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FRACTIONAL OPERATORS AND SOLUTION OF FRACTIONAL KINETIC EQUATIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract The objective of this paper is to explore certain properties involving the generalized hypergeometric function ${}_uF_v^{p,q;\lambda;\sigma,\tau}[z]$, as introduced by Khan et al. [27]. Several fractional derivatives and integral formulas involving the generalized hypergeometric function are computed. Further, we derive the solution of generalized fractional kinetic equations involving generalized hypergeometric function. The respective solution is given in terms of Mittag-Leffler function.

1 Introduction

The study of special functions and their applications has increasingly grown in the last 50 years. This comes together with the advent of powerful computation techniques and devices which have allowed applied scientists to envision real-world applications to this class of functions. Even though great work has been carried out in applying the existing theory of special functions to physical and engineering problems, theoretical studies have also been taken into account. Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during recent years. In particular, the special functions of more than one variable provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. The genesis of most special functions in mathematical physics, along with their extensions, stems from the exploration of physical problems. In mathematics, the generalized hypergeometric function represents a modified version of the confluent hypergeometric function of the first kind. Its applications span a wide array of fields including mathematical physics and numerous research domains, attracting the attention of mathematicians across various disciplines.

Lately, numerous authors have been providing extensions and generalizations for several special functions, including the beta function, gamma function, hypergeometric function, and confluent hypergeometric function (refer to [27] for details). Khan et al. [27] introduced novel extended hypergeometric and confluent hypergeometric functions utilizing the extended beta function. Inspired by the abovementioned work, in this paper, we investigate some properties of the generalized hypergeometric function ${}_uF_v^{p,q;\lambda;\sigma,\tau}[z]$. Several fractional derivatives and integral formulas involving the generalized hypergeometric function are computed. Further, we obtain the solution for generalized fractional kinetic equations incorporating the above-mentioned generalized hypergeometric function and the solution is obtained in terms of Mittag-Leffler function. We remember below the following basic definition and extension of special function. Some properties of the Mittag-Leffler function were considered in paper [8]. The Cauchy problem for matrix factorizations of the Helmholtz equation was addressed by approximating solutions using the Mittag-Leffler function as the kernel of the Carleman matrix, as seen in references [5]-[7] and [9]-[11].

Throughout the manuscript, let $\mathbb{C},\ \Re,\ \Re+,\ \mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers and positive integers, respectively, and let

$$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\},\ \ \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \ \mathbb{Z}_0^- = \mathbb{Z}/\mathbb{N}.$$

Euler introduced the beta function (see [13]) for a pair of complex numbers η_1 and η_2 with positive real part through the integral

$$B(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1}\,(1-t)^{\eta_2 - 1} dt \tag{1.1}$$

$$\Big(\Re(\eta_1) > 0\,,\ \Re(\eta_2) > 0;\ \eta_1, \eta_2 \notin \mathbb{Z}_0^-\Big).$$

The function of the form

$${}_2F_1\left[\begin{matrix}\alpha, & \beta; \\ \gamma; & \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k}\,\frac{z^k}{\Gamma(k+1)} \quad (|z|<1), \tag{1.2}$$

is known as the Gauss's hypergeometric function.

It has a more general form widely known as generalized hypergeometric series ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ defined by

$${}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\,\frac{z^k}{k!},$$

where,

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \quad (\alpha \in \mathbb{C})$$

is the well known Pochhammer symbol (see [18, p. 21 etc.]).

The confluent hypergeometric function (see [13]) is defined by the series

$${}_1F_1(\alpha; \beta;\ z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\,\frac{z^k}{k!}. \tag{1.3}$$

Chaudhry et al. [23] defined the extended form of beta function by introducing a parameter p

$$B(\eta_1, \eta_2;\, p) = \int_0^1 (1-t)^{\eta_2 - 1}\, t^{\eta_1 - 1}\, e^{\left[-\frac{p}{t(1-t)}\right]} dt \quad (0 < \Re(\eta_1)\,,\ (0 < \Re(\eta_2)), \tag{1.4}$$

where, $\Re(p) > 0$ and parameters η_1 and η_2 are arbitray complex numbers,
and applied the definition (1.4) to obtain the extended hypergeometric function as

$$F_p\left[\begin{matrix}\alpha, & \beta; \\ \gamma; & \end{matrix}\ z\right] = \frac{1}{B(\beta, \gamma-\beta)} \sum_{k=0}^{\infty} B(\beta+k, \gamma-\beta; p)\,(\alpha)_k\,\frac{z^k}{k!} \tag{1.5}$$

$$\Big(|z|<1;\ p \geqq 0;\ 0 > \Re(\beta) > \Re(\gamma)\Big).$$

Shadab et al. [26] introduced

$$B_p^{\lambda}(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1}\,(1-t)^{\eta_2 - 1}\, E_\lambda\left(-\frac{p}{t(1-t)}\right) dt \quad \Big(0 < \Re(\eta_1)\,,\, 0 < \Re(\eta_2)\Big), \tag{1.6}$$

where, $E_\lambda(z)$ is the long-familiar Mittag-Leffler function (see, [15, also Eq. (1.10)]) given in (1.10). In terms of (1.6), they also defined

$$F_{p,\lambda}\left[\begin{array}{ll}\alpha, & \beta; \\ \gamma; & \end{array} z\right] = \frac{1}{B(\beta,\gamma-\beta)}\sum_{k=0}^{\infty}(\alpha)_k\, B_\lambda^p\,(\beta+k,\gamma-\beta)\,\frac{z^k}{k!} \tag{1.7}$$

and

$$\Phi_{p,\lambda}(\beta,\gamma;\, z) = \frac{1}{B(\beta,\gamma-\beta)}\sum_{k=0}^{\infty} B_\lambda^p\,(\beta+k,\gamma-\beta)\,\frac{z^k}{k!}. \tag{1.8}$$

Recently, Khan et al. [27] introduced and investigated a new generalization of beta function as

$$B_{p,q}^{\lambda;\sigma,\tau}(\eta_1,\eta_2) = \int_0^1 t^{\eta_1-1}\,(1-t)^{\eta_2-1}\,E_\lambda\left(-\frac{p}{t^\sigma}\right)\,E_\lambda\left(-\frac{q}{(1-t)^\tau}\right)\,dt \tag{1.9}$$

$$\Big(\lambda,\sigma,\tau>0;\ \Re(p)\geq 0,\ \Re(q)\geq 0;\ \Re(\eta_1)>0,\ \Re(\eta_2)>0\Big),$$

where, E_λ (see [15]) is the Mittag-Leffler function

$$E_\mu(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\mu k)} \tag{1.10}$$

$$\Big(\Re(\mu)>0;\ z,\mu\in\mathbb{C};\ |z|<0\Big),$$

introduced by the swedish mathematician Mittag-Leffler (see also, application section) and its extension $E_{\mu,\eta}(x)$ was studied later by Wiman [4] which has the form

$$E_{\mu,\eta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\eta+\mu k)} \tag{1.11}$$

$$(\mu,\eta\in\mathbb{C};\ \Re(\mu)>0,\Re(\eta)>0).$$

In terms of (1.9), they also defined the generalized Gauss's hypergeometric function

$${}_2F_1^{p,q;\lambda;\sigma,\tau}\,(\alpha,\ \beta;\ \gamma;\ z) = \frac{1}{B(\beta,\gamma-\beta)}\sum_{k=0}^{\infty} B_{p,q}^{\lambda;\sigma,\tau}(\beta+k,\gamma-\beta)\,(\alpha)_k\,\frac{z^k}{k!} \tag{1.12}$$

$$\Big(\min\{\Re(p),\ \Re(q),\ \Re(\sigma),\ \Re(\tau)\}\geqq 0;\ \ \lambda>0;\ |z|<1;\ 0<\Re(\beta)<\Re(\gamma)\Big).$$

Now in view of the definition (1.9) and (1.12), we introduce a generalized hypergeometric function:

$${}_uF_v^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{l}\alpha_1,\ldots,\alpha_u; \\ \beta_1,\ldots,\beta_v;\end{array} z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{u}(\alpha_j)_k}{\prod_{j=1}^{v}(\beta_j)_k}\,\frac{z^k}{k!} = \sum_{k=0}^{\infty}\Omega_k^{p,q;\lambda;\sigma,\tau}\,\frac{z^k}{k!} \tag{1.13}$$

$$\Big(\min\{\Re(\sigma),\Re(\tau)\}\geqq 0;\ \lambda>0;\ \Re(p)\geq 0,\ \Re(q)\geq 0\Big),$$

where, the coefficients $\Omega_k^{p,q;\lambda;\sigma,\tau}$ are given by

$$\Omega_k^{p,q;\lambda;\sigma,\tau} := \begin{cases} (\alpha_1)_k\displaystyle\prod_{j=1}^{v}\frac{B_{p,q}^{\lambda;\sigma,\tau}(\alpha_{j+1}+k,\beta_j-\alpha_{j+1})}{B(\alpha_{j+1};\beta_j-\alpha_{j+1})} \\ (u=v+1;\Re(\beta_j)>\Re(\alpha_{j+1})>0;\ |z|<1) \\ \\ \displaystyle\prod_{j=1}^{v}\frac{B_{p,q}^{\lambda;\sigma,\tau}(\alpha_j+k,\beta_j-\alpha_j)}{B(\alpha_j;\beta_j-\alpha_j)} \\ (u=v;\ \Re(\beta_j)>\Re(\alpha_j)>0;\ \ z\in\mathbb{C}) \\ \\ \displaystyle\frac{1}{(\beta_1)_k\cdots(\beta_r)_k}\prod_{j=1}^{u}\frac{B_{p,q}^{\lambda;\sigma,\tau}(\alpha_j+k,\beta_{r+j}-\alpha_j)}{B(\alpha_j;\beta_{r+j}-\alpha_j)} \\ (r=v-u<0;\ \Re(\beta_{r+j})>\Re(\alpha_j)>0;\ \ z\in\mathbb{C}). \end{cases} \tag{1.14}$$

For $u-1=v=1$ in (1.13), the definition corresponds to the generalized Gauss's hypergeometric function in (1.12).

Remark 1.1. Former definitions of the gauss hypergeometric function and generalized hypergeometric function defined by other authors carry over naturally to our generalization and can be recovered from them as special cases of our definitions. To mention a few, we have:

For

$$\sigma=\tau=\lambda=1,\ p=q \text{ and } u-1=v=1,$$

the definition in (1.13) coincides with (1.5) .

The definition in (1.13) when

$$\sigma=\tau=\lambda=1 \text{ and } u-1=v=1,$$

coincides with the corresponding definition defined by Choi et. al. in [21, Eq. 7.1].

The case when

$$\lambda=1, p=q, u-1=v=1 \text{ and } \sigma=\tau$$

in (1.13), the corresponding definition in [12, Eq. 6.1] can be retrieved.

The Hadamard product for the power series

$$f(z):=\sum_{k=0}^{\infty} a_k\, z^k \quad (|z|<R_f) \quad \text{and} \quad g(z):=\sum_{k=0}^{\infty} b_k\, z^k \quad (|z|<R_g), \tag{1.15}$$

with radii of convergence R_F and R_g, respectively, can be defined by the series:

$$(f*g)(z)=\sum_{k=0}^{\infty} a_k\, b_k\, z^k \quad (|z|<R),$$

where

$$R:=\lim_{k\to\infty}\left|\frac{a_k b_k}{a_{k+1}b_{k+1}}\right|=\left(\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|\right)\left(\lim_{k\to\infty}\left|\frac{b_k}{b_{k+1}}\right|\right)=: R_f.R_g,$$

thereby, we have (see [33])

$$R \geqq R_f.R_g.$$

The Hadamard product, particularly, for an entire function can be written as: (see [17, Definition 6])

$$\begin{aligned} {}_uF_{r+u}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c} \alpha_1,\dots,\alpha_u; \\ \beta_1,\dots,\beta_{r+u}; \end{array} z\right] &= {}_1F_r\left[\begin{array}{c} 1; \\ \beta_1,\dots,\beta_r; \end{array} z\right] \\ &\times {}_uF_u^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c} \alpha_1,\dots,\alpha_u; \\ \beta_{r+1},\dots,\beta_{r+u}; \end{array} z\right] \quad (|z|<\infty). \end{aligned} \tag{1.16}$$

2 Operators of Fractional calculus and Generalized hypergeometric function

Here, we consider the compositions of certain fractional derivative formulas (see, for details, [25, 33]) with the generalized hypergeometric function (1.13).

For convenience, we dentote the left-sided hypergeometric fractional integral operator $I_{0+}^{w,\mu,\zeta}$ and hypergeometric fractional derivative operator $D_{0+}^{w,\mu,\zeta}$ by LHFI and LHFD.

They are defined, respectively, by

$$\begin{aligned} \left(I_{0+}^{w,\mu,\zeta} f\right)(x) &:= \frac{x^{-w-\mu}}{\Gamma(w)} \\ &\int_0^x (x-t)^{w-1}\, {}_2F_1\left[\begin{array}{cc} w+\mu, & -\zeta; \\ w; & \end{array} \frac{x-t}{x}\right] f(t)dt \end{aligned} \tag{2.1}$$

$$\Big(\Re(w) > 0\Big)$$

and

$$\begin{aligned}\left(D_{0+}^{w,\mu,\zeta} f\right)(x) &= \left(\frac{d}{dx}\right)^k \left\{\left(I_{0+}^{-w+\zeta,-\mu-\zeta,w+\zeta-k} f\right)(x)\right\} \\ &= \left(I_{0+}^{-w,-\mu,w+\zeta} f\right)(x)\end{aligned} \tag{2.2}$$

$$\Big(x > 0;\ w, \mu, \zeta \in \mathbb{C};\ \Re(w) \geqq 0;\ k = [\Re(w)] + 1\Big).$$

In particular, the operator $D_{0+}^{w,\mu,\zeta}$ (LHFD) is related to the Riemann-Liouville fractional derivative operator ${}_{RL}D_{0+}^{w}$ (RLFD) and the left-sided Erdélyi-Kober fractional derivative operator ${}_{EK}D_{0+}^{w,\zeta}$ (EKFD) as:

$${}_{RL}D_{0+}^{w} = D_{0+}^{w,-w,\zeta} \quad \text{and} \quad {}_{EK}D_{0+}^{w,\zeta} = D_{0+}^{w,0,\zeta}, \tag{2.3}$$

where, (see [3, Chapter 13])

$$\left({}_{RL}D_{0+}^{w} f\right)(x) := \left(\frac{d}{dx}\right)^k \left\{\frac{1}{\Gamma(k-w)} \int_0^x f(t)\,(x-t)^{k-w-1}\,dt\right\} \tag{2.4}$$

$$\Big(\Re(w) \geqq 0;\ [\Re(w)] = k-1;\ x > 0\Big)$$

and

$$\left({}_{EK}D_{0+}^{w,\zeta} f\right)(x) := x^{\zeta} \left(\frac{d}{dx}\right)^k \left\{\frac{1}{\Gamma(k-w)} \int_0^x \frac{t^{w+\zeta}\, f(t)}{(x-t)^{w-k+1}}\,dt\right\} \tag{2.5}$$

$$\Big(k = [\Re(w)] + 1;\ \Re(w) \geqq 0;\ x > 0\Big).$$

We denote the right sided hypergeometric fractional integral operator and hypergeometric fractional derivative operator by RHFI and RHFD, respectively. For $x > 0,\ w, \mu, \zeta \in \mathbb{C}$, the (RHFI) operator $I_{\infty-}^{w,\mu,\zeta}$ and the (RHFD) opeartor $D_{\infty-}^{w,\mu,\zeta}$ are

$$\begin{aligned}\left(I_{\infty-}^{w,\mu,\zeta}\right)(x) &:= \frac{1}{\Gamma(w)} \int_x^{\infty} \frac{(t-x)^w}{t^{w+\mu}} \\ &\cdot {}_2F_1\left[\begin{matrix}\mu+w, & -\zeta; \\ w; & \end{matrix}\ 1-\frac{t}{x}\right] f(t)dt \quad (0 < \Re(w))\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\left(D_{\infty-}^{w,\mu,\zeta}\right)(x) &= \left(I_{\infty-}^{-w-\mu,w+\zeta}\right)(x) \\ &= \left(-\frac{d}{dx}\right)^k \left\{I_{\infty-}^{-w+\zeta,-\mu-\zeta,w+\zeta-k}\right)(x)\Big\} \\ &\Big(k = [\Re(w)] + 1;\ 0 < \Re(w)\Big),\end{aligned} \tag{2.7}$$

where, ${}_2F_1[.]$ is given by (1.2).

The unfication of both the Weyl fractional derivative operator ${}_{W}D_{\infty-}^{w}$ and the Erdélyi-Kober fractional derivative operator ${}_{EK}D_{\infty-}^{w,\zeta}$ (right-sided) by the operator (RHFD) in (2.7) is given as:

$${}_{W}D_{\infty-}^{w} = D_{\infty-}^{w,-w,\zeta} \quad \text{and} \quad {}_{EK}D_{\infty-}^{w,\zeta} = D_{\infty-}^{w,0,\zeta}, \tag{2.8}$$

where, (see [2, Chapter 13])

$$\left({}_{W}D_{\infty-}^{w} f\right)(x) := \left(-\frac{d}{dx}\right)^k \left\{\frac{1}{\Gamma(k-w)} \int_0^x (t-x)^{k-w-1}\, f(t)\,dt\right\} \tag{2.9}$$

and

$$\left({}_{EK}D^{w}_{\infty-}\, f\right)(x) := x^{w+\zeta} \left(\frac{d}{dx}\right)^{k} \left\{\frac{1}{\Gamma(k-w)} \int_{x}^{\infty} t^{-\zeta}\,(t-x)^{k-w-1}\, f(t)\, dt\right\} \tag{2.10}$$

$$\Big(k = [\Re(w)] + 1;\ \Re(w) \geqq 0;\ x > 0\Big)$$

Lemma 2.1. *The undermentioned hypergeometric fractional derivative formulas are true:*

$$\left(D^{w,\mu,\zeta}_{0+}\, t^{\delta-1}\right)(x) = \frac{\Gamma(\delta)\,\Gamma(\delta+w+\mu+\zeta)}{\Gamma(\delta+\mu)\,\Gamma(\delta+\zeta)}\, x^{\delta+\mu-1} \tag{2.11}$$

$$\Big(x > 0;\ \Re(w) \geqq 0;\ \Re(\delta) + \min\{\Re(0, \Re(w+\mu+\zeta))\} > 0\Big)$$

and

$$\left(D^{w,\mu,\zeta}_{\infty-}\, t^{\delta-1}\right)(x) = \frac{\Gamma(1-\delta-\mu)\,\Gamma(1-\delta+w+\zeta)}{\Gamma(1-\delta)\,\Gamma(1-\delta+\zeta-\mu)}\, x^{\delta+\mu-1} \tag{2.12}$$

$$\Big(x > 0;\ \Re(w) \geqq 0;\ \Re(\delta) - \min\{\Re(-\mu-\zeta),\ \Re(w+\zeta)\} < 1\Big).$$

Theorem 2.2. *The undermentioned formula for LHFD holds true:*

$$\begin{aligned}\left(D^{w,\mu,\zeta}_{0+}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array}\ zt\right]\right)(x) &= x^{\delta+\mu-1}\,\frac{\Gamma(\delta)\,\Gamma(\delta+w+\mu+\zeta)}{\Gamma(\delta+\mu)\,\Gamma(\delta+\zeta)}\\ &\cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array}\ zx\right] * {}_{2}F_{2}\left[\begin{array}{c}\delta,\delta+w+\mu+\zeta;\\ \delta+\mu,\delta+\zeta;\end{array}\ zx\right]\end{aligned} \tag{2.13}$$

$$\Big(x > 0;\ \Re(w) \geqq 0;\ \Re(\delta) + \min\{0, \Re(w+\mu+\zeta)\} > 0\Big),$$

provided that the conditions with (1.13) *and the formula in* (2.13) *exists.*

Proof. The hypergeometric fractional derivative formula (2.13) is easily derivable with the help of the result in (2.11) and (1.13). We omit the details of the proof. □

By using the fractional derivative formula (2.12), Theorem 2.3 can easily be proved with similar approach as in Theorem 2.2. So we choose to skip all details.

Theorem 2.3. *The undermentioned formula for RHFD holds true:*

$$\begin{aligned}\left(D^{w,\mu,\zeta}_{\infty-}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array}\ \frac{z}{t}\right]\right)(x) &= x^{\delta+\mu-1}\,\frac{\Gamma(1-\delta-\mu)\,\Gamma(1-\delta+w+\zeta)}{\Gamma(1-\delta)\,\Gamma(1-\delta-\mu+\zeta)}\\ &\cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array}\ \frac{z}{x}\right] * {}_{2}F_{2}\left[\begin{array}{c}1-\delta-\mu, 1-\delta+w+\zeta;\\ 1-\delta, 1-\delta-\mu+\zeta;\end{array}\ \frac{z}{x}\right]\end{aligned} \tag{2.14}$$

$$\Big(x > 0;\ \Re(w) \geqq 0;\ \min\{\Re(-\mu-\zeta), \Re(w+\zeta)\} > \Re(\delta) - 1\Big),$$

provided that the conditions with (1.13) *and the formula in* (2.13) *exists.*

On setting $\mu = -w$ and $\mu = 0$ in formula (2.13) and in view of the relationships in (2.3), we construct below the following consequences of Theorem 2.2.

Corollary 2.4. *There holds the undermentioned Riemann-Liouville fractional derivative formula:*

$$\left({}_{RL}D^{w}_{0+}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} zt\right]\right)(x) = x^{\delta-w-1}\,\frac{\Gamma(\delta)}{\Gamma(\delta-w)} \cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} zx\right] * {}_1F_1\left[\begin{array}{c}\delta;\\ \delta-w;\end{array} zx\right] \tag{2.15}$$

$$\Big(x>0;\ \Re(w)\geqq 0;\ \Re(\delta)>0\Big).$$

Corollary 2.5. *There holds the undermentioned left-sided Erdélyi-Kober fractional derivative formula:*

$$\left({}_{EK}D^{w,\zeta}_{0+}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} zt\right]\right)(x) = x^{\delta-1}\,\frac{\Gamma(\delta+w+\zeta)}{\Gamma(\delta+\zeta)} \cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} zx\right] * {}_1F_1\left[\begin{array}{c}\delta+w+\zeta;\\ \delta+\zeta;\end{array} zx\right] \tag{2.16}$$

$$\Big(0<x;\ \Re(w)\geqq 0;\ \Re(\delta)+\min\{0,\Re(\zeta)\}>0\Big).$$

On setting $\mu=-w$ and $\mu=0$ in formula (2.14) and in view of the relationships in (2.8), we deduce the following consequences of Theorem 2.3.

Corollary 2.6. *There holds the undermentioned Weyl fractional derivative formula:*

$$\left({}_{W}D^{w}_{\infty-}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} \frac{z}{t}\right]\right)(x) = x^{\delta-w-1}\,\frac{\Gamma(1-\delta+w)}{\Gamma(1-\delta)} \cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} \frac{z}{x}\right] * {}_1F_1\left[\begin{array}{c}1-\delta+w;\\ 1-\delta;\end{array} \frac{z}{x}\right] \tag{2.17}$$

$$\Big(x>0;\ \Re(w)\geqq 0;\ \Re(w)>\Re(\delta)-1\Big).$$

Corollary 2.7. *There holds the undermentioned right-sided Erdélyi-Kober fractional derivative formula:*

$$\left({}_{EK}D^{w,\zeta}_{\infty-}\, t^{\delta-1}\, {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} \frac{z}{t}\right]\right)(x) = x^{\delta-1}\,\frac{\Gamma(1-\delta+w+\zeta)}{\Gamma(1-\delta+\zeta)} \cdot {}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{c}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{array} \frac{z}{x}\right] * {}_1F_1\left[\begin{array}{c}1-\delta+w+\zeta;\\ 1-\delta+\zeta;\end{array} \frac{z}{x}\right] \tag{2.18}$$

$$\Big(x>0;\ \Re(w)\geqq 0;\ \min\{\Re(-\eta),\Re(\omega+\eta)\}>\Re(\rho)-1\Big),$$

3 Fractional kinetic equations (FKE) with generalized hypergeometric function ${}_{u}F_{v}^{p,q;\lambda;\sigma,\tau}$

Here we compute the solution of generalized FKE involving the hypergeometric function in (1.13) (see also, [24, 31]). The results are derived using Laplace trnasform method and contains Mittag-leffler function (1.11) of two complex paramters and Bessel-Maitland function (see [14]). We need to recall the following definitions for our investigation:

For any measurable function $f : [0, \infty) \to \mathbb{R}$, the Laplace transform (see, [20]) is defined by:

$$\mathcal{L}[f(t); s] := \int_0^\infty e^{-st} f(t)\, dt. \tag{3.1}$$

The RL fractional integral operator (see, e.g., [1, 22]) is given by

$$\left({}_0D_t^{-\nu} f\right)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-p)^{\nu-1} f(p)\, dp \quad \left(\Re(\nu) > 0\right) \tag{3.2}$$

and its Laplace transform was studied in (see, [2, 19]) as

$$\mathcal{L}\{{}_0D_t^{-\nu} f(t); s\} = s^{-\nu}\, F(s) \tag{3.3}$$

where, $F(s)$ is the laplace transform given by (3.1).

The roots of the FKE were explored by Haubold and Mathai [16] during the study of biochemical reaction concerning the rate of change of production, destruction and reaction, which was:

$$\frac{dN}{dt} = pN(t) - dN(t), \tag{3.4}$$

where, $d = d(N)$ denotes the destruction rate, $p = p(N)$ denotes the production rate, $N = N(t)$ is the rate of reaction and N_t denotes the function:

$$N_t(t*) = N_t(t - t^*) \quad (t > 0).$$

Specific case of (3.4), for inhomogeneities or fluctuations in quantity $N(t)$, given by the equation

$$\frac{dN_i}{dt} = -c_i\, N_i(t), \tag{3.5}$$

where, at time $t = 0$, $(N_0 = N_i(0))$ trace the number of density of species i $(c_i > 0)$. Neglecting the index i and integrating (3.5), we have

$$N(t) - N_0 = -c_0\, {}_0D_t^{-1}\, N(t), \tag{3.6}$$

where ${}_0D_t^{-1}$ is the standard fractional integral operator.

A generalization of (3.6) introduced by Haubold and Mathai [16] is:

$$N(t) - N_0 = -c^\nu\, {}_0D_t^{-\nu}\, N(t), \tag{3.7}$$

where ${}_0D_t^{-\nu}$ is given by (3.2).

Further, Saxena and Kalla [31] studied the equation

$$N(t) - N_0 f(t) = -c^\nu . D_t^{-\nu}\, N(t) \quad (\Re(\nu) > 0,\, c > 0), \tag{3.8}$$

Theorem 3.1. *Let $c, t, \nu > 0, c \neq t$; $\lambda, \sigma, \tau \in \mathbb{R}^+$; then the equation*

$$N(t) - N_0\, {}_uF_v^{p,q;\lambda;\sigma,\tau} \left[\begin{matrix} \alpha_1, \ldots, \alpha_u; \\ \beta_1, \ldots, \beta_v; \end{matrix} \; t^\nu \right] := -c^\nu\, {}_0D_t^{-\nu}\, N(t) \tag{3.9}$$

has the solution

$$N(t) = N_0 \sum_{k=0}^{\infty} \Omega_k^{p,q;\lambda;\sigma,\tau}\, (t)^n\, E_{\nu,k+1}(-c^\nu t^\nu). \tag{3.10}$$

Proof. Applying above, the Laplace transform (3.1) and using (3.3) and (1.13), gives

$$N^*(s) = N_0 \left(\int_0^\infty e^{-st} \sum_{k=0}^{\infty} \Omega_k^{p,q;\lambda;\sigma,\tau}\, \frac{t^k}{k!} \right) dt - c^\nu\, s^{-\nu} N^*(s), \tag{3.11}$$

where, $N^*(s)$=$L\{N(t);s\}$.

Under the given assumption, integrating the integral in (3.11) term by term and using $L\{t^{\lambda};s\} = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}$, we have

$$\left(1+\left(\frac{c}{s}\right)^{\nu}\right)N^*(s) = N_0\sum_{k=0}^{\infty}\Omega_k^{p,q;\lambda;\sigma,\tau}\,\frac{1}{k!}\,\frac{\Gamma(1+k)}{s^{1+k}}.$$

Employing the geometric series expansion of $\left(1+\left(\frac{c}{s}\right)^{\nu}\right)^{-1}$ for $c<|s|$, we have

$$N^*(s) = N_0\sum_{k=0}^{\infty}\Omega_k^{p,q;\lambda;\sigma,\tau} \times \sum_{r=0}^{\infty}(-1)^r\,(c)^{\nu r}\,(s)^{-(\nu r+k+1)}. \tag{3.12}$$

Applying inverse Laplace transform and the relation

$$L^{-1}\{s^{-\nu};t\} = \frac{t^{\nu-1}}{\Gamma(\nu)},\quad \Re(\nu)>0,$$

gives

$$\begin{aligned} N(t) &= L^{-1}\{N^*(s);t\} \\ &= N_0\sum_{k=0}^{\infty}\Omega_k^{p,q;\lambda;\sigma,\tau}\,t^k\left\{\sum_{r=0}^{\infty}\frac{(-1)^r(ct)^{\nu r}}{\Gamma(\nu r+1+k)}\right\}. \\ &= N_0\sum_{k=0}^{\infty}\Omega_k^{p,q;\lambda;\sigma,\tau}\,t^k\,E_{\nu,k+1}(-c^{\nu}t^{\nu}), \end{aligned} \tag{3.13}$$

where, the coefficients $\Omega_k^{p,q;\lambda;\sigma,\tau}$ are given by (1.14).
Alternatively, we can also express (3.13) as

$$N(t) = N_0\sum_{k=0}^{\infty}{}_uF_v^{p,q;\lambda;\sigma,\tau}\left[\begin{matrix}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{matrix}\; t\right]W_{\nu,k+1}(-c^{\nu}t^{\nu}), \tag{3.14}$$

where, $W_{\nu,k+1}(t)$ is the Wright function (also known as Bessel Maitland function) [14] given by

$$W_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\quad(\beta\in\mathbb{C};\alpha>-1). \tag{3.15}$$

□

In view of relation (1.7), we straightforwardly deduce the following consequence of Theorem 3.1.

Corollary 3.2. *Let $c,t,\nu>0, c\neq t;\ \lambda,p\in\mathbb{R}^+$; then the equation*

$$N(t) - N_0\,F_{p,\lambda}\left[\begin{matrix}\alpha, & \beta;\\ \gamma; & \end{matrix}\; t^{\nu}\right] := -c^{\nu}\,{}_0D_t^{-\nu}\,N(t) \tag{3.16}$$

has the solution

$$\begin{aligned} N(t) =& N_0\frac{1}{B(\beta,\gamma-\beta)}\sum_{k=0}^{\infty}(\alpha)_n\,\mathbb{B}_{\lambda}^{p}\,(\beta+k,\gamma-\beta)\,t^k\,E_{\nu,k+1}(-c^{\nu}t^{\nu}) \\ =& N_0\frac{1}{B(\beta,\gamma-\beta)}\,F_{p,\lambda}\left[\begin{matrix}\alpha, & \beta;\\ \gamma; & \end{matrix}\; t\right]W_{\nu,k+1}(-c^{\nu}t^{\nu}). \end{aligned} \tag{3.17}$$

We now obtain the following result, which is more interesting and general in nature and can be derived using the similiar approach used in the proof of Theorem (3.1).

Theorem 3.3. *For all $\nu > 0,\ c > 0;\ \lambda, \sigma, \tau \in \mathbb{R}^+$; then the equation the equation*

$$N(t) - N_0\, {}_uF_v^{p,q;\lambda;\sigma,\tau}\left[\begin{array}{l}\alpha_1, \ldots, \alpha_u;\\ \beta_1, \ldots, \beta_v;\end{array} c^\nu t^\nu\right] := -c^\nu\, {}_0D_t^{-\nu}\, N(t) \tag{3.18}$$

has the solution

$$N(t) = N_0 \sum_{k=0}^{\infty} \Omega_k^{p,q;\lambda;\sigma,\tau}\, \frac{(ct)^{\nu k}\, \Gamma(\nu k + 1)}{k!}\, E_{\nu,\nu k+1}(-c^\nu t^\nu), \tag{3.19}$$

where the coefficients $\Omega_k^{p,q;\lambda;\sigma,\tau}$ are given by (1.14).

4 Conclusion

In this paper, we investigate some important properties involving the generalized hypergeometric function ${}_uF_v^{p,q;\lambda;\sigma,\tau}[z]$ such as fractional derivatives and integral formulas. Further, we have derived the solution of generalized fractional kinetic equations involving generalized hypergeometric function and the solution is obtained in terms of the Mittag-Leffler function. The extent to which beta and hypergeometric functions and their generalizations have contributed in mathematical physics and other fields have been a constant source of knowledge and help for researchers.

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