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SOME RESULTS ON ESSENTIAL PSEUDOSPECTRA AND $n \times n$ BLOCK OPERATOR MATRICES

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Abstract *The purpose of this paper is to give some new results on essential pseudospectra of bounded linear operators on a Banach space. More precisely, we prove the coincidence between the essential pseudospectra and essential spectra under some hypotheses. Also, we characterize the essential pseudospectra of the sum of two bounded linear operators. Moreover, we give results on essential pseudospectra for $n \times n$ block operator matrices.*

1 Introduction

The concept of pseudospectra was independently introduced by J. M. Varah, H. Landau, L. N. Trefethen and E. B. Davies. Due in particular to L. N. Trefethen, who advanced this approach for matrices and operators, and who applied it to a variety of really intriguing issues. Likewise, a number of mathematicians contributed to this area (see for example, [6, 7]).

For $\varepsilon > 0$, the following formula defines the pseudospectrum $\sigma_\varepsilon(A)$ of a closed, densely defined linear operator A

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ \lambda \in \mathbb{C} \text{ such that } \|(\lambda - A)^{-1}\| > \frac{1}{\varepsilon} \right\},$$

where $\sigma(A)$ represents the spectrum of A.

In [6], E. B. Davies has defined another pseudospectrum equivalent, for every closed operator A, by

$$\sigma_\varepsilon(A) = \bigcup_{\|D\|<\varepsilon} \sigma(A + D).$$

In this paper, we are interested by the study of the essential pseudospectra of bounded linear operators on a Banach space, these essential pseudospectra have several definitions, none of which are equivalent (see for instance, [7]). Our work focuses on the essential pseudospectra of Gustafson, Weidmann, Kato, Wolf, and Schechter. First, we give some results on the notion of the S-essential spectra in Theorem 3.1, this notion is of great important in several mathematical physics disciplines (see for example, [8, 9]). Moreover, in Theorem 3.4, we state a condition for which the essential pseuospectra coincide with the essential spectra (the particular case of S-essential spectra when S is the identity operator I). Next, we give a relationship between the essential pseudospectra of the sum of two bounded linear operators and the essential spectra of each operator in Theorem 3.5. Finally, when discussing block operator matrices, we recall many papers that have contributed to this field (see for instance, [4, 10, 11, 14]). The authors in [4] have specifically studied the essential pseudospecrtra of a 2×2 block operator matrix. Inspired by this work, the aim of Theorem 3.7 is to characterize the essential pseudospectra of $n \times n$ block operator matrices.

The paper is organized as follows. In Section 2, we recall some preliminary results which are fundamental for our purpose. Section 3 of this paper contains the main results.

2 Preliminaries

In this section, we collect some important definitions, notations and preliminary results which will be needed in the sequel. Throughout this paper, X will denote a Banach space and $\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) the set of all bounded linear (resp. compact) operators on X. For $A \in \mathcal{L}(X)$, we designate by $N(A)$ and $R(A)$ the null space and the range of A, respectively. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in X. The number $i(A) = \alpha(A) - \beta(A)$ is called the index of A.

The sets of upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm and Fredholm operators on X are respectively defined by $\Phi_+(X) := \{A \in \mathcal{L}(X) : \alpha(A) < \infty$ and $R(A)$ is closed$\}$, $\Phi_-(X) := \{A \in \mathcal{L}(X) : \beta(A) < \infty\}$, $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ and $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$.

Definition 2.1. Let $F \in \mathcal{L}(X)$.

(i) The operator F is called a Fredholm perturbation if $A + F \in \Phi(X)$ whenever $A \in \Phi(X)$.

(ii) The operator F is called an upper (resp. lower) semi-Fredholm perturbation if $A + F \in \Phi_+(X)$ (resp. $A + F \in \Phi_-(X)$) whenever $A \in \Phi_+(X)$ (resp. $A \in \Phi_-(X)$).

We write $\mathcal{F}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for the sets of all Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations respectively, we refer to [13] for more details about these notions.

The following lemma gives the stability of Fredholm operators under Fredholm perturbations.

Lemma 2.2. *[9, Lemma 2.1] Let $A, F \in \mathcal{L}(X)$, then.*

(i) If $A \in \Phi(X)$ and $F \in \mathcal{F}(X)$, then $A + F \in \Phi(X)$ and $i(A + F) = i(A)$.

(ii) If $A \in \Phi_+(X)$ and $F \in \mathcal{F}_+(X)$, then $A + F \in \Phi_+(X)$ and $i(A + F) = i(A)$.

(iii) If $A \in \Phi_-(X)$ and $F \in \mathcal{F}_-(X)$, then $A + F \in \Phi_-(X)$ and $i(A + F) = i(A)$.

In [1, Theorem 2.4], the authors have studied the Fredholm perturbation for a block operator matrix, we have the following proposition.

Proposition 2.3. *Let $F := (F_{ij})_{1 \le i,j \le n}$ be a block operator matrix where $F_{ij} \in \mathcal{L}(X), \forall (i,j) \in \{1, \ldots, n\}^2$, then*

$$F \in \mathcal{F}(X^n) \text{ if and only if } F_{ij} \in \mathcal{F}(X),\ \forall (i,j) \in \{1, \ldots, n\}^2.$$

Now, let us recall the notion of pseudo Fredholm operator.

Definition 2.4. Let $\varepsilon > 0$ and $A \in \mathcal{L}(X)$.

(i) An operator A is called pseudo Fredholm (respectively, pseudo semi-Fredholm) if $A + D$ is a Fredholm (respectively, semi-Fredholm) operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

(ii) An operator A is called an upper (resp. lower) pseudo semi-Fredholm if $A + D$ is an upper (resp. lower) semi-Fredholm operator for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$.

Denote by $\Phi^\varepsilon(X), \Phi^\varepsilon_\pm(X), \Phi^\varepsilon_+(X)$ and $\Phi^\varepsilon_-(X)$ the sets of pseudo Fredholm , pseudo semi-Fredholm, upper pseudo semi-Fredholm and lower pseudo semi-Fredholm operators respectively.

In this paper, we are concerned with the following S-essential spectra, where $S \in \mathcal{L}(X)$

Gustafson	$\sigma_{e1,S}(A)$	$:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi_+(X)\}$,
Weidmann	$\sigma_{e2,S}(A)$	$:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi_-(X)\}$,
Kato	$\sigma_{e3,S}(A)$	$:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi_\pm(X)\}$,
Wolf	$\sigma_{e4,S}(A)$	$:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi(X)\}$,
Schechter	$\sigma_{e5,S}(A)$	$:= \{\lambda \in \mathbb{C} : (\lambda S - A) \notin \Phi(X)$ *with* $i(\lambda S - A) = 0\}$.

For the essential pseudospectra, we are interested by

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} : (\lambda - A) \notin \Phi_+^\varepsilon(X)\},\\
\sigma_{e2,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} : (\lambda - A) \notin \Phi_-^\varepsilon(X)\},\\
\sigma_{e3,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} : (\lambda - A) \notin \Phi_\pm^\varepsilon(X)\},\\
\sigma_{e4,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} : (\lambda - A) \notin \Phi^\varepsilon(X)\},\\
\sigma_{e5,\varepsilon}(A) &:= \bigcap_{K \in \mathcal{K}(X)} \sigma_\varepsilon(A+K).
\end{aligned}$$

Note that if ε tends to 0 or S is the identity operator, we recover the usual definitions of Gustafson, Weidmann, Kato, Wolf and Schechter essential spectra denoted respectively by $\sigma_{e1}(A), \sigma_{e2}(A), \sigma_{e3}(A), \sigma_{e4}(A)$ and $\sigma_{e5}(A)$ of a bounded linear operator A. For more details, the reader is referred to [7]. It is well known that

$$\sigma_{e3,\varepsilon}(A) := \sigma_{e1,\varepsilon}(A) \cap \sigma_{e2,\varepsilon}(A) \subseteq \sigma_{e4,\varepsilon}(A) \subseteq \sigma_{e5,\varepsilon}(A),$$

and

$$\bigcap_{\varepsilon>0} \sigma_{ei,\varepsilon}(A) = \sigma_{ei}(A) \quad i \in \{1, \ldots, 5\}.$$

In [7], A. Jeribi has established the following results.

Proposition 2.5. *Let $\varepsilon > 0$ and $A \in \mathcal{L}(X)$. The following conditions are equivalent*

(i) $\lambda \in \sigma_\varepsilon(A)$.

(ii) There exists a bounded operator D such that $\|D\| < \varepsilon$ and $\lambda \in \sigma(A+D)$.

Proposition 2.6. *Let $\varepsilon > 0$ and $A \in \mathcal{L}(X)$, then $\lambda \notin \sigma_{e5,\varepsilon}(A)$ if and only if, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, $(A + D - \lambda) \in \Phi(X)$ and $i(A + D - \lambda) = 0$.*

3 Main results

In this section, we present our main results on the S-essential spectra, essential pseudospectra and $n \times n$ block operator matrices.

We begin our work by giving results on the S-essential spectra for the sum of two bounded linear operators in the following theorem.

Theorem 3.1. *Let $A, B, S \in \mathcal{L}(X)$ such that $S \neq B$.*

(i) If $AB \in \mathcal{F}_+(X)$, then

$$\sigma_{e1,S}(AS + B)\backslash\{0\} \subset [\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]\backslash\{0\}.$$

Moreover if, $BA \in \mathcal{F}_+(X)$ and $SA = AS$, then

$$\sigma_{e1,S}(AS + B)\backslash\{0\} = [\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]\backslash\{0\}. \tag{3.1}$$

(ii) If $AB \in \mathcal{F}_-(X)$, then

$$\sigma_{e2,S}(AS + B)\backslash\{0\} \subset [\sigma_{e2}(A) \cup \sigma_{e2,S}(B)]\backslash\{0\}.$$

Moreover if, $BA \in \mathcal{F}_-(X)$ and $SA = AS$, then

$$\sigma_{e2,S}(AS + B)\backslash\{0\} = [\sigma_{e2}(A) \cup \sigma_{e2,S}(B)]\backslash\{0\}. \tag{3.2}$$

(iii) If $AB \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, then

$$\begin{aligned}
\sigma_{e3,S}(AS + B)\backslash\{0\} \quad \subset \quad & [\sigma_{e3}(A) \cup \sigma_{e3,S}(B) \cup (\sigma_{e1}(A) \cap \sigma_{e2,S}(B)) \cup \\
& (\sigma_{e2}(A) \cap \sigma_{e1,S}(B))]\backslash\{0\}.
\end{aligned}$$

Moreover if, $BA \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ *and* $SA = AS$*, then*

$$\begin{aligned}\sigma_{e3,S}(AS+B)\backslash\{0\} &= [\sigma_{e3}(A) \cup \sigma_{e3,S}(B) \cup (\sigma_{e1}(A) \cap \sigma_{e2,S}(B)) \cup \\ &\quad (\sigma_{e2}(A) \cap \sigma_{e1,S}(B))]\backslash\{0\}.\end{aligned}$$

(iv) If $AB \in \mathcal{F}(X)$*, then*

$$\sigma_{ei,S}(AS+B)\backslash\{0\} \subset [\sigma_{ei}(A) \cup \sigma_{ei,S}(B)]\backslash\{0\}, \quad i = 4, 5. \tag{3.3}$$

Moreover if, $BA \in \mathcal{F}(X)$ *and* $SA = AS$*, then*

$$\sigma_{e4,S}(AS+B)\backslash\{0\} = [\sigma_{e4}(A) \cup \sigma_{e4,S}(B)]\backslash\{0\}.$$

Proof. For $\lambda \in \mathbb{C}$, we can write

$$(\lambda - A)(\lambda S - B) = AB + \lambda(\lambda S - AS - B). \tag{3.4}$$

$$(\lambda S - B)(\lambda - A) = BA + \lambda(\lambda S - SA - B). \tag{3.5}$$

(i) Suppose that $\lambda \notin [\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]\backslash\{0\}$, then $(\lambda - A) \in \Phi_+(X)$ and $(\lambda S - B) \in \Phi_+(X)$. By using [13, Theorem 5.26, p. 122], we get $(\lambda - A)(\lambda S - B) \in \Phi_+(X)$. Since $AB \in \mathcal{F}_+(X)$, applying equation (3.4), we infer that $(\lambda S - AS - B) \in \Phi_+(X)$, hence $\lambda \notin \sigma_{e1,S}(AS+B)$. Therefore

$$\sigma_{e1,S}(AS+B)\backslash\{0\} \subset [\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]\backslash\{0\}. \tag{3.6}$$

For the inverse inclusion of (3.6), let $\lambda \notin \sigma_{e4,S}(AS+B)\backslash\{0\}$, this implies that $(\lambda S - AS - B) \in \Phi_+(X)$ and $(\lambda S - SA - B) \in \Phi_+(X)$. Since $AB \in \mathcal{F}_+(X)$ and $BA \in \mathcal{F}_+(X)$, then using equations (3.4), (3.5) and Lemma 2.2, we get

$$(\lambda - A)(\lambda S - B) \in \Phi_+(X) \text{ and } (\lambda S - B)(\lambda - A) \in \Phi_+(X). \tag{3.7}$$

With the use of equation (3.7) and [12, Theorem 6, p. 157], we obtain $(\lambda - A) \in \Phi_+(X)$ and $(\lambda S - B) \in \Phi_+(X)$, hence $\lambda \notin [\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]$. So,

$$[\sigma_{e1}(A) \cup \sigma_{e1,S}(B)]\backslash\{0\} \subset \sigma_{e1,S}(AS+B)\backslash\{0\}.$$

(ii) The proof is similar to the previous one.

(iii) Since $\sigma_{e3,S}(.) = \sigma_{e1,S}(.) \cap \sigma_{e2,S}(.)$, $AB \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, $BA \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ and $SA = AS$, then by equations (3.1) and (3.2), we deduce that

$$\begin{aligned}\sigma_{e3,S}(AS+B)\backslash\{0\} &= [\sigma_{e3}(A) \cup \sigma_{e3,S}(B) \cup (\sigma_{e1}(A) \cap \sigma_{e2,S}(B)) \cup \\ &\quad (\sigma_{e2}(A) \cap \sigma_{e1,S}(B))]\backslash\{0\}.\end{aligned}$$

(iv) Suppose that $\lambda \notin [\sigma_{e5}(A) \cup \sigma_{e5,S}(B)]\backslash\{0\}$, then $(\lambda - A) \in \Phi(X)$ with $i(\lambda - A) = 0$ and $(\lambda S - B) \in \Phi(X)$ with $i(\lambda S - B) = 0$. It follows from [13, Theorem 5.7, p. 106] that $(\lambda - A)(\lambda S - B) \in \Phi(X)$ with $i((\lambda - A)(\lambda S - B)) = i(\lambda - A) + i(\lambda S - B) = 0$. Since $AB \in \mathcal{F}(X)$, using equation (3.4), we infer that $(\lambda S - AS - B) \in \Phi(X)$ with $i(\lambda S - AS - B) = 0$, hence $\lambda \notin \sigma_{e5,S}(AS+B)$. Therefore

$$\sigma_{e5,S}(AS+B)\backslash\{0\} \subset [\sigma_{e5}(A) \cup \sigma_{e5,S}(B)]\backslash\{0\}.$$

For $i = 4$, the proof is the same as for $i = 5$. To prove the opposite inclusion of (3.3) for $i = 4$, assume that $\lambda \notin \sigma_{e4,S}(AS+B)\backslash\{0\}$, this means that $(\lambda S - AS - B) \in \Phi(X)$ and $(\lambda S - SA - B) \in \Phi(X)$. Since $AB \in \mathcal{F}(X)$ and $BA \in \mathcal{F}(X)$, then by equations (3.4), (3.5) and Lemma 2.2, we have

$$(\lambda - A)(\lambda S - B) \in \Phi(X) \text{ and } (\lambda S - B)(\lambda - A) \in \Phi(X). \tag{3.8}$$

By using equation (3.8) and [12, Theorem 6, p. 157], we see that $(\lambda - A)$ and $(\lambda S - B) \in \Phi(X)$, hence $\lambda \notin [\sigma_{e4}(A) \cup \sigma_{e4,S}(B)]$. So,

$$[\sigma_{e4}(A) \cup \sigma_{e4,S}(B)]\backslash\{0\} \subset \sigma_{e4,S}(AS+B)\backslash\{0\}.$$

□

Remark 3.2. In the particular case when $S = I$, we get results proved in [2, Theorem 4.2].

The other main theorem provides results concerning the sum of square of two bounded linear operators for ST-essential spectra.

Theorem 3.3. *Let $S, T \in \mathcal{L}(X)$ such that $S \neq T$.*

(i) If $TS \in \mathcal{F}_+(X)$, then

$$\sigma_{e1,ST}(S^2+T^2)\backslash\{0\} \subset [\sigma_{e1,T}(S) \cup \sigma_{e1,S}(T)]\backslash\{0\}.$$

Moreover if, $TS = ST$, then

$$\sigma_{e1,ST}(S^2+T^2)\backslash\{0\} = [\sigma_{e1,T}(S) \cup \sigma_{e1,S}(T)]\backslash\{0\}. \tag{3.9}$$

(ii) If $TS \in \mathcal{F}_-(X)$, then

$$\sigma_{e2,ST}(S^2+T^2)\backslash\{0\} \subset [\sigma_{e2,T}(S) \cup \sigma_{e2,S}(T)]\backslash\{0\}.$$

Moreover if, $TS = ST$, then

$$\sigma_{e2,ST}(S^2+T^2)\backslash\{0\} = [\sigma_{e2,T}(S) \cup \sigma_{e2,S}(T)]\backslash\{0\}. \tag{3.10}$$

(iii) If $TS \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, then

$$\begin{aligned}\sigma_{e3,ST}(S^2+T^2)\backslash\{0\} \quad &\subset \quad [\sigma_{e3,T}(S) \cup \sigma_{e3,S}(T) \cup (\sigma_{e1,T}(S) \cap \sigma_{e2,S}(T)) \cup \\ &\qquad (\sigma_{e1,S}(T) \cap \sigma_{e2,T}(S))]\backslash\{0\}.\end{aligned}$$

Moreover if, $TS = ST$, then

$$\begin{aligned}\sigma_{e3,ST}(S^2+T^2)\backslash\{0\} \quad &= \quad [\sigma_{e3,T}(S) \cup \sigma_{e3,S}(T) \cup (\sigma_{e1,T}(S) \cap \sigma_{e2,S}(T)) \cup \\ &\qquad (\sigma_{e1,S}(T) \cap \sigma_{e2,T}(S))]\backslash\{0\}.\end{aligned}$$

(iv) If $TS \in \mathcal{F}(X)$, then

$$\sigma_{ei,ST}(S^2+T^2)\backslash\{0\} \subset [\sigma_{ei,T}(S) \cup \sigma_{ei,S}(T)]\backslash\{0\}, \quad i = 4, 5. \tag{3.11}$$

Moreover if, $TS = ST$, then

$$\sigma_{e4,ST}(S^2+T^2)\backslash\{0\} = [\sigma_{e4,T}(S) \cup \sigma_{e4,S}(T)]\backslash\{0\}.$$

Proof. For $\lambda \in \mathbb{C}$, we can write

$$(\lambda S - T)(\lambda T - S) = TS + \lambda(\lambda ST - (S^2+T^2)). \tag{3.12}$$

$$(\lambda T - S)(\lambda S - T) = ST + \lambda(\lambda TS - (S^2+T^2)). \tag{3.13}$$

(i) Suppose that $\lambda \notin [\sigma_{e1,T}(S) \cup \sigma_{e1,S}(T)]\backslash\{0\}$, then $(\lambda T - S) \in \Phi_+(X)$ and $(\lambda S - T) \in \Phi_+(X)$. It follows from [13, Theorem 5.26, p. 122] that $(\lambda S - T)(\lambda T - S) \in \Phi_+(X)$. Since $TS \in \mathcal{F}_+(X)$, then equation (3.12) gives $(\lambda ST - (S^2+T^2)) \in \Phi_+(X)$, hence $\lambda \notin \sigma_{e1,ST}(S^2+T^2)$. Therefore

$$\sigma_{e1,ST}(S^2+T^2)\backslash\{0\} \subset [\sigma_{e1,T}(S) \cup \sigma_{e1,S}(T)]\backslash\{0\}. \tag{3.14}$$

We prove now the inverse inclusion of (3.14). Let $\lambda \notin \sigma_{e4,ST}(S^2+T^2)\backslash\{0\}$, this implies that $(\lambda ST - (S^2+T^2)) \in \Phi_+(X)$ and $(\lambda TS - (S^2+T^2)) \in \Phi_+(X)$. Since $TS = ST \in \mathcal{F}_+(X)$, then by using equations (3.12), (3.13) and Lemma 2.2, we see that

$$(\lambda S - T)(\lambda T - S) \in \Phi_+(X) \text{ and } (\lambda T - S)(\lambda S - T) \in \Phi_+(X). \tag{3.15}$$

Using equation (3.15) and [12, Theorem 6, p. 157], we get $(\lambda S - T) \in \Phi_+(X)$ and $(\lambda T - S) \in \Phi_+(X)$, hence $\lambda \notin [\sigma_{e1,T}(S) \cup \sigma_{e1,S}(T)]$.

(ii) The proof is analogous to the previous one.

(iii) Since $\sigma_{e3,ST}(.) = \sigma_{e1,ST}(.) \cap \sigma_{e2,ST}(.)$ and $TS = ST \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, then by equations (3.9) and (3.10), we deduce that

$$\sigma_{e3,ST}(S^2+T^2)\backslash\{0\} = [\sigma_{e3,T}(S) \cup \sigma_{e3,S}(T) \cup (\sigma_{e1,T}(S) \cap \sigma_{e2,S}(T)) \cup (\sigma_{e1,S}(T) \cap \sigma_{e2,T}(S))]\backslash\{0\}.$$

(iv) Assume that $\lambda \notin [\sigma_{e5,T}(S) \cup \sigma_{e5,S}(T)]\backslash\{0\}$, then $(\lambda T - S) \in \Phi(X)$ with $i(\lambda T - S) = 0$ and $(\lambda S - T) \in \Phi(X)$ with $i(\lambda S - T) = 0$, the use of [13, Theorem 5.7, p. 106] implies that $(\lambda S - T)(\lambda T - S) \in \Phi(X)$ with $i((\lambda S - T)(\lambda T - S)) = 0$. Since $TS \in \mathcal{F}(X)$, it follows from equation (3.12) that $(\lambda ST - (S^2+T^2)) \in \Phi(X)$, hence $\lambda \notin \sigma_{e5,ST}(S^2+T^2)$. Therefore

$$\sigma_{e5,ST}(S^2+T^2)\backslash\{0\} \subset [\sigma_{e5,T}(S) \cup \sigma_{e5,S}(T)]\backslash\{0\}.$$

By the same argument, we obtain the statement for $i = 4$. For the opposite inclusion of (3.11), let $\lambda \notin \sigma_{e4,ST}(S^2+T^2)\backslash\{0\}$. So $(\lambda ST - (S^2+T^2)) \in \Phi(X)$ and $(\lambda TS - (S^2+T^2)) \in \Phi(X)$. Since $TS = ST \in \mathcal{F}(X)$, then by the use of equations (3.12), (3.13) and Lemma 2.2, we get

$$(\lambda S - T)(\lambda T - S) \in \Phi(X) \text{ and } (\lambda T - S)(\lambda S - T) \in \Phi(X). \tag{3.16}$$

Now, using [12, Theorem 6, p. 157], we see that $(\lambda S - T) \in \Phi(X)$ and $(\lambda T - S) \in \Phi(X)$, hence $\lambda \notin [\sigma_{e4,T}(S) \cup \sigma_{e4,S}(T)]$.

□

In the next main result, we state a condition under which the essential spectra coincide with the essential pseudospectra.

Theorem 3.4. *Let $\varepsilon > 0$ and $A, D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then*

(i) If there exists $\lambda \in \rho(A+D)$, then

$$\sigma_{ei}(A) = \sigma_{ei,\varepsilon}(A) \quad i = 1, 2, 3.$$

(ii) If there exists $\lambda \in \rho(A+D)$ such that $D \in \mathcal{F}(X)$, then

$$\sigma_{ei}(A) = \sigma_{ei,\varepsilon}(A) \quad i = 4, 5.$$

Proof. It suffices to prove the following inclusion $\sigma_{ei,\varepsilon}(A) \subset \sigma_{ei}(A)$, $i = 1, \ldots, 5$. Let $\lambda \in \rho(A+D)$, then for all bounded operator D such that $\|D\| < \varepsilon$, we can represent the operator $(\lambda - A)$ as

$$(\lambda - A) = (I + D(\lambda - A - D)^{-1})(\lambda - A - D). \tag{3.17}$$

$$(\lambda - A) = (\lambda - A - D)(I + (\lambda - A - D)^{-1}D). \tag{3.18}$$

We will prove only the statements for $i = 2, 5$. The other assertions can be checked by the same way.

(i) Let $\lambda \notin \sigma_{e2}(A)$, then $(\lambda - A) \in \Phi_-(X)$. The use of equation (3.18) and [12, Theorem 6, p. 157] gives $(\lambda - A - D) \in \Phi_-(X)$, hence $\lambda \notin \sigma_{e2,\varepsilon}(A)$.

(ii) Suppose that $\lambda \notin \sigma_{e5}(A)$, this implies that $(\lambda - A) \in \Phi(X)$ with $i(\lambda - A) = 0$. Since $D(\lambda - A - D)^{-1} \in \mathcal{F}(X)$ (using [7, Lemma 2.3.2, p. 50]), then $(I + D(\lambda - A - D)^{-1}) \in \Phi(X)$. It follows from (3.17) and [3, Theorem 4.43, p. 158] that $(\lambda - A - D) \in \Phi(X)$ with $i(\lambda - A - D) = 0$, hence $\lambda \notin \sigma_{e5,\varepsilon}(A)$.

□

As a consequence of Theorem 3.1 for $S = I$ and Theorem 3.4, we get a relation between the essential pseudospectra of the sum of two bounded linear operators and the essential spectra of each operator in the following theorem.

Theorem 3.5. *Let $\varepsilon > 0$ and $A, B, D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$, then*

(i) If there exists $\lambda \in \rho(A+B+D)$ and $AB \in \mathcal{F}_+(X)$, then

$$\sigma_{e1,\varepsilon}(A+B)\backslash\{0\} \subset [\sigma_{e1}(A) \cup \sigma_{e1}(B)]\backslash\{0\}.$$

(ii) If there exists $\lambda \in \rho(A+B+D)$ and $AB \in \mathcal{F}_-(X)$, then

$$\sigma_{e2,\varepsilon}(A+B)\backslash\{0\} \subset [\sigma_{e2}(A) \cup \sigma_{e2}(B)]\backslash\{0\}.$$

(iii) If there exists $\lambda \in \rho(A+B+D)$ and $AB \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$, then

$$\begin{aligned}\sigma_{e3,\varepsilon}(A+B)\backslash\{0\} \quad \subset \quad & [\sigma_{e3}(A) \cup \sigma_{e3}(B) \cup (\sigma_{e1}(A) \cap \sigma_{e2}(B)) \cup \\ & (\sigma_{e2}(A) \cap \sigma_{e1}(B))]\backslash\{0\}.\end{aligned}$$

(iv) If there exists $\lambda \in \rho(A+B+D)$ such that $D \in \mathcal{F}(X)$ and $AB \in \mathcal{F}(X)$, then

$$\sigma_{ei,\varepsilon}(A+B)\backslash\{0\} \subset [\sigma_{ei}(A) \cup \sigma_{ei}(B)]\backslash\{0\}, \quad i = 4, 5.$$

Proof. It follows from Theorem 3.4 that if there exists $\lambda \in \rho(A+B+D)$, then $\sigma_{ei}(A+B) = \sigma_{ei,\varepsilon}(A+B)$, $i = 1, 2, 3$, then according to Theorem 3.1 for $S = I$, we get assertions (i), (ii) and (iii). Statement (iv) can be checked by the same way. □

Now, in order to establish some results on essential pseudospectra of the $n \times n$ block operator matrix defined on X^n by

$$W = \begin{pmatrix} W_{11} & W_{12} & \cdots & \cdots & W_{1n} \\ W_{21} & W_{22} & \cdots & \cdots & W_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ W_{n1} & W_{n2} & \cdots & \cdots & W_{nn} \end{pmatrix} \tag{3.19}$$

where $W_{ij} \in \mathcal{L}(X), \forall (i,j) \in \{1,\dots,n\}^2$, we introduce the following main theorem.

Theorem 3.6. *Let $\varepsilon > 0$ and $U = \begin{pmatrix} W_{11} & W_{12} & \cdots & \cdots & W_{1n} \\ 0 & W_{22} & & & W_{2n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & W_{nn} \end{pmatrix}$ be the block operator matrix W such that $W_{ij} = 0$ for $i > j$, then*

(i) $\sigma_{e1,\varepsilon}(W_{11}) \subseteq \sigma_{e1,\varepsilon}(U) \subseteq \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk})$.

(ii) $\sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e2,\varepsilon}(U) \subseteq \bigcup_{k=1}^{n} \sigma_{e2,\varepsilon}(W_{kk})$.

(iii) $\sigma_{e1,\varepsilon}(W_{11}) \cap \sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e3,\varepsilon}(U) \subseteq \bigcup_{k,l=1}^{n} (\sigma_{e1,\varepsilon}(W_{kk}) \cap \sigma_{e2,\varepsilon}(W_{ll}))$.

(iv) $\sigma_{e1,\varepsilon}(W_{11}) \cup \sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e4,\varepsilon}(U) \subseteq \bigcup_{k=1}^{n} \sigma_{e4,\varepsilon}(W_{kk})$.

Proof. Let $D_k \in \mathcal{L}(X)$, $k \in \{1,\dots,n\}$ and consider $D = \begin{pmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ 0 & D_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & D_n \end{pmatrix}$ such that $\|D\| = \max_{k=1,\dots,n} \|D_k\| < \varepsilon$, (i.e, $\|D_k\| < \varepsilon$, for all $k \in \{1,\dots,n\}$), then for $\lambda \in \mathbb{C}$, we have

$$\lambda - U - D = \begin{pmatrix} \lambda - W_{11} - D_1 & -W_{12} & \cdots & \cdots & -W_{1n} \\ 0 & \lambda - W_{22} - D_2 & & & -W_{2n} \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda - W_{nn} - D_n \end{pmatrix}. \tag{3.20}$$

(i) We prove the first inclusion $\sigma_{e1,\varepsilon}(W_{11}) \subseteq \sigma_{e1,\varepsilon}(U)$. Let $\lambda \notin \sigma_{e1,\varepsilon}(U)$, then $(\lambda - U) \in \Phi_+^{\varepsilon}(X^n)$, this implies that $(\lambda - U - D) \in \Phi_+(X^n)$ for all $\|D\| < \varepsilon$. Applying [1, Proposition 2.1, p. 1190] and since $\|D_1\| < \varepsilon$, we get $(\lambda - W_{11} - D_1) \in \Phi_+(X)$, hence $(\lambda - W_{11}) \in \Phi_+^{\varepsilon}(X)$. Therefore $\lambda \notin \sigma_{e1,\varepsilon}(W_{11})$.

For the second inclusion $\sigma_{e1,\varepsilon}(U) \subseteq \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk})$, suppose that $\lambda \notin \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk})$, then $(\lambda - W_{kk}) \in \Phi_+^{\varepsilon}(X)$ for all $k \in \{1,\dots,n\}$. Since $\|D_k\| < \varepsilon$, so $(\lambda - W_{kk} - D_k) \in \Phi_+(X)$ for all $k \in \{1,\dots,n\}$. Using [1, Proposition 2.1, p. 1190], we get $(\lambda - U - D) \in \Phi_+(X^n)$ for all $\|D\| < \varepsilon$, hence $(\lambda - U) \in \Phi_+^{\varepsilon}(X^n)$. Therefore $\lambda \notin \sigma_{e1,\varepsilon}(U)$.

(ii) We prove the first inclusion $\sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e2,\varepsilon}(U)$. Let $\lambda \notin \sigma_{e2,\varepsilon}(U)$, this implies that $(\lambda - U) \in \Phi_-^{\varepsilon}(X^n)$, then $(\lambda - U - D) \in \Phi_-(X^n)$ for all $\|D\| < \varepsilon$. Applying again [1, Proposition 2.1, p. 1190] and since $\|D_n\| < \varepsilon$, we get $(\lambda - W_{nn} - D_n) \in \Phi_-(X)$, hence $(\lambda - W_{nn}) \in \Phi_-^{\varepsilon}(X)$. Therefore $\lambda \notin \sigma_{e2,\varepsilon}(W_{nn})$.

For the second inclusion $\sigma_{e2,\varepsilon}(U) \subseteq \bigcup_{k=1}^{n} \sigma_{e2,\varepsilon}(W_{kk})$, assume that $\lambda \notin \bigcup_{k=1}^{n} \sigma_{e2,\varepsilon}(W_{kk})$, then $(\lambda - W_{kk}) \in \Phi_-^{\varepsilon}(X)$ for all $k \in \{1,\dots,n\}$. Since $\|D_k\| < \varepsilon$, we have $(\lambda - W_{kk} - D_k) \in \Phi_-(X)$ for all $k \in \{1,\dots,n\}$. Using [1, Proposition 2.1, p. 1190], we get $(\lambda - U - D) \in \Phi_-(X^n)$ for all $\|D\| < \varepsilon$, hence $(\lambda - U) \in \Phi_-^{\varepsilon}(X^n)$. Therefore $\lambda \notin \sigma_{e2,\varepsilon}(U)$.

(iii) Assertions (iii) and (iv) follow immediately from assertions (i) and (ii).

□

In the last main theorem, we give a characterization for the essential pseudospectra of the $n \times n$ block operator matrix W, by giving conditions on operators W_{ij} such that $i > j$.

Theorem 3.7. *Let W be the block operator matrix defined above.*

(i) If $W_{ij} \in \mathcal{F}_+(X)$ for all $i > j$, then

$$\sigma_{e1,\varepsilon}(W_{11}) \subseteq \sigma_{e1,\varepsilon}(W) \subseteq \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk}).$$

(ii) If $W_{ij} \in \mathcal{F}_-(X)$ for all $i > j$, then

$$\sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e2,\varepsilon}(W) \subseteq \bigcup_{k=1}^{n} \sigma_{e2,\varepsilon}(W_{kk}).$$

(iii) If $W_{ij} \in \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$ for all $i > j$, then

$$\sigma_{e1,\varepsilon}(W_{11}) \cap \sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e3,\varepsilon}(W) \subseteq \bigcup_{k,l=1}^{n} \left(\sigma_{e1,\varepsilon}(W_{kk}) \cap \sigma_{e2,\varepsilon}(W_{ll})\right).$$

(iv) If $W_{ij} \in \mathcal{F}(X)$ for all $i > j$, then

$$\sigma_{e1,\varepsilon}(W_{11}) \cup \sigma_{e2,\varepsilon}(W_{nn}) \subseteq \sigma_{e4,\varepsilon}(W) \subseteq \bigcup_{k=1}^{n} \sigma_{e4,\varepsilon}(W_{kk}).$$

Proof. For $\lambda \in \mathbb{C}$, we can represent $(\lambda - W - D)$ as follows

$$(\lambda - W - D) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ -W_{21} & \ddots & & & \vdots \\ -W_{31} & -W_{32} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ -W_{n1} & \cdots & \cdots & \cdots & 0 \end{pmatrix} + \begin{pmatrix} \lambda - W_{11} - D_1 & -W_{12} & \cdots & \cdots & -W_{1n} \\ 0 & \lambda - W_{22} - D_2 & & & -W_{2n} \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda - W_{nn} - D_n \end{pmatrix}. \tag{3.21}$$

(i) We prove the first inclusion $\sigma_{e1,\varepsilon}(W_{11}) \subseteq \sigma_{e1,\varepsilon}(W)$. Let $\lambda \notin \sigma_{e1,\varepsilon}(W)$, then $(\lambda - W) \in \Phi_+^{\varepsilon}(X^n)$, this implies that $(\lambda - W - D) \in \Phi_+(X^n)$ for all $\|D\| < \varepsilon$. It follows from hypothesis $W_{ij} \in \mathcal{F}_+(X)$ for all $i > j$ and equation (3.21) that

$$\begin{pmatrix} \lambda - W_{11} - D_1 & -W_{12} & \cdots & \cdots & -W_{1n} \\ 0 & \lambda - W_{22} - D_2 & & & -W_{2n} \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda - W_{nn} - D_n \end{pmatrix} \in \Phi_+(X^n).$$

Then by the first inclusion of assertion (i) of Theorem 3.6, we have $(\lambda - W_{11} - D_1) \in \Phi_+(X)$ for all $\|D_1\| < \varepsilon$, hence $(\lambda - W_{11}) \in \Phi_+^{\varepsilon}(X)$. Therefore $\lambda \notin \sigma_{e1,\varepsilon}(W_{11})$.

We prove the second inclusion $\sigma_{e1,\varepsilon}(W) \subseteq \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk})$. Suppose that $\lambda \notin \bigcup_{k=1}^{n} \sigma_{e1,\varepsilon}(W_{kk})$, then $(\lambda - W_{kk}) \in \Phi_+^{\varepsilon}(X)$ for all $k \in \{1, \dots, n\}$. By using the fact that $\|D_k\| < \varepsilon$, we have $(\lambda - W_{kk} - D_k) \in \Phi_+(X)$ for all $k \in \{1, \dots, n\}$. Then by the second inclusion of assertion (i) of Theorem 3.6, we get

$$\begin{pmatrix} \lambda - W_{11} - D_1 & -W_{12} & \cdots & \cdots & -W_{1n} \\ 0 & \lambda - W_{22} - D_2 & & & -W_{2n} \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \lambda - W_{nn} - D_n \end{pmatrix} \in \Phi_+(X^n).$$

By using equation (3.21), we get $(\lambda - W - D) \in \Phi_+(X^n)$ for all $\|D\| < \varepsilon$, hence $(\lambda - W) \in \Phi_+^{\varepsilon}(X^n)$. Therefore $\lambda \notin \sigma_{e1,\varepsilon}(W)$.

(ii) The proof is analogous to the previous one.

(iii) Assertions (iii) and (iv) can be deduced from (i) and (ii).

□

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