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Solvability of a system of nonlinear variational inclusions involving Yosida approximation operator

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Abstract In this paper, we introduce the Yosida approximation operator associated with H-monotone operator in Hilbert spaces and prove some of its important properties. In addition, we consider a system of generalized nonlinear variational inclusions involving Yosida approximation operator and establish its equivalence with the fixed point problems using resolvent operator technique. This alternative equivalent formulation is employed to propose an iterative algorithm for solving this system of generalized nonlinear variational inclusions. Further, we discuss the existence of solution and convergence analysis of the proposed iterative algorithm under suitable conditions. Finally, we present a numerical example in support of the results proved and demonstrate the convergence result using MATLAB.

1 Introduction

In recent years, a significant interest has been shown by many researchers for the study of variational inclusions and their generalized forms, which plays a crucial role in bridging the gap between various fields such as analysis, elasticity, optimization, image processing, biomedical sciences and mathematical sciences. A wide range of problems arising in different domains of physical and applied sciences can be formulated in terms of variational inclusion problems. A typical variational inclusion problem is: Find $x \in \mathcal{H}$ such that $0 \in M(x)$, where $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued mapping on a Hilbert space $\mathcal{H}$. Several problems in different areas of research such as signal processing [1, 2], image recovery [3, 4] and machine learning [5, 6] can be well understood by modelling them as variational inclusions of the above form.

On the other hand, it is well known that monotonicity of the underlying operators plays an important role in the theory of variational inclusions. The notion of monotone operators was first introduced independently by Zarantonello [7] and Minty [8]. In sequel, many researchers have shown significant interest in monotone operators owing to their firm relation with the following evolution equation:

$$\begin{cases} \frac{dx}{dt} + M(x) = 0; \\ x(0) = x_0; \end{cases} \tag{1.1}$$

which is a model for many physical problems of practical applications. It is cumbersome to solve such types of models, if the involving function M is not continuous. To overcome this obstacle, Yosida introduced an idea to find a sequence of Lipschitz functions that approximate M in some sense. It is notable that set-valued monotone operators in Hilbert spaces can be regularized into single-valued Lipschitzian monotone operators via Yosida approximation and every such monotone operator can be associated with two quite useful single-valued lipschitz continuous operators, namely resolvent operator and Yosida approximation operator. The Yosida approximation operators are advantageous to estimate the solution of variational inclusion problem using resolvent operators. Several authors have utilized Yosida approximation of monotone operators to approximate the solution of variational inclusions, systems of variational inclusions and split variational inclusions. For more details, we refer [9–14].

The development of an efficient and implementable iterative methods is one of the most demanding and exciting areas in the theory of variational inequalities. Several types of splitting algorithms have been studied and modified for finding the approximate solution of variational inequality (inclusion) problems, see for example [15, 16], two-step forward-backward splitting method by Noor [17], generalized splitting method due to Kazmi and Bhat [18] and splitting method due to Peaceman and Rachford [19].

Keeping in view the excellent work referred above, in this paper, we introduce the Yosida approximation operator associated with H-monotone operator in Hilbert spaces and prove some of its important properties. In addition, we consider a system of generalized nonlinear variational inclusions involving Yosida approximation operator and establish its equivalence with the fixed point problems using resolvent operator technique. This alternative equivalent formulation is employed to propose an iterative algorithm for solving this system of generalized nonlinear variational inclusions. Further, we discuss the existence of solution and convergence analysis of the proposed iterative algorithm under suitable conditions. Finally, we present a numerical example in support of the results proved and demonstrate the convergence result using MATLAB. The results presented in this paper improve and extend many known results in the literature, see for example [6, 17, 18, 20–25] and the relevant references cited therein.

2 Preliminaries

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and $2^{\mathcal{H}}$ denotes the family of all the nonempty subsets of $\mathcal{H}$. First, we recall some well-known definitions and results that are critical to achieve the goal of this paper.

Definition 2.1 ([23]). Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued mappings and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. Then

(i) T is said to be monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall\, x, y \in \mathcal{H};$$

(ii) T is said to be strictly monotone if T is monotone and and equality holds if and only if $x = y$;

(iii) T is said to be δ-strongly monotone, if there exists $\delta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \delta\|x - y\|^2,\ \forall\, x, y \in \mathcal{H};$$

(iv) T is said to be γ-Lipschitz continuous, if there exists $\gamma > 0$ such that

$$\|Tx - Ty\| \leq \gamma\|x - y\|,\ \forall\, x, y \in \mathcal{H};$$

(v) M is said to be monotone, if

$$\langle u - v, x - y\rangle \geq 0,\ \forall\, x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(vi) M is said to be σ-strongly monotone, if there exists $\sigma > 0$ such that

$$\langle u - v, x - y\rangle \geq \sigma\|x - y\|^2,\ \forall\, x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(vii) M is said to be maximal monotone if M is monotone and $(I + \rho M)(\mathcal{H}) = \mathcal{H}$ for all $\rho > 0$, where I denotes the identity mapping on $\mathcal{H}$.

Remark 2.2. We note that the M is maximal monotone if and only if M is monotone and there is no other monotone operator whose graph contains strictly the graph Graph(M) of M, where Graph$(M) = \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in M(x)\}$.

Definition 2.3. Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. Then M is said to be H-monotone if M is monotone and $(H + \rho M)(\mathcal{H}) = \mathcal{H}$ holds for every $\rho > 0$.

Remark 2.4. If $H = I$, then the definition of I-monotone operators is that of maximal monotone operators.

Theorem 2.5 ([23]). *Let $H : \mathcal{H} \to \mathcal{H}$ be a strictly monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an H-monotone operator. Then the operator $(H + \rho M)^{-1}$ is single-valued.*

Definition 2.6 ([23]). Let the mappings H and M be same as in Theorem 2.5. Then the resolvent operator $\mathrm{R}_H^{M,\rho} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathrm{R}_H^{M,\rho}(x) = \left[H + \rho M\right]^{-1}(x), \ \forall\, x \in \mathcal{H}. \tag{2.1}$$

Remark 2.7. (i) If $\mathcal{H}$ is a Banach space, then the resolvent operator reduces to the proximal mapping considered by Xia and Huang [26].

(ii) If $M = \partial\varphi$, subdifferential of a proper convex lower semi-continuous functional $\varphi : \mathcal{H} \to (-\infty, \infty]$ and H is the identity mapping on $\mathcal{H}$, then the above resolvent operator reduces to the resolvent operator of φ.

Lemma 2.8 ([23]). *Let $H : \mathcal{H} \to \mathcal{H}$ be r-strongly monotone and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an H-monotone operator. Then the resolvent operator $\mathrm{R}_H^{M,\rho}$ is $\frac{1}{r}$-Lipschitz continuous, that is,*

$$\left\|\mathrm{R}_H^{M,\rho}(x) - \mathrm{R}_H^{M,\rho}(y)\right\| \le \frac{1}{r}\|x - y\|, \ \forall\, x, y \in \mathcal{H}.$$

Next, we shall introduce Yosida approximation operator associated with H-monotone operator and discuss some of its properties.

Definition 2.9. The Yosida approximation operator $\mathrm{J}_H^{M,\rho} : \mathcal{H} \to \mathcal{H}$ associated with H-monotone operator M is defined as

$$\mathrm{J}_H^{M,\rho}(x) = \frac{1}{\rho}\left[\mathrm{I} - \mathrm{R}_H^{M,\rho}\right](x), \ \forall\, x \in \mathcal{H} \text{ and } \rho > 0, \tag{2.2}$$

where I is the identity operator on $\mathcal{H}$.

Theorem 2.10. *Let the mappings H and M be same as in Lemma 2.8. Then the Yosida approximation operator defined by (2.2) is β-Lipschitz continuous, that is,*

$$\left\|\mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y)\right\| \le \beta\|x - y\|, \ \forall\, x, y \in \mathcal{H},$$

where $\beta = \frac{r+1}{\rho r}$.

Proof. In view of Lemma 2.8, we have for any $x, y \in \mathcal{H}$

$$\begin{aligned}
\left\|\mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y)\right\| &= \frac{1}{\rho}\left\|\left(x - \mathrm{R}_H^{M,\rho}(x)\right) - \left(y - \mathrm{R}_H^{M,\rho}(y)\right)\right\| \\
&\le \frac{1}{\rho}\left\{ \|x - y\| + \left\|\mathrm{R}_H^{M,\rho}(x) - \mathrm{R}_H^{M,\rho}(y)\right\|\right\} \\
&\le \frac{1}{\rho}\left\{\|x - y\| + \frac{1}{r}\|x - y\|\right\} \\
&= \frac{r+1}{\rho r}\|x - y\|.
\end{aligned}$$

That is,

$$\left\|\mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y)\right\| \le \beta\|x - y\|, \ \forall\, x, y \in \mathcal{H}, \text{ where } \beta = \frac{r+1}{\rho r}.$$

□

Theorem 2.11. *Let the mappings H and M be same as in Lemma 2.8. Then the Yosida approximation operator defined by (2.2) is δ-relaxed monotone, where $\delta = \frac{r-1}{\rho r}$ with $r > 1$.*

Proof. In view of Lemma 2.8 and Cauchy Schwatz, we have for any $x, y \in \mathcal{H}$

$$\begin{aligned}\big\langle \mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y), x-y\big\rangle &= \frac{1}{\rho}\big\langle \big(x - \mathrm{R}_H^{M,\rho}(x)\big) - \big(y - \mathrm{R}_H^{M,\rho}(y)\big), x-y\big\rangle \\ &= \frac{1}{\rho}\big\{\big\langle x-y, x-y\big\rangle - \big\langle \mathrm{R}_H^{M,\rho}(x) - \mathrm{R}_H^{M,\rho}(y), x-y\big\rangle\big\} \\ &\geq \frac{1}{\rho}\big\{\|x-y\|^2 - \|\mathrm{R}_H^{M,\rho}(x) - \mathrm{R}_H^{M,\rho}(y)\|\|x-y\|\big\} \\ &\geq \frac{1}{\rho}\big(1 - \frac{1}{r}\big)\|x-y\|^2 \\ &\geq \frac{r-1}{\rho r}\|x-y\|^2.\end{aligned}$$

That is,

$$\big\langle \mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y), x-y\big\rangle \geq -\delta\|x-y\|^2, \ \forall\, x, y \in \mathcal{H}, \text{ where } \ \delta = \frac{r-1}{\rho r}.$$

□

To illustrate the above concepts and results, we consider the following example.

Example 2.12. Let $\mathcal{H} = \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ be defined by $H(x) = 3x + 2, \ \forall\, x \in \mathbb{R}$. Then

$$\begin{aligned}\langle H(x) - H(y), x-y\rangle &= 3\|x-y\|^2 \\ &\geq 2\|x-y\|^2.\end{aligned}$$

Thus, H is r-strongly monotone with $r = 2$.

Let $M : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping defined by $M(x) = \{\frac{x}{2}\}, \ \forall\, x \in \mathbb{R}$. It can be easily verified that M is monotone. Also, for any $x \in \mathbb{R}$ and $\rho = 1$, we have

$$\big(H + \rho M\big)(x) = H(x) + M(x) = \frac{7x}{2} + 2.$$

Clearly the right hand side of above equation generates the whole space $\mathbb{R}$, that is,

$$\big(H + \rho M\big)(\mathbb{R}) = \mathbb{R}.$$

Hence, M is H-monotone.

Now, for $\rho = 1$, the resolvent operator and the Yosida approximation operator are given by

$$\begin{aligned}\mathrm{R}_H^{M,\rho}(x) &= \big[H + \rho M\big]^{-1}(x) = \frac{2x-4}{7} \\ \mathrm{J}_H^{M,\rho}(x) &= \frac{1}{\rho}\big[\mathrm{I} - \mathrm{R}_H^{M,\rho}\big](x) = \frac{5x+4}{7}.\end{aligned}$$

Further,

$$\begin{aligned}\big\|\mathrm{R}_H^{M,\rho}(x) - \mathrm{R}_H^{M,\rho}(y)\big\| &= \frac{2}{7}\|x-y\| \\ &\leq \frac{1}{2}\|x-y\|.\end{aligned}$$

This shows that the resolvent operator is $\frac{1}{r}$-Lipschitz continuous with $r = 2$. Also,

$$\begin{aligned}\big\|\mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y)\big\| &= \frac{5}{7}\|x-y\| \\ &\leq \frac{3}{2}\|x-y\|, \ \text{ where } \ \beta = \frac{r+1}{\rho r} = \frac{3}{2}.\end{aligned}$$

This shows that the Yosida approximation operator is $\frac{3}{2}$-Lipschitz continuous. Finally,

$$\begin{aligned}\langle \mathrm{J}_H^{M,\rho}(x) - \mathrm{J}_H^{M,\rho}(y), x - y\rangle &= \frac{5}{7}\|x - y\|^2 \\ &\geq \frac{1}{2}\|x - y\|^2, \text{ where } \delta = \frac{r-1}{\rho r} = \frac{1}{2}.\end{aligned}$$

Thus the generalized Yosida approximation operator is $\frac{1}{2}$-strongly monotone.

3 Formulation of the Problem and Iterative Algorithms

Let $H_1, H_2 : \mathcal{H} \to \mathcal{H}$ be a single-valued mappings and $M, N : \mathcal{H} \to 2^{\mathcal{H}}$ be H-monotone operators. We consider the following problem:

$$\begin{cases} 0 \in H_1(x) - H_1(y) + \rho_1\big(\mathrm{J}_{H_1}^{M,\rho_1}(y) + M(x)\big); \\ 0 \in H_2(y) - H_2(x) + \rho_2\big(\mathrm{J}_{H_2}^{N,\rho_2}(x) + N(y)\big), \end{cases} \tag{3.1}$$

Problem (3.1) is called system of generalized nonlinear variational inclusions involving Yosida approximation operators $\mathrm{J}_{H_1}^{M,\rho_1}$ and $\mathrm{J}_{H_2}^{M,\rho_2}$,

Some special cases:

1. If we take $H_1 = H_2, M = N, x = y$ and $\rho_1 = \rho_2 = \rho$, then SGNVI-Y (3.1) takes the following form: Find $x \in \mathcal{H}$ such that

$$0 \in \mathrm{J}_H^{M,\rho}(x) + M(x). \tag{3.2}$$

   Similar type of Yosida inclusion problems were investigated by Ahmad *et al.* [9] and Dilshad [11].

2. In case $\mathrm{J}_H^{M,\rho} = A$, a single-valued mapping, then Problem (3.2) reduces to the problem of finding $x \in \mathcal{H}$ such that

$$0 \in A(x) + M(x). \tag{3.3}$$

   Problem (3.3) was introduced and studied by Zeng *et al.* [27].

Here we remark that for appropriate choices of the mappings and the underlying space $\mathcal{H}$ in SGNVI-Y (3.1), one can obtain different classes of variational inclusions present in the literature, see for example [17, 18, 20, 21, 24] and the relevant literature mentioned therein.

Following lemma gives the equivalence of SGNVI-Y (3.1) with the fixed point problems which is vital for numerical point of view as it allows us to suggest different iterative methods for approximating the solution of SGNVI-Y (3.1).

Lemma 3.1. *The SGNVI-Y* (3.1) *admits a solution $x, y \in \mathcal{H}$ if and only if it satisfies*

$$x = \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1(y) - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}(y)\big], \qquad \rho_1 > 0, \tag{3.4}$$

where

$$y = \mathrm{R}_{H_2}^{N,\rho_2}\big[H_2(x) - \rho_2 \mathrm{J}_{H_2}^{N,\rho_2}(x)\big], \qquad \rho_2 > 0. \tag{3.5}$$

Proof. The proof immediately follows from the definition of the resolvent operator (2.1) and hence is omitted. □

Algorithm 3.2. For any initial point $x_0 \in \mathcal{H}$, compute $\{x_n\}, \{y_n\}$ by the iterative schemes:

$$x_{n+1} = \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1(y_n) - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big], \qquad \rho_1 > 0, \tag{3.6}$$

where

$$y_n = \mathrm{R}_{H_2}^{N,\rho_2}\big[H_2(x_n) - \rho_2 \mathrm{J}_{H_2}^{N,\rho_2}(x_n)\big], \qquad \rho_2 > 0. \tag{3.7}$$

Algorithm 3.2 is a named as the *generalized two-step forward-backward splitting method*.

By taking $H_1 = H_2 = H, M = N, \rho_1 = \rho_2 = \rho$ and $x_n = y_n$, the Iterative Algorithm 3.2 takes the following form.

Algorithm 3.3. For any initial point $x_0 \in \mathcal{H}$, compute $\{x_n\}$ by the iterative scheme:

$$x_{n+1} = \mathrm{R}_H^{M,\rho}\big[H(x_n) - \rho \mathrm{J}_H^{M,\rho}(x_n)\big], \qquad \rho > 0.$$

The Iterative Algorithm 3.3 gives the approximate solution to the Yosida inclusion problem (3.2).

We can rewrite (3.4) and (3.5) as

$$\begin{aligned} x &= \mathrm{R}_{H_1}^{M,\rho_1}\Big\{H_1\big[\mathrm{R}_{H_2}^{N,\rho_2}\big(H_2(x) - \rho_2 \mathrm{J}_{H_2}^{N,\rho_2}(x)\big)\big] - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}\big[\mathrm{R}_{H_2}^{N,\rho_2}\big(H_2(x) - \rho_2 \mathrm{J}_{H_2}^{N,\rho_2}(x)\big)\big]\Big\} \\ &= \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1 - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}\big]\mathrm{R}_{H_2}^{N,\rho_2}\big[H_2 - \rho_2 \mathrm{J}_{H_2}^{M,\rho_2}\big](x). \end{aligned} \tag{3.8}$$

This fixed point formulation allows us to propose the following iterative algorithm which we call the *modified resolvent method*.

Algorithm 3.4. For any initial point $x_0 \in \mathcal{H}$, compute $\{x_n\}$ by the iterative scheme:

$$x_{n+1} = \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1 - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}\big]\mathrm{R}_{H_2}^{N,\rho_2}\big[H_2 - \rho_2 \mathrm{J}_{H_2}^{M,\rho_2}\big](x_n), \quad \rho_1, \rho_2 > 0.$$

Iterative Algorithm 3.4 is a generalized version of Peaceman and Rachford's [19] splitting technique.

4 Existence Result and Convergence Criteria

This section is devoted to the existence and iterative approximation of the solution for SGNVI-Y (3.1) in view of Iterative Algorithm 3.2.

Theorem 4.1. *For $i = 1, 2$, let $H_i : \mathcal{H} \to \mathcal{H}$ be μ_i-Lipschitz continuous and r_i-strongly monotone, respectively and $M_i : \mathcal{H} \to 2^{\mathcal{H}}$ be an H_i-monotone operator. If there exist constants $\rho_1, \rho_2 > 0$ such that,*

$$\left|\rho_2 - \frac{\delta_2}{\beta_2^2}\right| < \frac{\sqrt{\delta_2^2 - \beta_2^2 \vartheta_2(2 - \vartheta_2)}}{\beta_2^2}; \quad \delta_2 > \beta_2\sqrt{\vartheta_2(2 - \vartheta_2)}; \quad \vartheta_2 < 1, \tag{4.1}$$

$$\left|\rho_1 - \frac{\delta_1}{\beta_1^2}\right| < \frac{\sqrt{\beta_1^2\left[(r_1 r_2 - \vartheta_1)^2 - 1\right] + \delta_1^2}}{\beta_1^2}; \quad \delta_1 > \beta_1\sqrt{(r_1 r_2 - \vartheta_1)^2 - 1}, \tag{4.2}$$

$$\textit{where} \quad \vartheta_i = \sqrt{\mu_i^2 - 2r_i + 1}, \quad \beta_i = \frac{r_i + 1}{\rho_i r_i}, \quad \delta_i = \frac{r_i - 1}{\rho_i r_i}, \ i = 1, 2.$$

Then the iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by the Iterative Algorithm 3.2 converge strongly to x, y, respectively, the solution of SGNVI-Y (3.1).

Proof. In view of Iterative Algorithm 3.2 and Lemma 2.8, we have

$$\begin{aligned} &\|x_{n+2} - x_{n+1}\| \\ &= \big\|\mathrm{R}_{H_1}^{M,\rho_1}\big[H_1(y_{n+1}) - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1})\big] - \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1(y_n) - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big]\big\| \\ &\le \frac{1}{r_1}\big\|H_1(y_{n+1}) - H_1(y_n) - \rho_1\big(\mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1}) - \mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big)\big\| \\ &\le \frac{1}{r_1}\big\{\big\|H_1(y_{n+1}) - H_1(y_n) - \big(y_{n+1} - y_n\big)\big\| + \big\|y_{n+1} - y_n - \rho_1\big(\mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1}) - \mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big)\big\|\big\}. \end{aligned} \tag{4.3}$$

Since H_1 is μ_1-Lipschitz continuous and r_1-strongly monotone, we have the following estimate:

$$\begin{aligned}\big\|H_1(y_{n+1})-H_1(y_n)-\big(y_{n+1}-y_n\big)\|^2\\ &=\big\|H_1(y_{n+1})-H_1(y_n)\big\|^2-2\big\langle H_1(y_{n+1})-H_1(y_n),y_{n+1}-y_n\big\rangle\\ &\quad+\|y_{n+1}-y_n\|^2\\ &\le\big(\mu_1^2-2r_1+1\big)\,\|y_{n+1}-y_n\|^2.\end{aligned}$$

Hence,

$$\big\|H_1(y_{n+1})-H_1(y_n)-(y_{n+1}-y_n)\,\big\|\le\sqrt{\mu_1^2-2r_1+1}\,\|y_{n+1}-y_n\|. \tag{4.4}$$

Using Theorems 2.10 and 2.11, we have the following estimate:

$$\begin{aligned}\big\|y_{n+1}-y_n-\rho_1\big(\mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1})-\mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big)\big\|\\ &=\|y_{n+1}-y_n\|^2-2\rho_1\big\langle \mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1})-\mathrm{J}_{H_1}^{M,\rho_1}(y_n),y_{n+1}-y_n\big\rangle\\ &\quad+\rho_1^2\big\|\mathrm{J}^{H_1}_{M,\rho_1}(y_{n+1})-\mathrm{J}^{H_1}_{M,\rho_1}(y_n)\big\|^2\\ &\le(1-2\rho_1\delta_1+\rho_1^2\beta_1^2)\|y_{n+1}-y_n\|^2.\end{aligned}$$

Hence,

$$\big\|y_{n+1}-y_n-\rho_1\big(\mathrm{J}_{H_1}^{M,\rho_1}(y_{n+1})-\mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big)\big\|\le\sqrt{1-2\rho_1\delta_1+\rho_1^2\beta_1^2}\,\|y_{n+1}-y_n\|\,. \tag{4.5}$$

Again, using Lemma 2.8, (3.7), (4.4), (4.5) and the fact that H_2 is μ_2-Lipschitz continuous and r_2-strongly monotone, we have the following estimate:

$$\begin{aligned}\|y_{n+1}-y_n\|&=\big\|\mathrm{R}_{H_2}^{N,\rho_2}\big[H_2(x_{n+1})-\rho_2\mathrm{J}_{H_2}^{N,\rho_2}(x_{n+1})\big]-\mathrm{R}_{H_2}^{N,\rho_2}\big[H_2(x_n)-\rho_2\mathrm{J}_{H_2}^{N,\rho_2}(x_n)\big]\big\|\\ &\le\frac{1}{r_2}\big\|H_2(x_{n+1})-H_2(x_n)-\rho_2\big(\mathrm{J}_{H_2}^{N,\rho_2}(x_{n+1})-\mathrm{J}_{H_2}^{N,\rho_2}(x_n)\big)\big\|\\ &\le\frac{1}{r_2}\big[\sqrt{\mu_2^2-2r_2+1}+\sqrt{1-2\rho_2\delta_2+\rho_2^2\beta_2^2}\big]\|x_{n+1}-x_n\|.\end{aligned} \tag{4.6}$$

Clubbing (4.3)-(4.6), we have

$$\|x_{n+2}-x_{n+1}\|\le\frac{1}{r_1r_2}\big((\vartheta_1+\vartheta_3)(\vartheta_2+\vartheta_4)\big)\|x_{n+1}-x_n\|, \tag{4.7}$$

where

$$\vartheta_1=\sqrt{\mu_1^2-2r_1+1},\qquad \vartheta_3=\sqrt{1-2\rho_1\delta_1+\rho_1^2\beta_1^2},$$
$$\vartheta_2=\sqrt{\mu_2^2-2r_2+1},\qquad \vartheta_4=\sqrt{1-2\rho_2\delta_2+\rho_2^2\beta_2^2}.$$

Hence, we have

$$\|x_{n+2}-x_{n+1}\|\le\vartheta\|x_{n+1}-x_n\|, \tag{4.8}$$

where $\vartheta=\dfrac{1}{r_1r_2}(\vartheta_1+\vartheta_3)(\vartheta_2+\vartheta_4)<\dfrac{1}{r_1r_2}(\vartheta_1+\vartheta_3),\quad$ if $\quad\vartheta_2+\vartheta_4<1.$
Conditions (4.1) and (4.2) ensures that $\vartheta_2+\vartheta_4<1$ and $\frac{1}{r_1r_2}(\vartheta_1+\vartheta_3)<1$ and thus $0\le\vartheta<1$. Therefore, from (4.8), it follows that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$ and consequently (4.6) implies that $\{y_n\}$ is a Cauchy sequence in $\mathcal{H}$. Hence, in view of completeness of $\mathcal{H}$, there exist $x,y\in\mathcal{H}$ such that $x_n\to x$ and $y_n\to y$. Further, since the mappings $H_1,H_2,\mathrm{R}_{H_1}^{M,\rho_1},\mathrm{R}_{H_2}^{N,\rho_2},\mathrm{J}_{H_1}^{M,\rho_1},\mathrm{J}_{H_2}^{N,\rho_2}$ are all continuous, hence in view of Iterative Algorithm 3.2 it follows that x,y satisfy (3.4), (3.5) and consequently by Lemma 3.1 $x,y\in\mathcal{H}$ is a solution of SGNVI-Y (3.1). Hence the result is established. □

5 Numerical Example

In this section, we construct a numerical example to illustrate that iterative sequences generated by the Iterative Algorithm 3.2 converge to the solution of SGNVI-Y (3.1). The computational table and the convergence graphs have been constructed using MATLAB R2016b.

Example 5.1. Let $\mathcal{H} = \mathbb{R}$, the set of real numbers. Let $H_1, H_2 : \mathbb{R} \to \mathbb{R}$ be single-valued mappings defined by $H_1(x) = x - 1$, $H_2(x) = 2x + 1, \forall x, y \in \mathbb{R}$ and let $M, N : \mathbb{R} \to 2^{\mathbb{R}}$ be set-valued mappings defined by $M_1(x) = \{2x\}, M_2(x) = \{\frac{x}{2}\}, \forall x \in \mathbb{R}$. Let the iterative sequences $\{x_n\}, \{y_n\}$ be computed by the following iterative scheme:

$$x_{n+1} = \mathrm{R}_{H_1}^{M,\rho_1}\big[H_1(y_n) - \rho_1 \mathrm{J}_{H_1}^{M,\rho_1}(y_n)\big], \tag{5.1}$$

$$\text{where} \qquad y_n = \mathrm{R}_{H_2}^{M,\rho_2}\big[H_2(x_n) - \rho_2 \mathrm{J}_{H_2}^{M,\rho_2}(x_n)\big]. \tag{5.2}$$

For $\rho_1 = \rho_2 = \frac{1}{3}$, we obtain the resolvent operators and Yosida approximation operators as

$$\mathrm{R}_{H_1}^{M,\rho_1}(x) = \big[H_1 + \rho_1 M\big]^{-1}(x) = \frac{3x}{5} + \frac{3}{5}; \tag{5.3}$$

$$\mathrm{R}_{H_2}^{N,\rho_2}(x) = \big[H_2 + \rho_2 N\big]^{-1}(x) = \frac{6x}{13} - \frac{6}{13}; \tag{5.4}$$

$$\mathrm{J}_{H_1}^{M,\rho_1}(x) = \frac{1}{\rho_1}\left[I - \mathrm{R}_{H_1}^{M,\rho_1}\right](x) = \frac{6x}{5} - \frac{9}{5}; \tag{5.5}$$

$$\mathrm{J}_{H_2}^{N,\rho_2}(x) = \frac{1}{\rho_2}\left[I - \mathrm{R}_{H_2}^{N,\rho_2}\right](x) = \frac{21x}{13} + \frac{18}{13}. \tag{5.6}$$

Again, for $\rho_1 = \rho_2 = \frac{1}{3}$, we obtain y_n and x_{n+1} using (5.1)-(5.6) as follows

$$y_n = \frac{114x_n}{169} - \frac{36}{169}$$

$$x_{n+1} = \frac{1026x_n}{4225} - \frac{1197}{4225}.$$

It is shown in Table 1 and Figure 1 that for initial values $x_0 = -5, x_0 = -2, x_0 = 4$ and $x_0 = 6.5$, the sequence $\{x_n\}$ converges to $x = 0.3742$, the solution of SGNVI-Y (3.1). One can easily check that the fixed point formulations (3.4) and (3.5) are satisfied for $x = 0.3742$. All the graphs plotted in Figure 1 are combined in Figure 2.

Table 1: Computational results for different initial values $x_0 = -5, x_0 = -2, x_0 = 4, x_0 = 6.5$.

No. of iteration	For $x_0 = -5$ x_n	For $x_0 = -2$ x_n	For $x_0 = 4$ x_n	For $x_0 = 6.5$ x_n
$n = 0$	-5.0000	-2.0000	4.0000	6.5000
$n = 1$	-0.9309	-0.2024	1.2547	1.8618
$n = 2$	0.0573	0.2342	0.5880	0.7354
$n = 3$	0.2972	0.3402	0.4261	0.4619
$n = 4$	0.3555	0.3659	0.3868	0.3955
$n = 5$	0.3696	0.3722	0.3772	0.3794
$n = 6$	0.3731	0.3737	0.3749	0.3754
$n = 7$	0.3739	0.3741	0.3744	0.3745
$n = 8$	0.3741	0.3742	0.3742	0.3743
$n = 9$	0.3742	0.3742	0.3742	0.3742
$n = 10$	0.3742	0.3742	0.3742	0.3742

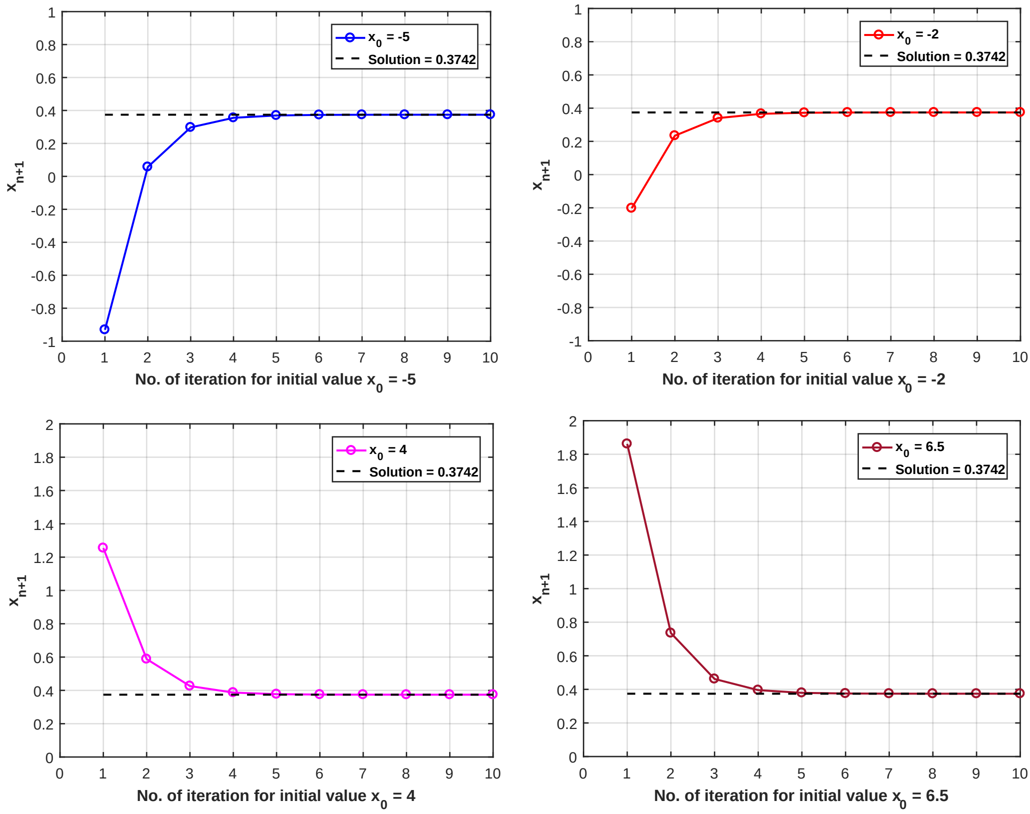

Figure 1: Convergence of $\{x_n\}$ with initial values $x_0 = -5, x_0 = -2, x_0 = 4, x_0 = 6.5$.

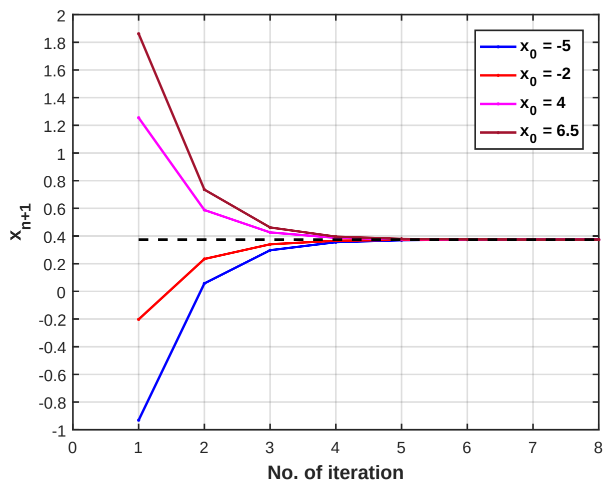

Figure 2: Conjoining graph of convergence of $\{x_n\}$.

Conclusion

In this paper, the Yosida approximation operator associated to H-monotone operator has been discussed. Further a system of generalized nonlinear variational inclusions involving Yosida approximation operator have been investigated. In addition, some iterative techniques for studying this system of variational inclusions have been formulated. The iterative algorithms considered in this paper unify, extend and improve some known algorithms available in the literature. Moreover, the convergence analysis of the generalized two-step forward-backward splitting method have been analyzed. A numerical example has been constructed and using MATLAB, a computational table and convergence graphs have been plotted in support of the convergence of the sequences generated by the Iterative Algorithm 3.2. The work presented in this paper extend and improve several known results in the literature.

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