Palestine Journal of Mathematics

Vol 13(3)(2024) , 690–700

SUBCLASS OF MULTIVALENT FUNCTIONS INVOLVING JACKSON'S $(r, q)-$DERIVATIVE OPERATOR

Ma'moun I. Y. Alharayzeh

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 30C45; Secondary 30C50.

Keywords and phrases: New subclass, $p-$valent function, Quantum or (r, q)-Calculus, (r, q)-Derivative operator.

The author would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract. This paper investigates a new subclass of multivalent analytic functions in the open unit disk, characterized using Jackson's derivative operator. Before obtaining coefficient characterization, we examine certain sufficient requirements for the functions belonging to this class. Several fascinating features, including coefficient estimates, the growth and distortion theorem, extreme points, and the radius of starlikeness and convexity of functions belonging to the subclass are shown using this technique.

1 Introduction and Definition

Let $\hat{\mathbb{A}}$ represent the set of all analytic functions defined on the open unit disk $\underline{\mathcal{U}} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$, with the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, where a_n is a complex number. Now, lets $\hat{\mathbb{A}}(p)$ denote the class consisting of functions f that have a Taylor series expansion of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, where a_n is a complex number and $p \in \mathbb{N} = \{1, 2, 3, ...\}$. These functions are analytic and p-valent within the open unit disk $\underline{\mathcal{U}}$. We observe that $\hat{\mathbb{A}}(1) = \hat{\mathbb{A}}$. Additionally, lets $S(p)$ represent the subclass of $\hat{\mathbb{A}}(p)$ consisting of p-valent functions that are univalent within $\underline{\mathcal{U}}$. Furthermore, we define $S_p^*(\alpha)$ and $C_p(\alpha)$ as the classes of p-valent functions that are respectively starlike of order α and convex of order α, where $0 \leq \alpha < p$. In particular, the classes $S_p^*(0) = S_p^*$ and $C_p(0) = C_p$ correspond respectively to the familiar classes of starlike and convex p-valent functions in $\underline{\mathcal{U}}$.

Let $\mathcal{T}(p)$, where $p \in \mathbb{N} = \{1, 2, 3, ...\}$, be a subclass of $S(p)$. It consists of functions with the following form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad a_n > 0, \tag{1.1}$$

defined on the open unit disk $\underline{\mathcal{U}} = \{z \in \mathbb{C} : |z| < 1\}$ A function f belonging to $\mathcal{T}(p)$ is referred to as a p-valent function with negative coefficients. We can define subclasses of $\mathcal{T}(p)$ denoted by $S^*_{\mathcal{T},p}(\alpha)$ and $C_{\mathcal{T},p}(\alpha)$ for $0 \leq \alpha < p$. These subclasses consist of p-valent functions that are respectively starlike of order α and convex of order α. The class $\mathcal{T}(1)$, denoted as $\mathcal{T}$, was initially introduced and studied by Silverman [23]. In his work, Silverman investigated the subclasses of $\mathcal{T}(1)$ denoted by $S^*_{\mathcal{T},1}(\alpha) = S^*_{\mathcal{T}}(\alpha)$, and $C_{\mathcal{T},1}(\alpha) = C_{\mathcal{T}}(\alpha)$, where $0 \leq \alpha < 1$. These subclasses represent p-valent functions that are respectively starlike of order α and convex of order α. Let

$\underline{\mathcal{M}}(a,b,c)$ represent the subset of $\hat{\mathbb{A}}(1)$ comprising functions $q \in \hat{\mathbb{A}}(1)$ that satisfy the inequality:

$$\left|\frac{zq'(z)-q(z)}{azq'(z)+(1-b)q(z)}\right| < c, \text{ where } 0 \le \mathrm{a} \le 1,\ 0 \le \mathrm{b} < 1 \text{ and } 0 < \mathrm{c} \le 1,$$

for all $z \in \underline{\mathcal{U}}$. This particular of functions has been examined by Darus [9].

In this section, we review established concepts and fundamental findings of (r,q)-calculus. Throughout this paper, we assume that r and q are constantly satisfying $0 < q < r \le 1$. We provide definitions and theorems related to (r,q)-calculus, which will be referenced in the following papers [18, 20, 21, 22, 24, 25, 26, 27] and [15].

For $0 < q < r \le 1$ jackson‘s (r,q)-derivative of a function $f \in \hat{\mathbb{A}}(p)$ is, by definition, given as follows

$$\mathcal{D}_{r,q}f(z) := \begin{cases} \dfrac{f(rz)-f(qz)}{(r-q)\,z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \tag{1.2}$$

From (1.2), we have

$$\mathcal{D}_{r,q}f(z) = [p]_{r,q}\, z^{p-1} + \sum_{n=p+1}^{\infty} [n]_{r,q}\, a_n\, z^{n-1} \qquad (0 < q < r \le 1),$$

where $[p]_{r,q} = \frac{r^p-q^p}{r-q}$ and $[n]_{r,q} = \frac{r^n-q^n}{r-q}$.

Note that, for $r = 1$ the jackson (r,q)-derivative reduces to the jackson q-derivative operator of the function f, $\mathcal{D}_q f(z)$ (refer to [1], [12], [13] and [14]). Note also that $\mathcal{D}_{1,q}f(z) \to f'(z)$ when $q \to 1_-$, where f' is the classical derivative of the function f.

Obviously, for a function $g(z) = z^n$, we obtain

$$\mathcal{D}_{r,q}g(z) = \mathcal{D}_{r,q}z^n = \frac{r^n-q^n}{r-q} z^{n-1} = [n]_{r,q}\, z^{n-1}.$$

And

$$\lim_{q\to 1-} \mathcal{D}_{1,q}g(z) = \lim_{q\to 1-} \frac{1-q^n}{1-q} z^{n-1} = nz^{n-1} = g'(z),$$

where g' is the ordinary derivative.

The application of q-calculus extends to a wide range of fields within applied sciences, encompassing ordinary fractional calculus, quantum physics, optimal control, hypergeometric series, operator theory, q-difference and q-integral equations, as well as geometric function theory in complex analysis. The pioneering use of q-calculus was introduced by Jackson [13]. Additionally, Kanas and Raducanu [16] employed fractional q-calculus operators to explore distinct categories of functions that exhibit analytic properties within the domain $\underline{\mathcal{U}}$. For comprehensive information on q-calculus, references such as [6, 7, 10, 13, 16, 17, 19, 27], and the relevant citations therein can be consulted.

In addition to the advancement of q-calculus theory and its applications, the theory of (r,q)-calculus, based on two parameters, has also gained significant attention in recent decades. In 1991, R. Chakrabarti and R. Jagannathan [8] introduced (r,q)-calculus, which was further studied by P. N. Sadjang [21], who investigated the fundamental theorem of (r,q)-calculus and derived some (r,q)-Taylor formulas. More recently, M. Tunc¸ and E. G"ov [27] defined the (r,q)-derivative and (r,q)-integral on finite intervals, along with studying various properties of (r,q)-calculus and the (r,q)-analog of important integral inequalities. The (r,q)-derivative has garnered significant attention and undergone rapid development during this period, with contributions from multiple authors.

The field of geometric function theory has explored various subclasses within the class $\hat{\mathbb{A}}(p)$ using the (r,q)-calculus mentioned earlier. Ismail et al. [11] were pioneers in utilizing the q-derivative operator $\mathcal{D}_q$ to investigate the q-calculus counterpart of the class S^* of starlike functions in $\underline{\mathcal{U}}$. This application of (r,q)-calculus has opened up new avenues for studying and understanding geometric function theory, providing valuable insights into specific subclasses and their properties.

From now on we introduce some general subclass of analytic and multivalent functions associated with (r,q)-derivative operator as follows.

Definition 1.1. For $0 \le h \le 1$, $0 \le m < 1$, $0 \le s < 1$, $k \ge 0$, $0 < q < r \le 1$ and $p \in \mathbb{N} = \{1,2,3,...\}$, we let $\Upsilon(h,m,s,k,r,q,p)$ consist of functions $f \in \mathcal{T}(P)$ satisfying the condition

$$\operatorname{Re}\left(\frac{z\left(\mathcal{D}_{r,q}f(z)\right)' - \mathcal{D}_{r,q}f(z)}{hz\left(\mathcal{D}_{r,q}f(z)\right)' + (1-s)\mathcal{D}_{r,q}f(z)}\right)$$
$$> \quad k\left|\frac{z\left(\mathcal{D}_{r,q}f(z)\right)' - \mathcal{D}_{r,q}f(z)}{hz\left(\mathcal{D}_{r,q}f(z)\right)' + (1-s)\mathcal{D}_{r,q}f(z)} - 1\right| + m. \tag{1.3}$$

Our initial finding consists of coefficient inequalities for functions $f \in \Upsilon(h,m,s,k,r,q,p)$. Also, our results encompass the growth and distortion theorem, as well as the determination of extreme points. Lastly, we establish the radius of starlikeness and convexity for functions belonging to the class $\Upsilon(h,m,s,k,r,q,p)$. The technique which was studied by Aqlan et al. [5] and also in [2][3][4].

Now, let's delve into the coefficient inequalities.

2 Coefficient Inequalities

In this section, we establish a necessary and sufficient condition for a function f belonging to the class $\Upsilon(h,m,s,k,r,q,p)$. Our first result is presented below:

Theorem 2.1. *Let* $0 \le h \le 1$, $0 \le m < 1$, $0 \le s < 1$, $k \ge 0$, $0 < q < r \le 1$, *and* $p \in \mathbb{N} = \{1,2,3,...\}$. *A function f defined by (1.1) is in the class* $\Upsilon(h,m,s,k,r,q,p)$ *if and only if*

$$\sum_{n=p+1}^{\infty} \mu_n \, a_n \le \mu_p, \tag{2.1}$$

where

$$\mu_n = \left|\left(h\,(n-1) + 1 - s\right)\left(2 + k - m\right) + (k+1)\,(2-n)\right|\left(r^n - q^n\right). \tag{2.2}$$

Proof. We have $f \in \Upsilon(h,m,s,k,r,q,p)$ if and only if the condition (1.3) is obtained. Let

$$\varpi = \frac{z\left(\mathcal{D}_{r,q}f(z)\right)' - \left(\mathcal{D}_{r,q}f(z)\right)}{hz\left(\mathcal{D}_{r,q}f(z)\right)' + (1-s)\left(\mathcal{D}_{r,q}f(z)\right)},$$

upon the fact that,

$$Re\,(\,\varpi\,) \;\ge\; k\,|\,\varpi - 1\,| + m \quad \text{if and only if} \quad (\,k+1\,)\;|\,\varpi - 1\,| \;\le\; 1 - m.$$

Now

$$\begin{aligned}
&(k+1)\left|\varpi-1\right| \\
=\ &(k+1)\left|\frac{(p-2)[p]_{r,q}z^{p-1}+\sum\limits_{n=p+1}^{\infty}(2-n)\,[n]_{r,q}a_n z^{n-1}}{(h(p-1)+1-s)\,[p]_{r,q}z^{p-1}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}a_n z^{n-1}}-1\right| \\
\leq\ &1-m,
\end{aligned}$$

is equivalent to

$$(k+1)\left|\frac{(p-2)\,[p]_{r,q}+\sum\limits_{n=p+1}^{\infty}(2-n)\,[n]_{r,q}\,a_n z^{n-p}}{(h(p-1)+1-s)\,[p]_{r,q}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n z^{n-p}}-1\right|\leqslant 1-m.$$

So

$$\begin{aligned}
&(k+1)\left|\frac{\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s+2-n)\,[n]_{r,q}\,a_n z^{n-p}-(h(p-1)+1-s+2-p)\,[p]_{r,q}}{(h(p-1)+1-s)\,[p]_{r,q}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n z^{n-p}}\right| \\
\leq\ &1-m. \qquad (2.3)
\end{aligned}$$

The inequality can be simplified to,

$$\begin{aligned}
&(k+1)\,\frac{\sum\limits_{n=p+1}^{\infty}\left|h\,(n-1)+1-s+2-n\right|[n]_{r,q}\,a_n-\left|h(p-1)+1-s+2-p\right|[p]_{r,q}}{(h(p-1)+1-s)\,[p]_{r,q}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n} \\
\leq\ &1-m. \qquad (2.4)
\end{aligned}$$

Then

$$\begin{aligned}
&(k+1)\left[\sum_{n=p+1}^{\infty}\left|h\,(n-1)+1-s+2-n\right|[n]_{r,q}a_n-\left|h(p-1)+1-s+2-p\right|[p]_{r,q}\right] \\
\leq\ &(h\,(p-1)+1-s)\,(1-m)\,[p]_{r,q}-(1-m)\sum_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n. \qquad (2.5)
\end{aligned}$$

Thus

$$\begin{aligned}
&(k+1)\sum_{n=p+1}^{\infty}\left|h\,(n-1)+1-s+2-n\right|[n]_{r,q}a_n+(1-m)\sum_{n=p+1}^{\infty}(h\,(n-1)+1-s)[n]_{r,q}a_n \\
\leq\ &(h\,(p-1)+1-s)(1-m)\,[p]_{r,q}+\left|h\,(p-1)+1-s+2-p\right|(k+1)\,[p]_{r,q}\,, \qquad (2.6)
\end{aligned}$$

then, we get

$$\begin{aligned}
&\sum_{n=p+1}^{\infty}\left|(2+k-m)\,(h\,(n-1)+1-s)+(k+1)\,(2-n)\right|[n]_{r,q}\,a_n \\
\leq\ &\left|(2+k-m)\,(h\,(p-1)+1-s)+(k+1)\,(2-p)\right|[p]_{r,q}\,.
\end{aligned}$$

Which (2.1).

Given that inequality (2.1) is satisfied, our objective is to prove the validity of (1.3). Importantly, it should be noted that the inequality described in (1.3) is equivalent to inequality (2.3). From condition (2.1) we have (2.6) and then (2.5), after that (2.5) is equivalent to (2.4).

Now it is suffices to show that,

$$\left|\frac{\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s+2-n)\,[n]_{r,q}\,a_n z^{n-p}-(h(p-1)+1-s+2-p)\,[p]_{r,q}}{(h(p-1)+1-s)\,[p]_{r,q}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n z^{n-p}}\right|$$
$$\leq \frac{\sum\limits_{n=p+1}^{\infty}|h(n-1)+1-s+2-n|\,[n]_{r,q}\,a_n-|h(p-1)+1-s+2-p|\,[p]_{r,q}}{(h(p-1)+1-s)\,[p]_{r,q}-\sum\limits_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n}. \tag{2.7}$$

Since,

$$\left|(h(p-1)+1-s)\,[p]_{r,q}-\sum_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n z^{n-p}\right|$$
$$\geq |h(p-1)+1-s|\,[p]_{r,q}-\left|\sum_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n z^{n-p}\right|,$$
$$=(h(p-1)+1-s)\,[p]_{r,q}-\sum_{n=p+1}^{\infty}(h(n-1)+1-s)\,[n]_{r,q}\,a_n,\qquad \text{where } |z|<1,$$

and hence, we obtain (2.7). □

Theorem 2.2. *Let* $0 \leq h \leq 1$, $0 \leq m < 1$, $0 \leq s < 1$, $k \geq 0$, $0 < q < r \leq 1$, *and* $p \in \mathbb{N} = \{1, 2, 3, ...\}$. *If the function* f *defined by (1.1) be in the class* $\Upsilon(h, m, s, k, r, q, p)$ *then*

$$a_n \leq \frac{\mu_p}{\mu_n}, \qquad n = p+1, p+2, p+3, ..., \tag{2.8}$$

where μ_n *is given by (2.2).*

Equality achieved for the following functions,

$$f(z) = z^P - \frac{\mu_p\, z^n}{\mu_n}. \tag{2.9}$$

Proof. Since $f \in \Upsilon(h, m, s, k, r, q, p)$ Theorem 2.1 holds.

Now

$$\sum_{n=p+1}^{\infty} \mu_n a_n \leq \mu_p,$$

we have,

$$a_n \leq \frac{\mu_p}{\mu_n}.$$

□

It is evident that the function defined in equation (2.9) fulfills the condition stated in equation (2.8). Consequently, the function f defined in (2.9) belongs to the class $\Upsilon(h, m, s, k, r, q, p)$. For this particular function, it is evident that the obtained result is sharp.

3 Growth and Distortion Theorems for the Subclass $\Upsilon(h, m, s, k, r, q, p)$

In the following section, we will examine the growth and distortion theorem along with the covering property of functions belonging to the class $\Upsilon(h, m, s, k, r, q, p)$.

Theorem 3.1. *Let* $0 \leq h \leq 1$, $0 \leq m < 1$, $0 \leq s < 1$, $k \geq 0$, $0 < q < r \leq 1$, *and* $p \in \mathbb{N} = \{1, 2, 3, ...\}$. *If the function* f *defined by (1.1) be in the class* $\Upsilon(h, m, s, k, r, q, p)$ *then for* $0 < |z| = l < 1$, *we have*

$$l^p - \frac{\mu_p}{\mu_{p+1}} l^{p+1} \leqslant |f(z)| \leqslant l^p + \frac{\mu_p}{\mu_{p+1}} l^{p+1}. \tag{3.1}$$

Equality achieved for the function,

$$f(z) = z^p - \frac{\mu_p}{\mu_{p+1}} z^{p+1}, \quad (z = \pm l, \ \pm il),$$

where μ_p *and* μ_{p+1} *are found by (2.2).*

Proof. We will focus our proof on establishing the inequality on the right-hand side of inequality (3.1), and Using similar arguments, we can derive the other inequality.

Since $f \in \Upsilon(h, m, s, k, r, q, p)$ by Theorem 2.1, we have,

$$\sum_{n=p+1}^{\infty} \mu_n a_n \leq \mu_p.$$

Now

$$\mu_{p+1} \sum_{n=p+1}^{\infty} a_n = \sum_{n=p+1}^{\infty} \mu_{p+1} a_n \leq \sum_{n=p+1}^{\infty} \mu_n a_n \leq \mu_p.$$

And therefore

$$\sum_{n=p+1}^{\infty} a_n \leqslant \frac{\mu_p}{\mu_{p+1}}, \tag{3.2}$$

since

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n,$$

we have,

$$|f(z)| = \left| z^p - \sum_{n=p+1}^{\infty} a_n z^n \right| \leq |z|^p + |z|^{p+1} \sum_{n=p+1}^{\infty} a_n |z|^{n-(p+1)} \leq l^p + l^{p+1} \sum_{n=p+1}^{\infty} a_n.$$

By utilizing inequality (3.2), we obtain the right-hand side inequality in (3.1). □

Theorem 3.2. *If the function* f *defined by (1.1) belongs to the class* $\Upsilon(h, m, s, k, r, q, p)$ *for* $0 < |z| = l < 1$, *then the following holds:*

$$p\, l^{p-1} - \frac{(p+1)\mu_p}{\mu_{p+1}} l^p \leqslant |f'(z)| \leqslant p\, l^{p-1} + \frac{(p+1)\mu_p}{\mu_{p+1}} l^p. \tag{3.3}$$

Equality satisfied for the function f *given by*

$$f(z) = z^p - \frac{\mu_p}{\mu_{p+1}} z^{p+1}, \qquad (z = \pm l, \ \pm il),$$

where μ_p *and* μ_{p+1} *are given by (2.2).*

Proof. Since $f \in \Upsilon(h, m, s, k, r, q, p)$ by Theorem 2.1 we have

$$\sum_{n=p+1}^{\infty} \mu_n \, a_n \leq \mu_p.$$

Now,

$$\mu_{p+1} \sum_{n=p+1}^{\infty} n \, a_n \leq (p+1) \sum_{n=p+1}^{\infty} \mu_n a_n \leq (p+1)\mu_p.$$

Hence

$$\sum_{n=p+1}^{\infty} n a_n \leqslant \frac{(p+1)\mu_p}{\mu_{p+1}}, \tag{3.4}$$

since

$$f'(z) = p z^{p-1} - \sum_{n=p+1}^{\infty} n \, a_n \, z^{n-1}.$$

Then, we have

$$p \, |z|^{p-1} - |z|^p \sum_{n=p+1}^{\infty} n a_n \, |z|^{n-1-p} \leqslant |f'(z)| \leqslant p \, |z|^{p-1} + |z|^p \sum_{n=p+1}^{\infty} n a_n \, |z|^{n-1-p}, \quad \text{where } |z| < 1.$$

Applying inequality (3.4), we obtain Theorem 3.2, thus concluding the proof. □

Theorem 3.3. *If the function f defined by (1.1) belongs to the class $\Upsilon(h, m, s, k, r, q, p)$, then f is starlike of order δ, where*

$$\delta = 1 - \frac{\mu_p p}{-\mu_p + \mu_{p+1}}.$$

The result is sharp with

$$f(z) = z^p - \frac{\mu_p}{\mu_{p+1}} z^{p+1},$$

where μ_p and μ_{p+1} are found by (2.2).

Proof. It is enough to establish that (2.1) implies the following.

$$\sum_{n=p+1}^{\infty} a_n(n - \delta) \leq 1 - \delta. \tag{3.5}$$

That is,

$$\frac{n-\delta}{1-\delta} \leq \frac{\mu_n}{\mu_p}, \qquad n \geq p+1. \tag{3.6}$$

From (3.6) we have the inequality

$$\delta \leqslant 1 - \frac{\mu_p(n-1)}{-\mu_p + \mu_n} = \psi(n),$$

where $n \geq p+1$. And $\psi(n) \geq \psi(p+1)$, (3.6) holds for any $0 \leq h \leq 1$, $0 \leq m < 1$, $0 \leq s < 1$, $k \geq 0$, $0 < q < r \leq 1$ and $p \in \mathbb{N} = \{1, 2, 3, ...\}$. This completes the proof of Theorem 3.3. □

4 Extreme Points of the Class $\Upsilon(h, m, s, k, r, q, p)$

The extreme points of the class $\Upsilon(h, m, s, k, r, q, p)$ is determined by the following theorem.

Theorem 4.1. *Let* $f_p(z) = z^p$,
and

$$f_n(z) = z^p - \frac{\mu_p}{\mu_n} z^n, \quad n = p+1, p+2, p+3, ...,$$

where μ_n *is given by (2.2).*

Then $f \in \Upsilon(h, m, s, k, r, q, p)$ *if and only if it can be expressed in the following form:*

$$f(z) = \sum_{n=p}^{\infty} y_n f_n(z) \tag{4.1}$$

where $y_n \geq 0$ and $\sum_{n=p}^{\infty} y_n = 1.$

Proof. Suppose f can be expressed as in (4.1). Our goal is to show that $f \in \Upsilon(h, m, s, k, r, q, p)$.

By (4.1) we have

$$f(z) = \sum_{n=p}^{\infty} y_n \left\{ z^p - \frac{\mu_p z^n}{\mu_n} \right\}.$$

Then

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n = z^p - \sum_{n=p+1}^{\infty} \frac{\mu_p y_n}{\mu_n} z^n.$$

So that

$$a_n = \frac{\mu_p \, y_n}{\mu_n}, \qquad n \geqslant p+1.$$

Now, we have

$$\sum_{n=p+1}^{\infty} y_n = 1 - y_p \leq 1.$$

Setting

$$\sum_{n=p+1}^{\infty} y_n \frac{\mu_p}{\mu_n} \times \frac{\mu_n}{\mu_p} = \sum_{n=p+1}^{\infty} y_n = 1 - y_p \leqslant 1.$$

By applying Theorem 2.1, we can conclude that the function f belongs to the class $\Upsilon(h, m, s, k, r, q, p)$.

Conversely, it is enough to show that

$$a_n = \frac{\mu_p}{\mu_n} y_n.$$

Now we have $f \in \Upsilon(h, m, s, k, r, q, p)$ then by previous Theorem 2.2.

$$a_n \leqslant \frac{\mu_p}{\mu_n}, \qquad n \geqslant p+1.$$

That is,

$$\frac{\mu_n a_n}{\mu_p} \leqslant 1,$$

but $y_n \leq 1$.
Setting,

$$y_n = \frac{\mu_n a_n}{\mu_p}, \qquad n \geqslant p+1.$$

This leads to the desired outcome, thereby concluding the proof of the theorem. □

Corollary 4.2. *The extreme point of the class* $\Upsilon(h, m, s, k, r, q, p)$ *are the function*

$$f_p(z) = z^p,$$

and

$$f_n(z) = z^p - \frac{\mu_p}{\mu_n} z^n, \quad n = p+1, p+2, p+3, ...,$$

where μ_n *is given by (2.2).*

Finally, in this paper, we study the radius of starlikeness and convexity.

5 Radius of Starlikeness and Convexity

The theorems presented below provide the radius of starlikeness and convexity for the class $\Upsilon(h, m, s, k, r, q, p)$.

Theorem 5.1. *If the function* f *defined by (1.1) belongs to the class* $\Upsilon(h, m, s, k, r, q, p)$*, then* f *is starlike of order* δ $(0 \le \delta < p)$*, within the disk* $|z| < R$ *where*

$$R \quad = \quad \inf \left[\frac{\mu_n}{\mu_p} \times \left(\frac{p-\delta}{n-\delta} \right) \right]^{\frac{1}{n-p}}, n = p+1, p+2, p+3, ..., \tag{5.1}$$

where μ_n *is given by (2.2).*

Proof. Here (5.1) implies

$$\mu_p (n-\delta) |z|^{n-P} \le \mu_n (p-\delta).$$

It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le p - \delta,$$

for $|z| < R$, we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le \frac{\sum\limits_{n=p+1}^{\infty} (n-p) a_n |z|^{n-p}}{1 - \sum\limits_{n=p+1}^{\infty} a_n |z|^{n-p}}. \tag{5.2}$$

By aid of (2.8), we have

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leqslant \frac{\sum\limits_{n=p+1}^{\infty} \frac{\mu_p (n-p) |z|^{n-p}}{\mu_n}}{1 - \sum\limits_{n=p+1}^{\infty} \frac{\mu_p |z|^{n-p}}{\mu_n}}.$$

The last expression is bounded above by $p - \delta$ if.

$$\sum_{n=p+1}^{\infty} \frac{\mu_p (n-p) |z|^{n-p}}{\mu_n} \le \left[1 - \sum_{n=p+1}^{\infty} \frac{\mu_p |z|^{n-p}}{\mu_n} \right] (p-\delta),$$

and it follows that

$$|z|^{n-p} \leqslant \left[\frac{\mu_n}{\mu_p} \left(\frac{p-\delta}{n-\delta} \right) \right], \quad n \geqslant p+1,$$

the given expression is equivalent to our condition stated in (5.1) of the theorem. □

Theorem 5.2. *If the function f defined by (1.1) belongs to the class $\Upsilon(h, m, s, k, r, q, p)$, then f is convex of order ε $(0 \leq \varepsilon < p)$, within the disk $|z| < w$ where*

$$w = \inf\left[\frac{\mu_n}{\mu_p} \times \left(\frac{p(p-\varepsilon)}{n(n-\varepsilon)}\right)\right]^{\frac{1}{n-p}}, \qquad n = p+1, p+2, p+3, ...,$$

and μ_n is given by (2.2).

Proof. By employing the same methodology utilized in the proof of Theorem 5.1, we can establish that within the region $|z| \leq w$, and with the assistance of inequality (2.8), the following inequality holds:

$$\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \leq p - \varepsilon.$$

Hence, we can conclude the statement of Theorem 5.2. □

6 Conclusion

The primary objective of this article is to discover a novel subclass of multivalent analytic functions within the open unit disk. These functions are characterized using Jackson's derivative operator. Additionally, the article explores specific sufficient conditions that must be satisfied by functions belonging to this class, focusing on coefficient characterization. This approach offers numerous fascinating features and potential implications.

Funding
No funds were received.

Availability of data and materials
No data were used or available upon request or included within the article.

Conflicts of Interest
The author declare that they have no conflicts of interest.

Acknowledgments
The author would like to extend their sincere gratitude to the reviewers for their valuable and insightful comments on the paper. They also express their thanks to Dr. Abdullah Algunmeeyn for his valuable time and invaluable contributions.

Author contributions
The author has read and approved the final version of the manuscript.

References

[1] M. I. Alharayzeh. *On a subclass of k-uniformly analytic and multivalent functions defined by q-calculus operator*, Far East Journal of Mathematical Sciences., **132(1)**, 1-20, (2021).

[2] M. I. Alharayzeh and H.S. Alzboon. *On a subclass of k-uniformly analytic functions with negative coefficients and their properties,* Nonlinear Functional Analysis and Applications., Vol. **28**, No. 2 , pp. 589-599, (2023).

[3] M. Alharayzeh and M. Darus. *On subclass of analytic univalent functions associated with negative coefficients*, Inter. Jour. of Math. and Math. Scie., 1-11, (2010).

[4] M. I. Alharayzeh and F. Ghanim. *New subclass of k-uniformly univalent analytic functions with negative coefficients defined by multiplier transformation,* Abstract and Applied Analysis., Volume 2022, Article ID 4593799, 6 pages, (2022).

[5] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni. *Classes of k-uniformly convex and starlike functions*, Tamk. Jour. of Math., **35(3)**, 261-266, (2004).

[6] S. Araci, U. Duran, M. Acikgoz and H. M. Srivastava, *A certain (p, q)- derivative operator and associated divided differences*, J. Inequal. Appl., 2016:301, (2016).

[7] A. Aral, V. Gupta and R. P. Agarwal, *Applications of q-calculus in operator theory,* Springer, New York., (2013).

[8] R. Chakrabarti, R. Jagannathan, *A (p,q)-oscillator realization of two parameter quantum algebras*, J. Phys. A Math. Gen., **24**, L711-L718, (1991).

[9] M. Darus. *Some subclasses of analytic functions*, Jour. of Inst. of Math. & Comp. Sci., **16**(3), 121-126, (2003).

[10] M. Govindaraj and S. Sivasubramanian, *On a class of analytic function related to conic domains involving q-calculus,* Analysis Math., **43** (3), no. 5, 475-487, (2017).

[11] M. E. H. Ismail, E. Merkes and D. Steyr. *A generalization of starlike functions*, Complex Var. Theory., **14**, 77-84, (1990).

[12] F. H. Jackson. *On a q-definite integrals*, Quart. J. Pure Appl. Math., **41**, 193-203, (1910).

[13] F. H. Jackson. *On q-functions and a certain difference operator*, Earth Env. Sci. T. R. So., **46**, 253-281, (1909).

[14] F. H. Jackson. *q-difference equations,* Am. J. Math., **32**, 305-314, (1910).

[15] H. Kalsoom, M. Amer, M. D. Junjua, S. Hassain, G. Shahzadi, *Some (p,q)-estimates of Hermite-Hadamard-type inequalities for coordinated convex and quasi convex functions,* Mathematics., **7**, 1-22, (2019).

[16] S. Kanas and D. R¢aducanu, *Some subclass of analytic functions related to conic domains,* Math. Slovaca., **64**, no. **5**, 1183-1196, (2014).

[17] Z. Karahuseyin, S. Altinkaya, and S.Yalcin, *On H_3(1) Hankel determinant for univalent functions defined by using q-derivative operator* , TJMM., **9**, No. 1, 25-33, (2017).

[18] M. Kunt, I. Iscan, N. Alp, M. Z. Sarikaya, *(p,q)-Hermite-Hadamard inequalities and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat., **112**, 969-992, (2018).

[19] A. O. Lasode and T. O. Opoola. *Some new results on a certain subclass of analytic functions associated with q-differential operator and subordination*, Palestine Journal of Mathematics., Vol. **12(3)**, 115–127, (2023).

[20] J. Prabseang, K. Nonlaopon, J. Tariboon, *(p,q)-Hermite-Handamard inequalities for double integral and (p,q)-differentiable convex function*, Axioms., **8**, 1-10, (2019).

[21] P. N. Sadjang, *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas*, Results Math., **73**, 1-21, (2013).

[22] P. N. Sadjang, *On the (p,q)-gamma and the (p,q)-beta functions*, arXiv:1506.07394., (2015).

[23] H. Silverman. *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**, 109-116, (1975).

[24] H. M. Srivastava, *Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci., **5**(34), 390-444, (2011).

[25] M. Tunc¸, E. Göv, *(p,q)-Integral inequalities*, RGMIA Res. Rep. Coll., **19**, 1-13, (2016).

[26] M. Tunc¸, E. Göv, *(p,q)-Integral inequalities for convex functions*, RGMIA Res. Rep. Coll., **19**, 1-12, (2016).

[27] M. Tunc¸, E. Göv, *Some integral inequalities via (p,q)-calculus on finite intervals*, RGMIA Res. Rep. Coll., **19**, 1-12, (2016).

Author information

Ma'moun I. Y. Alharayzeh, Department of Scientific Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan.
E-mail: mamoun@bau.edu.jo, mamoun.math@gmail.com

Received: 2023-07-12
Accepted: 2023-12-05