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Solutions of Pell equation $x^2 - dy^2 = \pm 1$ via linearly recurring sequences and continued fractions

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Abstract *In this paper, we provide new combinatorial and analytic expressions of the solutions of the Pell equation* $x^2 - dy^2 = \pm 1$. *Our approach is based on some properties of recurring sequences with periodic coefficients and continued fraction expansion of* $\sqrt{\mathrm{d}}$. *Furthermore, using the established results, we build some algorithms to calculate the solutions. As an application, we investigate some specific values of* d, *notably* $\mathrm{d} = k^2 \pm l$ *when* l *divides* k. *Yet, to enhance our approach, some numerical examples are studied.*

1 Introduction

Let d *be a positive integer that is not a perfect square. The classical Pell equation is given by*

$$x^2 - dy^2 = \pm 1 \tag{1.1}$$

Several authors have extensively studied the Pell equation (1.1) (see, for instance, [3], [4], [8], [13], [14]). We first recall some known facts from the fascinating history of Pell equations. Ironically, Pell (1611-1685) was not the first to work on this problem, nor did he contribute to our knowledge of solving it. It is said that Euler (1707-1783) mistakenly attributed Brouncher's work (1620-1684) on this equation to Pell, and the name stuck.
The Pell's equation was first studied by Brahmagupta (598-670) and Bhaskara (1114-1185). Its complete theory was worked out by Lagrange (1736-1813).
Recently, the method followed by Teckan in [11] depends essentially on continued fraction expansion of $\sqrt{d}$ *for some specific values of* d. *On the other hand, K. Conrad treated the problem by the elementary method and triangular-square numbers (see, for instance, [10]).*

Continued fractions play an important role in solutions to the Pell equation (1.1). That is, for a non-square positive integer d, *the continued fraction expansion of* $\sqrt{\mathrm{d}}$ *is a simple* p-*periodic continued fraction defined by*

$$\sqrt{d} = [b_0, \overline{b_1, b_2, \cdots, b_p}] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots + \cfrac{1}{b_p + \cdots}}}, \tag{1.2}$$

where the sequence eventually repeats, i.e $b_{p+i} = b_i$ *for all* $i \geq 1$, *and* $b_1, b_2, ..., b_{p-1}$ *is a palindrome, namely*

$$\sqrt{d} = \left[b_0; \overline{b_1, b_2, ..., b_2, b_1, 2b_0}\right].$$

We denote by $A_n/B_n = [b_0, b_1, b_2, \cdots, b_n]$ *the* n^{th} *convergent of the continued fraction* (1.2) *such that*

$$\begin{aligned} A_n &= b_n A_{n-1} + A_{n-2} \\ B_n &= b_n B_{n-1} + B_{n-2} \end{aligned} \tag{1.3}$$

where $A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1$.

It is well known in the literature that all positive integer solutions $\{(X_k, Y_k)\}_{k\geq 1}$ *of the Pell equation (1.1) are given by the following theorem.*

Theorem 1.1. *(see [12], Theorem A) Let* d *be a positive integer that is not a perfect square, and let* p *be the minimal period length of the simple periodic continued fraction* $\sqrt{d} = [b_0; \overline{b_1, ..., b_p}]$. *Let* A_n/B_n *be the* n^{th} *convergent of this simple continued fraction. Then*

(a) the positive solutions of the positive Pell's equation $x^2 - dy^2 = 1$ *are given by*

$$(X_k, Y_k) = \begin{cases} (A_{kp-1}, B_{kp-1}) & \text{if } p \text{ is even} \\ (A_{2kp-1}, B_{2kp-1}) & \text{if } p \text{ is odd} \end{cases} \quad \text{for all} \quad k = 1, 2, 3, ..$$

(b) the positive solutions of the negative Pell's equation $x^2 - dy^2 = -1$ *are given by*

$$\begin{aligned} X_k &= A_{(2k-1)p-1} \qquad k = 1, 2, 3, ... \\ Y_k &= B_{(2k-1)p-1} \qquad k = 1, 2, 3, ... \end{aligned}$$

when p *is odd, and there is no solution when* p *is even.*

The solution (X_k, Y_k) *is called the* k^{th} *solution, and the solution* (X_1, Y_1) *with the smallest value among all solutions* $\{(X_k, Y_k)\}_{k\geq 1}$ *of Pell equation (1.1) is called fundamental solution.*

Now, let us recall some properties on linearly recurring sequences (see for instance [1, 2, 6, 7]). A linear homogeneous difference equation of order $r \geq 2$ *with variable coefficients is given by*

$$V_{n+r} = c_0(n)V_{n+r-1} + \cdots + c_{r-1}(n)V_n, \tag{1.4}$$

where $c_j(n)$ *are real functions, and* $V_0, \cdots, V_{r-1}$ *are the initial values. When the coefficients are constant, the sequence* $\{V_n\}_{n\geq 0}$ *is nothing else but the linearly recurring sequence, also known as the r-Generalized Fibonacci Sequence. Let's elaborate further on this case. We consider the polynomial* $P(X) = X^r - c_0X^{r-1} - \cdots - c_{r-2}X - c_{r-1}$, *called the characteristic polynomial, and its roots* $\lambda_1, ..., \lambda_s$ *with multiplicities* $m_1, ..., m_s$ *respectively, called the characteristic roots. The classical Analytic formula of the terms of* $\{V_n\}_{n\geq 0}$ *(known also as Binet formula) is given by*

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j} n^j \right) \lambda_i^n \qquad \text{for } n \geq 0, \tag{1.5}$$

where $\beta_{i,j}$ *are determined uniquely from the initial conditions* $\{V_j\}_{0\leq j\leq r-1}$, *by solving the system of* r *linear equations*

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j} n^j \right) \lambda_i^n = V_n, \quad n = 0, 1, \cdots, r-1.$$

Meanwhile, the combinatorial expression of the terms of $\{V_n\}_{n\geq 0}$ *was established in [5] as follows:*

$$V_n = \rho(n, r)w_0 + \rho(n-1, r)w_1 + \cdots + \rho(n-r+1, r)w_{r-1}, \text{ for } n \geq r, \tag{1.6}$$

where $w_s = c_{r-1}V_s + \cdots + c_sV_{r-1}$ *for* $s = 0, 1, \cdots, r-1$ *and* $\rho(n, r)$ *is given by*

$$\rho(n, r) = \sum_{k_0 + 2k_1 + ... + rk_{r-1} = n-r} \frac{(k_0 + ... + k_{r-1})!}{k_0! ... k_{r-1}!} c_0^{k_0} ... c_{r-1}^{k_{r-1}}, \text{ for } n \geq r. \tag{1.7}$$

with $\rho(r,r) = 1$ *and* $\rho(n,r) = 0$ *if* $n \leq r-1$.

The novel method that we deal with in this paper is based on some recent results established in [1] regarding the linear difference equations with periodic coefficients, where some initial conditions are considered. The study of this paper is structured as follows: Section 2 involves two parts. In the first one, we formulate the positive integer solutions emanating from combinatorial expressions of the numerator A_n *and denominator* B_n *of convergents of* $\sqrt{d}$ *considered as two difference equations with periodic coefficients by converting these equations to equivalent difference equations with constant coefficients. Moreover, we provide some algorithms constructed from the established results that enable us to compute the solutions of the Pell equation (1.1). Similarly, in the second part, we formulate the analytic expressions of solutions emanating from the analytic formula (Binet formula) of* A_n *and* B_n*. In section 3, we give the continued fraction expansion of* $\sqrt{k^2 \pm l}$ *when* l *divides* k*, then we apply the results established in section 2 to solve the Pell equation* $x^2 - (k^2 \pm l)y^2 = 1$.

2 Combinatorial and analytic solutions of Pell equation $x^2 - dy^2 = \pm 1$

In this section, we study the recurrence relations of the sequences $\{A_n\}_{n \geq -1}$ *and* $\{B_n\}_{n \geq -1}$*, released from the continued fractions expansion of* $\sqrt{d}$*, to formulate combinatorial and analytic expressions of solutions of the Pell equation (1.1), and we only consider positive integer solutions.*

Let A_n/B_n *be the* n^{th} *convergent of the continued fraction* $[b_0, \overline{b_1, ..., b_p}]$ *such that* (1.3) *is satisfied. The fundamental recurrence formulas (1.3) can be viewed as two linear difference equations of order* 2 *with periodic coefficients of period* p.

First, let's consider the sequence $\{A_n\}_{n \geq -1}$*, then we have*

$$\begin{cases} A_n = b_n A_{n-1} + A_{n-2}, & (n \geq 1) \\ A_{-1} = 1\ , \quad A_0 = b_0 \end{cases}$$

where $b_{p+i} = b_i$ *for all* $i \geq 1$*, then*

$$\begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ A_{n-2} \end{pmatrix}, \quad (n \geq 1).$$

Put $U_n = \begin{pmatrix} A_n \\ A_{n-1} \end{pmatrix}$ *and* $C_n = \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}$*, then we get*

$$U_p = C_p C_{p-1} ... C_1 U_0.$$

Put $B = C_p C_{p-1} ... C_1$*, then for every* $k \geq 1$ *and using the periodicity conditions of* b_n*, we obtain*

$$U_{kp} = B^k U_0$$

Now, we point out that $C_p C_{p-1} ... C_1 C_0 = \begin{pmatrix} A_p & B_p \\ A_{p-1} & B_{p-1} \end{pmatrix}$.

That is $B = \begin{pmatrix} A_p & B_p \\ A_{p-1} & B_{p-1} \end{pmatrix} C_0^{-1} = \begin{pmatrix} A_p & B_p \\ A_{p-1} & B_{p-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -b_0 \end{pmatrix}$.

Hence

$$B = \begin{pmatrix} B_p & A_p - b_0 B_p \\ B_{p-1} & A_{p-1} - b_0 B_{p-1} \end{pmatrix}$$

Let $P_B(x) = x^2 - c_0 x - c_1$ *be the characteristic polynomial of the matrix B, we have*

$$\begin{aligned} c_0 &= B_p + A_{p-1} - b_0 B_{p-1} \qquad (c_0 \in \mathbb{N}^\star) \\ c_1 &= A_p B_{p-1} - B_p A_{p-1} = (-1)^{p-1} \qquad (c_1 \in \{-1, 1\}) \end{aligned} \tag{2.1}$$

.

Furthermore, from [9], we can check that

$$A_{p-1} = b_0 B_{p-1} + B_{p-2}$$

Then we get

$$c_0 = B_p + A_{p-1} - b_0 B_{p-1} = 2A_{p-1} \tag{2.2}$$

It follows from the Cayley-Hamilton Theorem that $B^2 - c_0 B - c_1 I_2 = \Theta_2$. *Thereafter, for every* $k \geq 2$ *we have*

$$\left(B^k - c_0 B^{k-1} - c_1 B^{k-2}\right) \begin{pmatrix} A_0 \\ A_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and since $\begin{pmatrix} A_{kp} \\ A_{kp-1} \end{pmatrix} = B^k \begin{pmatrix} A_0 \\ A_{-1} \end{pmatrix}$, *we get for every* $k \geq 2$

$$A_{kp-1} = c_0 A_{(k-1)p-1} + c_1 A_{(k-2)p-1} \tag{2.3}$$

We observe that $\{A_{kp-1}\}_{k\geq 0}$ *is a linearly recurring sequence with constant coefficients* c_0, c_1 *and initial conditions* $A_{-1} = 1$ *and* A_{p-1}. *Similar reasoning can be used for the sequence* $\{B_{kp-1}\}_{k\geq 0}$ *(only the initial conditions are changed). Now, we are ready to exploit formulas* (1.5)-(1.6) *in the aim to provide handled expressions of the solutions of the Pell equation* (1.1).

2.1 Combinatorial expressions of solutions of Pell equation $x^2 - dy^2 = \pm 1$

Consider the linearly recurring sequence $\{A_{kp-1}\}_{k\geq 0}$ *given by* (2.3), *with initial conditions* $A_{-1} = 1$ *and* A_{p-1}. *It ensues from formula* (1.6) *that the combinatorial expression of* $\{A_{kp-1}\}_{k\geq 0}$ *is given by,*

$$A_{kp-1} = \rho(k,2)[c_1 A_{-1} + c_0 A_{p-1}] + \rho(k-1,2) c_1 A_{p-1} \qquad (k \geq 2)$$

where $\rho(n,2)$ *is given by*

$$\rho(n,2) = \sum_{h=0}^{[\frac{n-2}{2}]} \binom{n-2-h}{h} c_0^{n-2-2h}\, c_1^h \qquad (n > 2)$$

with $\rho(1,2) = 0$ *and* $\rho(2,2) = 1$. *Making use of Theorem 1.1 and since the solutions of the Pell equation* $x^2 - dy^2 = \pm 1$ *are expressed in terms of* $(p-1)^{th}$ *convergent of* $\sqrt{d}$, *we get the following result.*

Proposition 2.1. *Let* $[b_0, \overline{b_1, ..., b_p}]$ *be the simple periodic continued fraction of* $\sqrt{d}$ *and* A_{p-1}/B_{p-1} *its* $(p-1)^{th}$ *convergent, then the combinatorial expressions of* (X_k, Y_k), *solutions of Pell's equation* $x^2 - dy^2 = 1$, *are given as follows:*

- *When p is even :*

$$\begin{cases} X_k = -\rho(k,2) + A_{p-1}\rho(k+1,2) \\ Y_k = B_{p-1}\rho(k+1,2) \end{cases} \qquad (k \geq 1)$$

- *When p is odd :*

$$\begin{cases} X_k = \rho(2k,2) + A_{p-1}\rho(2k+1,2) \\ Y_k = B_{p-1}\rho(2k+1,2) \end{cases} \qquad (k \geq 1),$$

where $\rho(1,2) = 0$, $\rho(2,2) = 1$ *and for* $n > 2$,

$$\rho(n,2) = \sum_{h=0}^{[\frac{n-2}{2}]} \binom{n-2-h}{h} (-1)^{h(p-1)} \left(2A_{p-1}\right)^{n-2-2h}.$$

For the negative Pell's equation $x^2 - dy^2 = -1$, *the combinatorial solutions* (X_k, Y_k) *are given by*

$$\begin{cases} X_k = \rho(2k-1,2) + A_{p-1}\rho(2k,2) \\ Y_k = B_{p-1}\rho(2k,2) \end{cases} \qquad (k \geq 1)$$

when p *is odd, and there is no solution when* p *is even.*

To derive numerical formulas from the above proposition, which leads to the development of algorithms for computing the solutions, straightforward calculations permit us to formulate them as follows:

Corollary 2.2. *Let* $[b_0, \overline{b_1, ..., b_p}]$ *be the simple periodic continued fraction of* $\sqrt{d}$ *and* A_{p-1}/B_{p-1} *its* $(p-1)^{th}$ *convergent. Suppose that* p *is even, then for* $n \geq 2$, *the positive integer solutions* (X_n, Y_n) *of* $x^2 - dy^2 = 1$ *are given by :*
Case $n = 2j$

$$\begin{cases} X_{2j} = \displaystyle\sum_{h=0}^{j-1} (-1)^h \binom{2j-2-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2j-1-2h}\right) - 1\right] (2A_{p-1})^{2j-2-2h} \\ Y_{2j} = B_{p-1} \displaystyle\sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h} (2A_{p-1})^{2j-1-2h} \end{cases}$$

Case $n = 2j+1$

$$\begin{cases} X_{2j+1} = (-1)^j A_{p-1} + \displaystyle\sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2j-2h}\right) - 1\right] (2A_{p-1})^{2j-1-2h} \\ Y_{2j+1} = B_{p-1} \displaystyle\sum_{h=0}^{j} (-1)^h \binom{2j-h}{h} (2A_{p-1})^{2j-2h} \end{cases}$$

Thus, according to Corollary 2.2, we construct the following algorithms.

Algorithm 1 : Calculate even solutions X_{2j} and Y_{2j} of Pell equation $x^2 - dy^2 = 1$ when p is even.

```
Input : A positive integers $A_{p-1}, B_{p-1}, N$
Output : The values of $X_{2j}, Y_{2j}$
for $j \leftarrow 0$ To $N$ do
    $X_{2j} \leftarrow 0$
    $Y_{2j} \leftarrow 0$
    if $j \geq 1$ then
        for $h \leftarrow 0$ To $j-1$ do
            $X_{2j} \leftarrow (-1)^h \binom{2j-2-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2j-1-2h}\right) - 1\right] (2A_{p-1})^{2j-2-2h}$
            $Y_{2j} \leftarrow B_{p-1}(-1)^h \binom{2j-1-h}{h} (2A_{p-1})^{2j-1-2h}$
            $X_{2j} \leftarrow X_{2j} + 0$
            $Y_{2j} \leftarrow Y_{2j} + 0$
            $h \leftarrow h + 1$
        end for
    else
        $X_{2j} \leftarrow 1$
        $Y_{2j} \leftarrow 0$
    end if
end for
```

Algorithm 2 *: Calculate odd solutions* X_{2j+1} *and* Y_{2j+1} *of Pell equation* $x^2 - dy^2 = 1$ *when* p *is even.*

```
Input : A positive integers $A_{p-1}, B_{p-1}, N$
Output : The values of $X_{2j+1}, Y_{2j+1}$
```

```
for $j \leftarrow 0$ To $N$ do
    $sum \leftarrow 0$
    $Y_{2j+1} \leftarrow 0$
    if $j \geq 1$ then
        for $h \leftarrow 0$ To $j-1$ do
            $sum \leftarrow (-1)^h \binom{2j-1-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2j-2h}\right) - 1\right] (2A_{p-1})^{2j-1-2h}$
            $X_{2j+1} \leftarrow sum + (-1)^j A_{p-1}$
            $X_{2j+1} \leftarrow X_{2j+1} + 0$
            $h \leftarrow h + 1$
        end for
        for $h \leftarrow 0$ To $j$ do
            $Y_{2j+1} \leftarrow (-1)^h B_{p-1} \binom{2j-h}{h} (2A_{p-1})^{2j-2h}$
            $Y_{2j+1} \leftarrow Y_{2j+1} + 0$
            $h \leftarrow h + 1$
        end for
    else
        $X_{2j+1} \leftarrow A_{p-1}$
        $Y_{2j+1} \leftarrow B_{p-1}$
    end if
end for
```

Example 2.3. Our purpose here is to find the 20^{th} until 29^{th} solutions of the Pell equation $x^2 - 15y^2 = 1$ via Corollary 2.2.
We have $\sqrt{15} = [3, \overline{1, 6}]$, $A_1 = 4$ and $B_1 = 1$. To facilitate the calculations we will use the Python code A constructed from Algorithm 1, and we will give some even solutions in the following table.

j	X_{2j}	Y_{2j}
10	418558976041008000	108071462907496880
11	25943903806170873856	6698687158460467200
12	1608103477006553055232	415210532361641459712
13	99676471670600118566912	25736354319263309234176
14	6178333140100200867561472	1595238757261963530600448

For $n = 2j + 1$, using the Python code B constructed from Algorithm 2, we get the odd solutions given in the following table.

j	X_{2j+1}	Y_{2j+1}
10	3295307847776485376	850844827670995456
11	25943903806170873856	52738652440012742656
12	12660571893450835755008	3268945606453119418368
13	784751201471350242803712	202621888947653389058048
14	48641913919330265287622656	12559288169148054989963264

Corollary 2.4. *Let $[b_0, \overline{b_1, ..., b_p}]$ be the simple periodic continued fraction of $\sqrt{d}$ and A_{p-1}/B_{p-1} its $(p-1)^{th}$ convergent. Suppose that p is odd, then for $n \geq 1$, the positive integer solutions (X_n, Y_n) of $x^2 - dy^2 = 1$ are given by*

$$\begin{cases} X_n &= \sum_{h=0}^{n-1} \binom{2n-2-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2n-1-2h}\right) + 1\right] (2A_{p-1})^{2n-2-2h} \\ Y_n &= B_{p-1} \sum_{h=0}^{n-1} \binom{2n-1-h}{h} (2A_{p-1})^{2n-1-2h} \end{cases}$$

In this case, we provide the following iterative Algorithm 3 to calculate the solutions X_n and Y_n for $n = 1, ..., N$.

Algorithm 3 : *Calculate solutions X_n and Y_n of Pell equation $x^2 - dy^2 = 1$ when p is odd.*

Input : *A positive integers A_{p-1}, B_{p-1}, N*
Output : *The values of X_n, Y_n*

```
for $n \leftarrow 0$ To $N$ do
    $X_n \leftarrow 0$
    $Y_n \leftarrow 0$
    if $n \geq 2$ then
        for $h \leftarrow 0$ To $n-1$ do
            $X_n \leftarrow \binom{2n-2-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2n-1-2h}\right) + 1\right] (2A_{p-1})^{2n-2-2h}$
            $Y_n \leftarrow B_{p-1}\binom{2n-1-h}{h}(2A_{p-1})^{2n-1-2h}$
            $X_n \leftarrow X_n + 0$
            $Y_n \leftarrow Y_n + 0$
            $h \leftarrow h + 1$
        end for
    else if n=1 then
        $X_n \leftarrow 2A_{p-1}^2 + 1$
        $Y_n \leftarrow 2A_{p-1}B_{p-1}$
    else
        $X_n \leftarrow 1$
        $Y_n \leftarrow 0$
    end if
end for
```

Example 2.5. The case when p is odd will be treated in this example.
Let us consider the Pell equation $x^2 - 29y^2 = 1$. Since $\sqrt{29} = [5, \overline{2, 1, 1, 2, 10}]$, a simple calculation shows that $A_4 = 70$ and $B_4 = 13$. Using the Python code C emanated from Algorithm 3, we classify the 5^{th} until 8^{th} solutions of $x^2 - 29y^2 = 1$ in the following table

n	X_n	Y_n
5	1447011301184484147200	268703252919468621824
6	2836431545199868550473184	526712115001947337418342 4
7	5559973100430669258653664542722	1032461085139784748176402022240
8	10898659243099883877320527020294144	2023830213823884979276957714743296

Pursuing the same method as described above, we establish the combinatorial solutions of the negative Pell equation $x^2 - dy^2 = -1$ as follows:

Corollary 2.6. *Let $[b_0, \overline{b_1, ..., b_p}]$ be the simple periodic continued fraction of $\sqrt{d}$ and A_{p-1}/B_{p-1} its $(p-1)^{th}$ convergent. Then, for all $n \geq 2$, the positive integer solutions (X_n, Y_n) of $x^2 - dy^2 = -1$ are given by*

$$\begin{cases} X_n = A_{p-1} + \displaystyle\sum_{h=0}^{n-2} \binom{2n-3-h}{h} \left[2A_{p-1}^2 \left(1 + \frac{h}{2n-2-2h}\right) + 1\right] (2A_{p-1})^{2n-3-2h} \\ Y_n = B_{p-1} \displaystyle\sum_{h=0}^{n-1} \binom{2n-2-h}{h} (2A_{p-1})^{2n-2-2h} \end{cases}$$

When p is odd and there is no solution when p is even.

The following iterative Algorithm 4 constructed from Corollary 2.6 gives us the solutions X_n and Y_n for $n = 1, ..., N$ of Pell equation $x^2 - dy^2 = -1$ when the period length of continued fraction expansion of $\sqrt{d}$ is odd since it has no solutions when p is even.

Algorithm 4 : *Calculate solutions X_n and Y_n of Pell equation $x^2 - dy^2 = -1$ when p is odd.*

Input : *A positive integers A_{p-1}, B_{p-1}, N*
Output : *The values of X_n, Y_n*

```
for $n \leftarrow 1$ To $N$ do
    $sum \leftarrow 0$
    $Y_n \leftarrow 0$
    if $n \geq 2$ then
        for $h \leftarrow 0$ To $n-2$ do
            $sum \leftarrow \binom{2n-3-h}{h}\left[2A_{p-1}^2\left(1+\frac{h}{2n-2-2h}\right)+1\right](2A_{p-1})^{2n-3-2h}$
            $sum \leftarrow sum + 0$
            $X_n \leftarrow sum + A_{p-1}$
            $h \leftarrow h+1$
        end for
        for $h \leftarrow 0$ To $n-1$ do
            $Y_n \leftarrow B_{p-1}\binom{2n-2-h}{h}(2A_{p-1})^{2n-2-2h}$
            $Y_n \leftarrow Y_n + 0$
            $h \leftarrow h+1$
        end for
    else
        $sum \leftarrow 0$
        $X_n \leftarrow sum + A_{p-1}$
        $Y_n \leftarrow B_{p-1}$
    end if
end for
```

Example 2.7. In this example, we aim to determine positive integer solutions of the Pell equation $x^2 - 41y^2 = -1$.
We have $\sqrt{41} = [6, \overline{2, 2, 12}]$, $A_2 = 32$ and $B_2 = 5$. Using the Python code D constructed from Algorithm 4, we calculate the 5^{th} until 9^{th} solutions of $x^2 - 41y^2 = -1$

n	X_n	Y_n
5	9027004963488032	1409781323735045
6	36992664137590767616	5777283520649277440
7	151595928608841991716864	23675306457839414804480
8	621240078446370327829151744	97021400086942409591619584
9	2545841689877297610151216283648	397593673880983491558577799168

2.2 Analytic expressions of solutions of Pell equation $x^2 - dy^2 = \pm 1$

In the aim to solve the Pell equation $x^2 - dy^2 = \pm 1$*, we provide here a different method by using the analytic formula (Binet formula) of the linearly recurring sequences* $\{A_{kp-1}\}_{k\geq 0}$ *and* $\{B_{kp-1}\}_{k\geq 0}$.
Let $P_B(x) = x^2 - c_0x - c_1$ *be the characteristic polynomial of the matrix* $B = C_pC_{p-1}...C_1$ *given by (2.1). We point out that* P_B *has always two distinct quadratic irrational roots* $\lambda_1 \neq \lambda_2$*. This reality comes from the fact that the discriminant* $\Delta = c_0^2 + 4(-1)^{p-1}$ *of* P_B *is a strictly positive integer, not a perfect square, namely, we distinguish two cases.*

- *(i) If p is even,* $\Delta = c_0^2 - 4$ *and since* $\mathrm{Tr}\,(C_2C_1) = b_1b_2 + 2$*, where* $b_1 \geq 1$ *and* $b_2 \geq 1$*, then* $\mathrm{Tr}\,(C_2C_1)^2 > 4$*. By induction, we get* $\mathrm{Tr}\,(C_k...C_2C_1)$ *increases when* k *increases.*
- *(ii) If p is odd,* $\Delta = \mathrm{Tr}\,(B)^2 + 4 > 0$.

Thence $P_B(x) = x^2 - c_0x - c_1$ *admits two distinct real roots* λ_1 *and* λ_2*. Furthermore, since* $c_0 = 2A_{p-1}$ *and* $c_1 = (-1)^{p-1}$*, we get*

$$\begin{aligned} \lambda_1 &= A_{p-1} + \sqrt{A_{p-1}^2 + (-1)^{p-1}} \\ \lambda_2 &= A_{p-1} - \sqrt{A_{p-1}^2 + (-1)^{p-1}} \end{aligned} \tag{2.4}$$

Consider the linearly recurring sequence $\{A_{kp-1}\}_{k\geq 0}$ *given by (2.3), namely*

$$A_{kp-1} = c_0A_{(k-1)p-1} + c_1A_{(k-2)p-1} \qquad (k \geq 2)$$

with initial conditions $A_{-1} = 1$ *and* A_{p-1}. *The associated characteristic polynomial is* $P_B(x) = x^2 - c_0 x - c_1 = (x - \lambda_1)(x - \lambda_2)$. *Then the analytic formula of* $\{A_{kp-1}\}_{k\geq 0}$ *is given by*

$$A_{kp-1} = \beta_1 \lambda_1^k + \beta_2 \lambda_2^k \qquad (k \geq 0)$$

where β_1 *and* β_2 *are determined from the initial conditions, by solving the system*

$$\begin{cases} \beta_1 + \beta_2 = 1 \\ \beta_1 \lambda_1 + \beta_2 \lambda_2 = A_{p-1} \end{cases}$$

Therefore we get $\beta_1 = \beta_2 = \frac{1}{2}$.
Similarly, we consider $\{B_{kp-1}\}_{k\geq 0}$ *as a linearly recurring sequence of order 2 with initial conditions* $B_{-1} = 0$ *and* B_{p-1}. *Furthermore, from [9] we have*

$$A_{p-1}^2 + (-1)^{p-1} = dB_{p-1}^2.$$

Summarizing, we obtain the following results.

Proposition 2.8. *Let* $[b_0, \overline{b_1, ..., b_p}]$ *be the simple periodic continued fraction of* $\sqrt{d}$ *and* A_{p-1}/B_{p-1} *its* $(p-1)^{th}$ *convergent. Then, for all* $k \geq 1$, *the analytic expressions of solutions* (X_k, Y_k) *of Pell equation* $x^2 - dy^2 = 1$ *are given as follows:*
- *When p is even :*

$$\begin{cases} X_k = \frac{1}{2}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^k + \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^k\right] \\ Y_k = \frac{1}{2\sqrt{d}}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^k - \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^k\right] \end{cases}$$

- *When p is odd :*

$$\begin{cases} X_k = \frac{1}{2}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^{2k} + \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^{2k}\right] \\ Y_k = \frac{1}{2\sqrt{d}}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^{2k} - \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^{2k}\right] \end{cases}$$

For the negative Pell equation $x^2 - dy^2 = -1$, *the analytic expressions of solutions* (X_k, Y_k) *are given by*

$$\begin{cases} X_k = \frac{1}{2}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^{2k-1} + \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^{2k-1}\right] \\ Y_k = \frac{1}{2\sqrt{d}}\left[\left(A_{p-1} + B_{p-1}\sqrt{d}\right)^{2k-1} - \left(A_{p-1} - B_{p-1}\sqrt{d}\right)^{2k-1}\right] \end{cases}$$

when p is odd, and there is no solution when p is even.

Example 2.9. Consider the Pell equation $x^2 - 13y^2 = \pm 1$, we have $\sqrt{13} = [3, \overline{1, 1, 1, 1, 6}]$ $(p = 5)$ and $A_4 = 18$, $B_4 = 5$. By Proposition 2.8, the analytic expressions of solutions of $x^2 - 13y^2 = 1$ are given as follows:

$$\begin{cases} X_k = \frac{1}{2}\left[(18 - 5\sqrt{13})^{2k} + (18 + 5\sqrt{13})^{2k}\right] \\ Y_k = \frac{1}{2\sqrt{13}}\left[(18 + 5\sqrt{13})^{2k} - (18 - 5\sqrt{13})^{2k}\right] \end{cases} \qquad (k \geq 1)$$

For the negative Pell equation $x^2 - 13y^2 = -1$, we get

$$\begin{cases} X_k = \frac{1}{2}\left[(18 - 5\sqrt{13})^{2k-1} + (18 + 5\sqrt{13})^{2k-1}\right] \\ Y_k = \frac{1}{2\sqrt{13}}\left[(18 + 5\sqrt{13})^{2k-1} - (18 - 5\sqrt{13})^{2k-1}\right] \end{cases} \qquad (k \geq 1)$$

3 Solving the Pell equation $x^2 - (k^2 \pm l)y^2 = 1$

Continued fractions are essential tools for many authors who have dealt with positive solutions of the Pell equation $x^2 - dy^2 = \pm 1$ for some specific values of d. For example, in [11], the author considers the continued fraction expansion of $\sqrt{d}$ for $d = k^2 \pm 1,\ k^2 \pm 2$ and $k^2 \pm k$, where k is a positive integer. In this section, as an application of Section 2, we give the continued fraction expansion of $\sqrt{k^2 \pm l}$ where l divides k, then we consider the combinatorial expressions of solutions of the Pell equation $x^2 - (k^2 \pm l)y^2 = 1$.

Theorem 3.1. *Let $k > 1$ and $l \geq 1$ be integers such that l divides k. Then*

(i) The continued fraction expansion of $\sqrt{k^2 + l}$ is

$$\sqrt{k^2 + l} = \left[k, \overline{\frac{2k}{l}, 2k}\right]$$

(ii) The fundamental solution of $x^2 - (k^2 + l)y^2 = 1$ is $(X_1, Y_1) = \left(\frac{2k^2}{l} + 1, \frac{2k}{l}\right)$. Moreover, for $n \geq 2$, the combinatorial expressions of the solutions (X_n, Y_n) of $x^2 - (k^2 + l)y^2 = 1$ are given by

$\star$ case 1: $n = 2j$

$$\begin{cases} X_{2j} = \sum_{h=0}^{j-1} (-1)^h \binom{2j-2-h}{h} \left[\frac{4j-2-2h}{2j-1-2h}\left(\frac{2k^2}{l}+1\right)^2 - 1\right]\left(\frac{4k^2}{l}+2\right)^{2j-2-2h} \\ Y_{2j} = \frac{2k}{l}\sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h}\left(\frac{4k^2}{l}+2\right)^{2j-1-2h} \end{cases}$$

$\star$ case 2: $n = 2j + 1$

$$\begin{cases} X_{2j+1} = (-1)^j\left(\frac{2k^2}{l}+1\right) + \sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h}\left[\frac{4j-2h}{2j-2h}\left(\frac{2k^2}{l}+1\right)^2 - 1\right]\left(\frac{4k^2}{l}+2\right)^{2j-1-2h} \\ Y_{2j+1} = \frac{2k}{l}\sum_{h=0}^{j} (-1)^h \binom{2j-h}{h}\left(\frac{4k^2}{l}+2\right)^{2j-2h} \end{cases}$$

Proof.

(i) Let $k > 1$ and $l \geq 1$ be integers such that l divides k. Then, after a simple calculation, we get

$$\begin{aligned} \sqrt{k^2+l} &= k + (\sqrt{k^2+l} - k) = k + \cfrac{1}{\cfrac{\sqrt{k^2+l}+k}{l}} \\ &= k + \cfrac{1}{\cfrac{2k}{l} + \cfrac{1}{2k + (\sqrt{k^2+l} - k)}} \end{aligned}$$

Then $\sqrt{k^2+l} = \left[k, \overline{\frac{2k}{l}, 2k}\right]$.

(ii) Since the period $p = 2$, it is given by Theorem 1.1 that the fundamental solution of equation $x^2 - (k^2 + l)y^2 = 1$ is $(X_1, Y_1) = (A_1, B_1)$. Therefore, using (1.3) we get $(X_1, Y_1) = \left(\frac{2k^2}{l} + 1, \frac{2k}{l}\right)$. For $n \geq 2$, the combinatorial expressions of the solutions are derived directly from Corollary 2.2.

□

Example 3.2. In this example, we aim to find the consecutive combinatorial expressions of solutions (X_5, Y_5) and (X_6, Y_6) of $x^2 - 39y^2 = 1$.

Since $39 = 6^2 + 3$ and 3 divides 6 then $\sqrt{39} = \left[6, \overline{4, 12}\right]$. It follows that $A_1 = \dfrac{2k^2}{l} + 1 = 25$ and $B_1 = \dfrac{2k}{l} = 4$. Thus

$$\begin{cases} X_5 = 25 + \displaystyle\sum_{h=0}^{1} \binom{3-h}{h} \left[\frac{8-2h}{4-2h} \times 25^2 - 1\right] (-1)^h \, 50^{3-2h} \\ Y_5 = 4 \times \displaystyle\sum_{h=0}^{2} \binom{4-h}{h} (-1)^h \, 50^{4-2h} \end{cases}$$

and

$$\begin{cases} X_6 = \displaystyle\sum_{h=0}^{2} \binom{4-h}{h} \left[\frac{10-2h}{5-2h} \times 25^2 - 1\right] (-1)^h \, 50^{4-2h} \\ Y_6 = 4 \times \displaystyle\sum_{h=0}^{2} \binom{5-h}{h} (-1)^h \, 50^{5-2h} \end{cases}$$

Hence $(X_5, Y_5) = (155937625, 24970004)$ and $(X_6, Y_6) = (7793761249, 1248000600)$

Now, we give the consecutive combinatorial expressions of solutions of $x^2 - (k^2 - l)y^2 = 1$ *in the following theorem;*

Theorem 3.3. *Let* $k > 1$ *and* $l \geq 1$ *be integers such that* l *divides* k *and* $l \neq k$. *Then*

(i) The continued fraction expansion of $\sqrt{k^2 - l}$ *is*

$$\sqrt{k^2 - l} = \left[k - 1, \overline{1, \frac{2(k-l)}{l}, 1, 2(k-1)}\right]$$

(ii) The fundamental solution of $x^2 - (k^2 - l)y^2 = 1$ *is* $(X_1, Y_1) = \left(\dfrac{2k^2}{l} - 1, \dfrac{2k}{l}\right)$. *Moreover, for* $n \geq 2$, *the combinatorial expressions of the solutions* (X_n, Y_n) *of* $x^2 - (k^2 - l)y^2 = 1$ *are given by*

$\star$ *case 1:* $n = 2j$

$$\begin{cases} X_{2j} = \displaystyle\sum_{h=0}^{j-1} (-1)^h \binom{2j-2-h}{h} \left[\frac{4j-2-2h}{2j-1-2h} \left(\frac{2k^2}{l} - 1\right)^2 - 1\right] \left(\frac{4k^2}{l} - 2\right)^{2j-2-2h} \\ Y_{2j} = \dfrac{2k}{l} \displaystyle\sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h} \left(\frac{4k^2}{l} - 2\right)^{2j-1-2h} \end{cases}$$

$\star$ *case 2:* $n = 2j + 1$

$$\begin{cases} X_{2j+1} = (-1)^j \left(\frac{2k^2}{l} - 1\right) + \sum_{h=0}^{j-1} (-1)^h \binom{2j-1-h}{h} \left[\frac{4j-2h}{2j-2h} \left(\frac{2k^2}{l} - 1\right)^2 - 1\right] \left(\frac{4k^2}{l} - 2\right)^{2j-1-2h} \\ Y_{2j+1} = \dfrac{2k}{l} \displaystyle\sum_{h=0}^{j} (-1)^h \binom{2j-h}{h} \left(\frac{4k^2}{l} - 2\right)^{2j-2h} \end{cases}$$

Proof.

(i) Let $k > 1$ and $l \geq 1$ be integers such that l divides k and $l \neq k$. Then, a straightforward computation gives

$$\sqrt{k^2-l} = k-1+(\sqrt{k^2-l}-(k-1)) = k-1+\cfrac{1}{\cfrac{\sqrt{k^2-l}+(k-1)}{2k-l-1}}$$

$$= k-1+\cfrac{1}{1+\cfrac{1}{\cfrac{2(k-l)}{l}+\cfrac{1}{1+\cfrac{1}{2(k-1)+\sqrt{k^2-l}-(k-1)}}}}$$

Thus $\sqrt{k^2-l} = \left[k-1, \overline{1, \frac{2(k-l)}{l}, 1, 2(k-1)}\right]$

(ii) Since the period $p=4$, then it is given by Theorem 1.1 that the fundamental solution of equation $x^2-(k^2-l)y^2=1$ is $(X_1,Y_1)=(A_3,B_3)$. Therefore, using (1.3) we get $(X_1,Y_1)=\left(\frac{2k^2}{l}-1, \frac{2k}{l}\right)$. For $n \geq 2$, the combinatorial expressions of the solutions are derived directly from Corollary 2.2.

□

Example 3.4. Using Theorem 3.3, we give the consecutive combinatorial expressions of solutions (X_3,Y_3) and (X_4,Y_4) of Pell equation $x^2-14y^2=1$. Since $14=4^2-2$ and 2 divides 4, we get $\sqrt{14}=[3,\overline{1,2,1,6}]$.
From (1.3) we can see that $A_3=15$ and $B_3=4$. Thence,

$$\begin{cases} X_3 = -15+\left[2\times 15^2-1\right]\times 30 \\ Y_3 = 4\times \sum_{h=0}^{1}\binom{2-h}{h}(-1)^h \times 30^{2-2h} \end{cases}$$

and

$$\begin{cases} X_4 = \sum_{h=0}^{1}\binom{2-h}{h}\left[\frac{6-2h}{3-2h}\times 15^2-1\right](-1)^h\ 30^{2-2h} \\ Y_4 = 4\times \sum_{h=0}^{1}\binom{3-h}{h}(-1)^h\ 30^{3-2h} \end{cases}$$

Therefore, $(X_3,Y_3)=(13455,3596)$ and $(X_4,Y_4)=(403201,107760)$.

A Python code of solutions to Pell equation $x^2-dy^2=1$ when $n=2j$

```
import math as m
numbers =[10,11,12,13,14]
A_p1 = int(input("Enter␣A_p-1␣:"))
B_p1 = int(input("Enter␣B_p-1␣:"))
for j in numbers :
    X2j = 0
    Y2j = 0
    if j >= 1 :
        for h in range(0, j) :
            X2j+= m.comb(2*j-2-h, h) * m.pow(2*A_p1,2*j-2-2*h)
                * m.pow(-1,h) * (A_p1 * 2*A_p1 * ((2*j-1-h)
              /(2*j-1-2*h)) -1)
            Y2j+= B_p1 * m.comb(2*j-1-h, h) * m.pow(2*A_p1,2*
              j-1-2*h) * m.pow(-1,h)
```

```
        else  :
            X2j = 1
            Y2j = 0

        print("the␣value␣of␣X",2*j ,"is␣:", int(X2j))
        print("the␣value␣of␣Y",2*j ,"is␣:", int(Y2j))
```

B Python code of solutions to Pell equation $x^2 - dy^2 = 1$ when $n = 2j + 1$

```
import math as m
numbers =[10,11,12,13,14]
A_p1 = int(input("Enter␣A_p-1␣:"))
B_p1 = int(input("Enter␣B_p-1␣:"))
for j in numbers :
        sum = 0
        Y2j1 = 0
        if j >= 1 :
           for h in range(0, j) :
                sum += m.comb(2*j-1-h, h) * m.pow(2*A_p1,2*j-1-2*h)
                     * m.pow(-1,h) * (2*m.pow(A_p1,2)*((2*j-h)/(2*j-2*
                     h)) -1)
                X2j1 = sum + A_p1 * m.pow(-1,j)
        else :
            X2j1 = 0
        if j >= 1 :
            for h in range(0, j+1) :
                Y2j1 += B_p1 *  m.comb(2*j-h, h) * m.pow(2*A_p1,2*j
                     -2*h) * m.pow(-1,h)
        elif j == 0 :
             Y2j1 = B_p1
        else :
             Y2j1 = 0
        print("the␣value␣of␣X",2*j+1 ,"is␣:", int(X2j1))
        print("the␣value␣of␣Y",2*j+1 ,"is␣:", int(Y2j1))
```

C Python code of solutions to Pell equation $x^2 - dy^2 = 1$ when p is odd

```
import math as m
numbers =[5,6,7,8]
A_p1 = int(input("Enter␣A_p-1␣:"))
B_p1 = int(input("Enter␣B_p-1␣:"))
for n in numbers :
        Xn =0
        Yn = 0
        if n >= 2 :
            for h in range(0, n) :
                  Xn+= m.comb(2*n-2-h, h) * m.pow(2*A_p1,2*n-2-2*h)
                        * (2*m.pow(A_p1,2) * ((2*n-1-h)/(2*n-1-2*h))
                       +1)
                  Yn+=  B_p1 * m.comb(2*n-1-h, h) * m.pow(2*A_p1,2*n
                     -1-2*h)
        elif n == 1:
             Xn = 2*m.pow(A_p1,2) + 1
             Yn = 2*A_p1 * B_p1
```

```
        else :
            Xn = 1
            Yn = 0
        print("the␣value␣of␣X",n ,"is␣:", int(Xn))
        print("the␣value␣of␣Y",n ,"is␣:", int(Yn))
```

D Python code of solutions to Pell equation $x^2 - dy^2 = -1$ when p is odd

```
 import math as m
numbers =[5,6,7,8,9]
A_p1 = int(input("Enter␣A_p-1␣:"))
B_p1 = int(input("Enter␣B_p-1␣:"))
for n in numbers :
        sum =0
        Yn = 0
        if n >= 2 :
                for h in range(0, n-1) :
                     sum += m.comb(2*n-3-h,h) * m.pow(2*A_p1,2*n-3-2*h)
                            * (2* m.pow(A_p1,2)* ((2*n-2-h)/(2*n-2-2*h))
                         +1)
        elif n == 1:
              sum = 0
        else :
           print("There␣is␣no␣solution")
        if n >= 2 :
              for h in range(0, n) :
                   Yn += B_p1 * m.comb(2*n-2-h, h) * m.pow(2*A_p1,2*n
                        -2-2*h)
        elif n == 1:
             Yn = B_p1
        else :
             print("There␣is␣no␣solution")
        Xn = sum + A_p1
        print("the␣value␣of␣X",n ,"is␣:", int(Xn))
        print("the␣value␣of␣Y",n ,"is␣:", int(Yn))
```

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