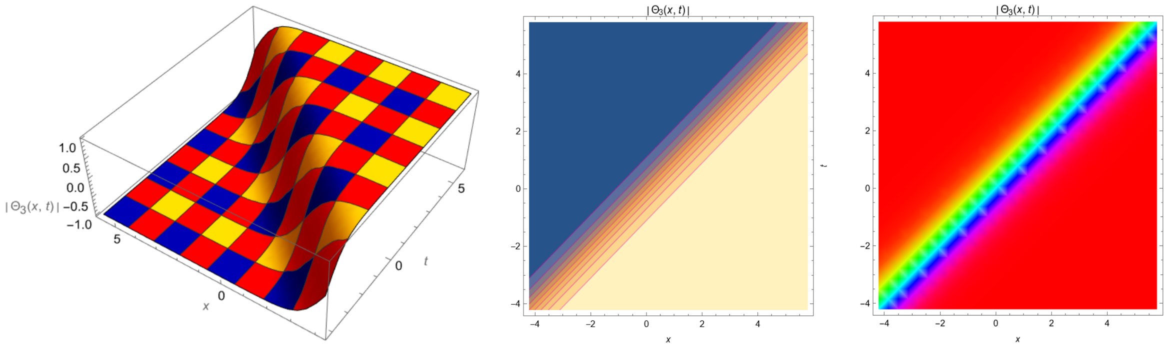

Figure 3. The 3-dimensional, contour and density figures of $|\Theta_3(x,t)|$ in (3.56), when $k = 1,\ w = 1$.

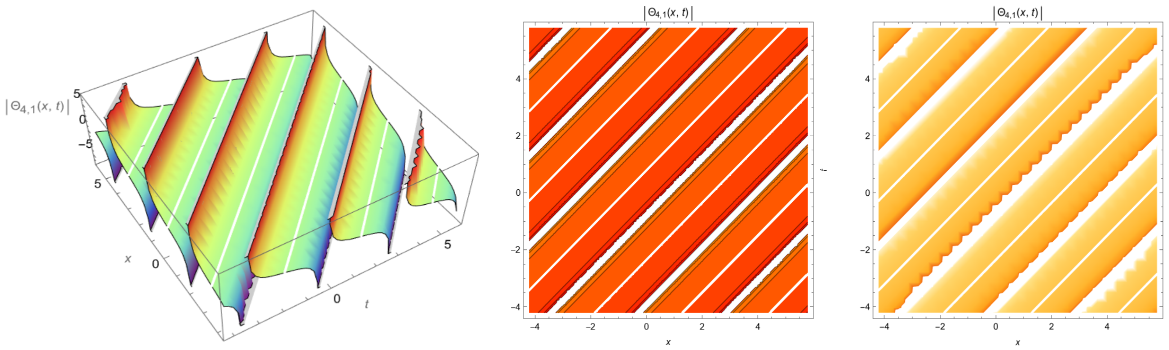

Figure 4. The 3-dimensional, contour and density figures of $|\Theta_{4,1}(x,t)|$ in (3.59), when $k = 1,\ w = 1$.

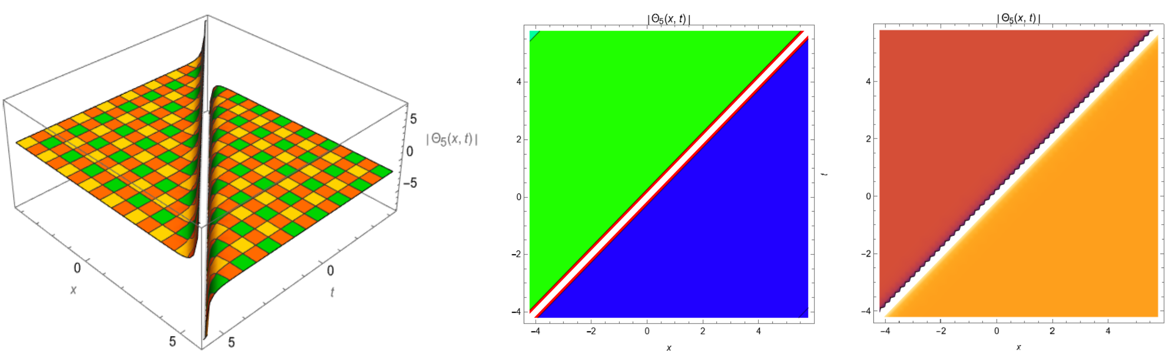

Figure 5. The 3-dimensional, contour and density figures of $|\Theta_5(x,t)|$ in (3.64), when $k = 1,\ w = 1$.

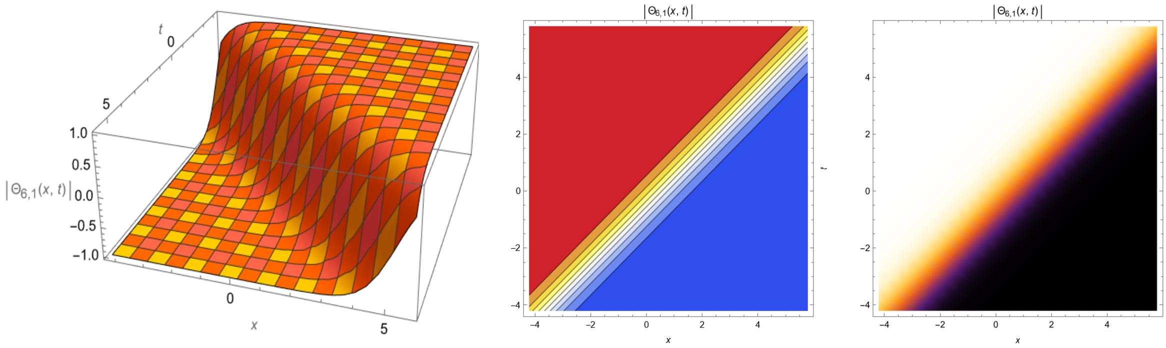

Figure 6. The 3-dimensional, contour and density figures of $|\Theta_{6,1}(x,t)|$ in (3.67), when $k = 1,\ w = 1$.

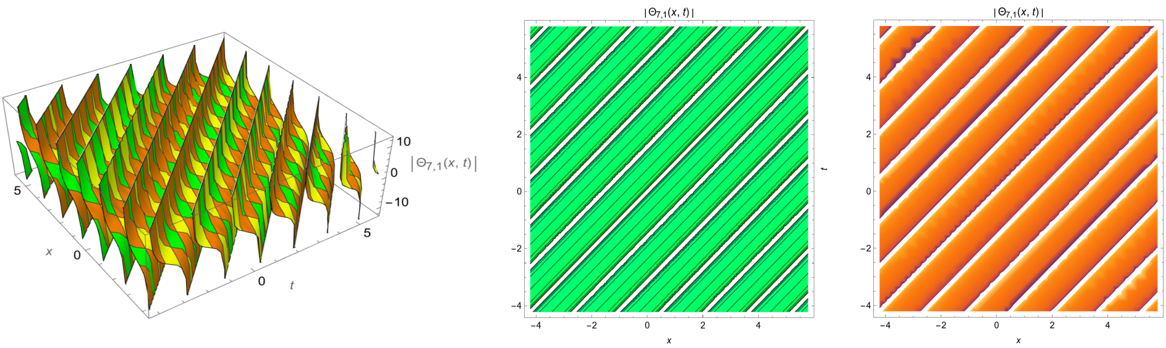

Figure 7. The 3-dimensional, contour and density figures of $|\Theta_{7,1}(x,t)|$ in (3.72), when $k = 1,\ w = 1$.

4 Lagrangian of the MEWE in potential form using the variational problem method

It is crucial that if an equation has odd order, it does not accept a variational problem, but thanks to the potential form Δ_v, this equation accepts a variational problem [8, 10, 11].

The MEWE

$$\Delta_\Theta : \Theta_t + \Theta^2\Theta_x - \Theta_{xxt} = 0$$

is in odd order. Frechet derivative of Δ_Θ is

$$\wp_{\Delta_\Theta} : \wp_t + \Theta^2\wp_x + 2\Theta\Theta_x - \wp_x^2\wp_t. \tag{4.1}$$

Note that $\wp_{\Delta_\Theta} \neq \wp^*_{\Delta_\Theta}$. We say that the MEWE does not accept a variational problem. The MEWE in potential form Δ_v is obtained by the well-known differential substitution $\Theta = v_x$,

$$\Delta_v = v_{xt} + v_x^2 v_{xx} - v_{xxxt} = 0. \tag{4.2}$$

Eq.(4.2) is named "the MEWE in the potential form" and its Frechet derivative is

$$\wp_{\Delta_v} = \wp_x\wp_t + v_x^2\wp_x^2 + 2v_xv_{xx}\wp_x - \wp_x^3\wp_t. \tag{4.3}$$

Note that Eq.(4.3) is self-adjoint.

Thanks to the **Theorem 2.2**, the MEWE in potential form Δ_v has a Lagrangian of the form

$$L[v] = \int_0^1 v\Delta_v[\lambda v]d\lambda$$
$$= -\frac{1}{2}v_x v_t - \frac{1}{12}v_x^4 - \frac{1}{2}v_{xx}v_{xt} + DivP. \tag{4.4}$$

Thus, we have

$$\mathcal{L}_{\Delta_v}[v] = -\frac{1}{2}(v_x v_t + \frac{1}{6}v_x^4 + v_{xx}v_{xt}). \tag{4.5}$$

5 Significant outcomes of solving model (3.2) utilizing the $\mu-$conservation laws for the MEWE

First of all, we will compute the $\mu-$conservation laws for the MEWE as Δ_v. Consider the second order Lagrangian Eq.(4.5) for the MEWE as Δ_v

$$\Delta_v = v_{xt} + v_x^2 v_{xx} - v_{xxxt}$$
$$= \grave{E}(\mathcal{L}_{\Delta_v}). \tag{5.1}$$

Surmise that $\Upsilon = \varphi\partial_v$ be a vector field for $\mathcal{L}_{\Delta_v}[v]$. Let $\mu = \lambda_1 dx + \lambda_2 dt$ be a semi-basic one-form such that $\wp_x\lambda_2 = \wp_t\lambda_1$ when $\Delta_v = 0$.

Thanks to the Eq.(2.4), Ω and its coefficients are

$$\Omega = \varphi\frac{\partial}{\partial v} + \psi^x\frac{\partial}{\partial v_x} + \psi^t\frac{\partial}{\partial v_t} + \psi^{xx}\frac{\partial}{\partial v_{xx}} + \psi^{xt}\frac{\partial}{\partial v_{xt}}, \tag{5.2}$$

in which

$$\psi^x = (\wp_x + \lambda_1)\varphi,\ \ \psi^t = (\wp_t + \lambda_2)\varphi,\ \ \psi^{xx} = (\wp_x + \lambda_1)\psi^x,\ \ \psi^{xt} = (\wp_t + \lambda_2)\psi^x. \tag{5.3}$$

By applying the $\mu-$prolongation Ω acts on the $\mathcal{L}_{\Delta_v}[v]$ and substituting $\frac{1}{v_x}(-\frac{1}{6}v_x^4 - v_{xx}v_{xt})$ for v_t, we get

$$-\frac{2}{3}\varphi_{vv} = 0, \quad -\varphi_v = 0,$$
$$\varphi_v\lambda_2 + \varphi_{vt} = 0,$$
$$-\frac{2}{3}\varphi\lambda_{1v} - \frac{7}{6}\varphi_v\lambda_1 - \frac{7}{6}\varphi_{vx} = 0,$$
$$\varphi_{xt} + \varphi_t\lambda_1 + \varphi_x\lambda_2 + \lambda_1\lambda_2\varphi = 0,$$
$$\frac{\varphi}{2}\lambda_1 + \frac{\varphi_x}{2} = 0,$$
$$-\frac{\varphi_{xx}}{2} - \frac{\varphi}{2}\lambda_{1x} - \lambda_1\varphi_x - \frac{\varphi}{2}\lambda_1^2 = 0. \tag{5.4}$$

Consider $\varphi = \varrho$, and $\mathcal{L}_{\Delta_v}[v] = 0$. A special solution of the system (5.4) is given by

$$\lambda_1 = -\frac{\varrho_x}{\varrho}, \quad \lambda_2 = -\frac{\varrho_t}{\varrho}. \tag{5.5}$$

Therefore, for $\mathcal{L}_{\Delta_v}[v]$, $\Upsilon = \varrho\frac{\partial}{\partial v}$ is a μ-symmetry. Then, by using **Theorem 2.2**, there exists an $\aleph$-vector P^i which is conservation law of μ, that is, $(\wp_i + \lambda_i)P^i = 0$. Then, by of Eq.(2.6), the $\aleph$-vector P^i for $\mathcal{L}_{\Delta_v}[v]$ is got

$$P^1 = -\varrho(\frac{1}{2}v_t + \frac{1}{3}v_x^3 - v_{xxt}),$$

$$P^2 = -\frac{v_x}{2}\varrho. \tag{5.6}$$

So, for $\mathcal{L}_{\Delta_v}[v]$, conservation law of μ is the form $\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$.

Corollary 5.1. *Conservation law of μ for the MEWE in potential form $\Delta_v = \grave{E}(\mathcal{L}_{\Delta_v})$ is as*

$$\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0, \tag{5.7}$$

where P^1 and P^2 are the $\aleph$-vector P^i of Eq.(5.6)

Remark 5.2. Conservation law of μ for the MEWE in potential form Δ_v, satisfying to the Noether's Theorem for $\mu-$symmetry, that is to say

$$\begin{aligned}(\wp_i + \lambda_i)P^i &= -\varrho(v_{xt} + v_x^2 v_{xx} - v_{xxxt}) \\ &= Q\grave{E}(\mathcal{L}_{\Delta_v}).\end{aligned} \tag{5.8}$$

Secondly, let us consider the MEWE as Δ_v

$$\Delta_v = v_{xt} + v_x^2 v_{xx} - v_{xxxt} = 0. \tag{5.9}$$

Eq.(5.9) corresponds to

$$\wp_x(v_t + \frac{1}{3}v_x^3 - v_{xxt}) = 0,$$

or equivalently

$$v_t + \frac{1}{3}v_x^3 - v_{xxt} = \Pi_1(t),$$

where $\Pi_1 = \Pi_1(t)$ is an arbitrary function. If we substitute

$$\Pi_1 - \frac{1}{3}v_x^3 + v_{xxt}$$

for v_t and substitute Θ for v_x in the $\aleph$-vector P^i of Eq.(5.6), then, we get the $\aleph$-vectors P^1 and P^2 as:

$$P^1 = -\varrho(\frac{1}{2}\Pi_1 + \frac{1}{6}\Theta^3 - \frac{1}{2}\Theta_{xt}),$$

$$P^2 = -\frac{\Theta}{2}\varrho. \tag{5.10}$$

Corollary 5.3. *Conservation law of μ for the MEWE Δ_Θ is*

$$\wp_x P^1 + \wp_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0, \tag{5.11}$$

where P^1 and P^2 are the $\aleph$-vector P^i of Eq.(5.10).

Remark 5.4. The MEWE Δ_Θ satisfies the characteristic form, that is to say

$$\begin{aligned}(\wp_i + \lambda_i)P^i &= -\frac{\varrho}{2}(\Theta_t + \Theta_x^2\Theta_x - \Theta_{xxt}) \\ &= Q\Delta_\Theta.\end{aligned} \tag{5.12}$$

6 Conclusions

In this paper, we considered the MEWE to scrutinize the $\mu-$symmetries, symmetry reductions, invariant solutions, exact solutions, and conservation laws. Firstly, we offered some essential properties of the $\mu-$symmetries, $\mu-$conservation law, and a modified version of GERFM. The vital object of the $\mu-$symmetry is a semi-basic one-form $\mu = \lambda_i dx_i$, which must satisfy compatibility conditions. Then we demonstrated that the approach of the $\mu-$symmetry reduction could also be analyzed in terms of the formulation of the Noether theorem when $\mu-$symmetries were regarded to discover the invariant solutions of PDEs named the $\mu-$invariant solutions. With the help of these infinitesimal generators, we get similarity variables and similarity functions. By utilizing these similarity variables and functions, we reduced nonlinear PDE into ODE. We employed mGERFM to this reduced ODE to get the soliton solution. We discovered various families of optical solutions, such as exponential function solutions, combined periodic soliton solution, singular periodic wave solution, shock wave solutions, trigonometric function solutions, mixed form soliton solution, hyperbolic solution in mixed form, and periodic soliton solution the prototype using the mGERFM. In this context, exponential function solutions Eq.(3.46), and Eq.(3.48), combined periodic soliton solution Eq.(3.51), singular periodic wave solution Eq.(3.53), shock wave solutions Eq.(3.56), and Eq.(3.69), trigonometric function solutions Eq.(3.59), and Eq.(3.72), mixed form soliton solution Eq.(3.64), hyperbolic solution in mixed form Eq.(3.67), and periodic soliton solution Eq.(3.74). One of the primary benefits of such approaches is that awaited configurations for solutions are determined from the beginning of the process. Moreover, we obtained Lagrangian potential by using the variational problem method and the Frechet derivative. In this context, the equation must have Lagrangian necessary and sufficient condition that its Frechet derivative is self-adjoint. Finally, the $\mu-$conservation law was investigated. The main novelty of this paper is that the MEWE equation is first studied using the $\mu-$symmetry method, a modified version of the GERFM, and $\mu-$conservation law. The 3-dimensional, contour, and density figures of the reached solutions were drawn with the aid of the Mathematica package program. The accuracy of the solutions obtained was tested and verified in the Maple package program.

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