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ON ULAM STABILITY AND HYPERSTABILITY OF A FUNCTIONAL EQUATION IN m-BANACH SPACES

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Abstract In this paper, we study the Ulam stability and hyperstability of a functional equation in m-Banach spaces. Multi-additive and multi-Jensen functions are particular cases of this functional equation. We also improve the main result of [Ciepliński, K. On Ulam Stability of a Functional Equation. *Results Math.* **2020**, 75, Paper No. 151, 11 pp.] and its consequences..

1 Introduction

Assume that X is a linear space over the field $\mathbb{F}$, and Y is a linear space over the field $\mathbb{K}$. Let $a_{11}, a_{12}, \ldots, a_{n1}, a_{n2} \in \mathbb{F}$ and $A_{i_1,\ldots,i_n} \in \mathbb{K}$ for $i_1, \ldots, i_n \in \{1, 2\}$ be given scalars. Ciepliński [6] studied the Ulam stability of the following quite general functional equation

$$f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) = \sum_{i_1,\ldots,i_n\in\{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1}, \ldots, x_{ni_n}). \tag{1.1}$$

in m-Banach spaces. Some special cases of the functional equation have been investigated by some authors (see for example [2, 5, 9]). Functional equation (1.1) with $a_{11} = a_{12} = \cdots = a_{n1} = a_{n2} = 1$ and $A_{i_1,\ldots,i_n} = 1$ for $i_1, \ldots, i_n \in \{1, 2\}$ gives the multi-additive functional equation

$$f(x_{11} + x_{12}, \cdots, x_{n1} + x_{n2}) = \sum_{i_1,\ldots,i_n\in\{1,2\}} f(x_{1i_1}, \ldots, x_{ni_n}).$$

The multi-Jensen functional equation

$$f\left(\frac{x_{11} + x_{12}}{2}, \cdots, \frac{x_{n1} + x_{n2}}{2}\right) = \frac{1}{2^n} \sum_{i_1,\ldots,i_n\in\{1,2\}} f(x_{1i_1}, \ldots, x_{ni_n}),$$

is a special case of (1.1) with $a_{11} = a_{12} = \cdots = a_{n1} = a_{n2} = \frac{1}{2}$ and $A_{i_1,\ldots,i_n} = \frac{1}{2^n}$ for $i_1, \ldots, i_n \in \{1, 2\}$. In this note, we prove the stability and hyperstability of the functional equation (1.1) which improve Ciepliński's result [6, Theorem 7] and its consequences.

First, let us recall some basic definitions and facts concerning m-normed spaces (see for instance [3, 8, 10]).

Definition 1.1. Let $m \in \mathbb{N}$ be such that $m \geq 2$ and $\mathcal{Y}$ be an at least m-dimensional real linear space. A function $\|\cdot, \ldots, \cdot\| : \mathcal{Y}^m \to \mathbb{R}$ is called a *m-norm* on $\mathcal{Y}^m$ if it fulfils the following four conditions:

(i) $\|x_1, \dots, x_m\| = 0$ if and only if $x_1, \dots, x_m$ are linearly dependent;

(ii) $\|x_1, \dots, x_m\|$ is invariant under permutation;

(iii) $\|\alpha x_1, \dots, x_m\| = |\alpha|\|x_1, \dots, x_m\|$;

(iv) $\|x + y, x_2, \dots, x_m\| \leq \|x, x_2, \dots, x_m\| + \|y, x_2, \dots, x_m\|$,

for any $\alpha \in \mathbb{R}$ and $x, y, x_1, \dots, x_m \in \mathcal{Y}$. The pair $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ is called an *m-normed space*.

It follows from (i), (iii) and (iv) that the function $\|\cdot, \dots, \cdot\|$ is non-negative.

We say that a sequence $\{y_n\}_n$ of elements of an m-normed space $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ is *Cauchy sequence* provided

$$\lim_{n,k\to\infty} \|y_n - y_k, x_2, \dots, x_m\| = 0, \qquad x_2, \dots, x_m \in \mathcal{Y}.$$

The sequence $\{y_n\}_n$ is called *convergent* if there is a $y \in \mathcal{Y}$ such that

$$\lim_{n\to\infty} \|y_n - y, x_2, \dots, x_m\| = 0, \qquad x_2, \dots, x_m \in \mathcal{Y}.$$

In this case we say that y is the limit of $\{y_n\}_n$ and it is denoted by $\lim_{n\to\infty} y_n = y$.

By an *m-Banach space* we mean an m-normed space such that each its Cauchy sequence is convergent.

Example 1.2. Let $\mathbb{R}$ be the set of real numbers and $X = \mathbb{R}^3$. For $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$ in X, define

$$\|x, y\| := \sqrt{(b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2 + (a_1b_2 - a_2b_1)^2}.$$

Then $(X, \|., .\|)$ is a 2-Banach space.

Example 1.3. For $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2)$ in $\mathbb{R}^3$, define

$$\|x, y\| := |b_1c_2 - b_2c_1| + |a_1c_2 - a_2c_1| + |a_1b_2 - a_2b_1|.$$

Then $(X, \|., .\|)$ is a 2-Banach space.

Example 1.4. A trivial example of an m-normed space is $\mathbb{R}^m$ equipped with the following m-norm:

$$\|x_1, \cdots, x_m\| = \left| \det \begin{pmatrix} x_{11} & . & . & . & x_{1m} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ x_{m1} & . & . & . & x_{mm} \end{pmatrix} \right|,$$

where $x_i = (x_{i1}, \cdots, x_{im}) \in \mathbb{R}^m$ for all $i = 1, \cdots, m$.

We will use the following lemmas.

Lemma 1.5. *Let $(\mathcal{Y}, \|\cdot, \dots, \cdot\|)$ be an $(m+1)$-normed space and $x \in \mathcal{Y}$. If $\|x, y\| = 0$ for all $y \in \mathcal{Y}^m$, then $x = 0$.*

Proof. Let $y, z \in \mathcal{Y}$ be linearly independent elements. Since $\|x, x, \cdots, x, y\| = 0$ and $\|x, x, \cdots, x, z\| = 0$, there exist scalars λ, μ such that $x = \lambda y$ and $x = \mu z$. Then $\lambda y - \mu z = 0$, and we conclude that $\lambda = \mu = 0$. Hence $x = 0$. □

By Lemma 1.5 and (iv), it is obvious that each convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product hold true.

Finally, it should be noted that more information on m-normed spaces as well as on some problems investigated in them can be found for example in [1, 3, 4, 7, 8, 10].

2 Main Results

For convenience, we set

$$Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) : \quad = f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2})$$

$$-\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} f(x_{1i_1}, \ldots, x_{ni_n}).$$

The following theorem presents a more general result than Theorem 7 of [6]. Indeed, Theorem 7 of [6] states the Ulam stability of functional equation (1.1) in $(m + 1)$-Banach spaces. Now, we prove the Ulam hyperstability of functional equation (1.1) in $(m + 1)$-normed spaces, where $m \in \mathbb{N}$.

Theorem 2.1. *Assume that $\mathcal{Y}$ is an $(m + 1)$-normed space. Let $\varphi : X^{2n} \to [0, +\infty)$ and $f : X^n \to \mathcal{Y}$ be functions such that*

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \varphi(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) \tag{2.1}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in \mathcal{Y}^m$. Then f fulfills equation (1.1).

Proof. Replacing z by kz in (2.1) and dividing the resultant inequality by k^m, we obtain

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \tfrac{1}{k^m}\varphi(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) \tag{2.2}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X,\ z \in \mathcal{Y}^m$ and $k \in \mathbb{N}$. Allowing k tending to infinity, we get

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| = 0$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in \mathcal{Y}^m$. Hence by Lemma 1.5, f satisfies (1.1). □

Corollary 2.2. *[6, Theorem 7] Assume $\varepsilon > 0$ and $\mathcal{Y}$ is an $(m+1)$-normed space. If $f : X^n \to \mathcal{Y}$ is a function satisfying*

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \varepsilon \tag{2.3}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $z \in \mathcal{Y}^m$, then f fulfills equation (1.1) for $x_1, \ldots, x_n \in X$.

Proof. The result follows from Theorem 2.1 by letting $\varphi(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) = \varepsilon$. □

It should be noted that Corollary 2.2 proves the Ulam hyperstability of functional equation (1.1), which improves Theorem 7 of [6].

In the following results, $\mathcal{X}$ is a normed linear space.

Theorem 2.3. *Assume that $\varepsilon, \theta \geqslant 0$ and $\mathcal{Y}$ is an $(m + 1)$-Banach space. Let $g : \mathcal{X} \to \mathcal{Y}^m$ be a surjective function and*

$$\left| \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \right| > 1. \tag{2.4}$$

If $f : \mathcal{X}^n \to \mathcal{Y}$ is a function satisfying

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), g(z)\| \leqslant \varepsilon + \theta\|z\| \tag{2.5}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, z \in \mathcal{X}$, then there is a unique function $F : \mathcal{X}^n \to \mathcal{Y}$ fulfilling equation (1.1) and

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), g(z)\| \leqslant \frac{\varepsilon + \theta\|z\|}{|\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n}| - 1} \tag{2.6}$$

for $x_1, \ldots, x_n, z \in \mathcal{X}$.

Proof. Put

$$A := \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}, \qquad a_i := a_{i1} + a_{i2}, \quad i \in \{1,\dots,n\}.$$

Let us first note that (2.5) with $x_{i2} = x_{i1}$ for $i \in \{1,\dots,n\}$ gives

$$\|f(a_1x_{11},\dots,a_nx_{n1}) - Af(x_{11},\dots,x_{n1}), g(z)\| \leqslant \varepsilon + \theta\|z\|, \qquad (x_{11},\dots,x_{n1},z) \in \mathcal{X}^{n+1},$$

and consequently

$$\begin{aligned}&\left\|\tfrac{f(a_1^{k+1}x_{11},\dots,a_n^{k+1}x_{n1})}{A^{k+1}} - \tfrac{f(a_1^kx_{11},\dots,a_n^kx_{n1})}{A^k}, g(z)\right\| \leqslant \tfrac{\varepsilon+\theta\|z\|}{|A|^{k+1}},\\ &(x_{11},\dots,x_{n1},z) \in \mathcal{X}^{n+1},\ k \in \mathbb{N}_0.\end{aligned} \tag{2.7}$$

Fix $l, p \in \mathbb{N}_0$ such that $l < p$. Then

$$\begin{aligned}&\left\|\tfrac{f(a_1^{p}x_{11},\dots,a_n^{p}x_{n1})}{A^{p}} - \tfrac{f(a_1^lx_{11},\dots,a_n^lx_{n1})}{A^l}, g(z)\right\| \leqslant \textstyle\sum_{j=l}^{p-1} \tfrac{\varepsilon+\theta\|z\|}{|A|^{j+1}},\\ &(x_{11},\dots,x_{n1},z) \in \mathcal{X}^{n+1},\end{aligned} \tag{2.8}$$

and thus for each $(x_{11},\dots,x_{n1}) \in \mathcal{X}^n$, the sequence $\left\{\tfrac{f(a_1^kx_{11},\dots,a_n^kx_{n1})}{A^k}\right\}_{k\in\mathbb{N}_0}$ is a Cauchy sequence. Using the fact that $\mathcal{Y}$ is a Banach space we conclude that this sequence is convergent, which allows us to define

$$F(x_{11},\dots,x_{n1}) := \lim_{k\to\infty} \frac{f(a_1^kx_{11},\dots,a_n^kx_{n1})}{A^k}, \qquad (x_{11},\dots,x_{n1}) \in \mathcal{X}^n. \tag{2.9}$$

Putting now $l = 0$ and letting $p \to \infty$ in (2.8) we see that

$$\|f(x_{11},\dots,x_{n1}) - F(x_{11},\dots,x_{n1}), g(z)\| \leqslant \frac{\varepsilon+\theta\|z\|}{|A|-1}, \qquad (x_{11},\dots,x_{n1},z) \in \mathcal{X}^{n+1},$$

i.e., condition (2.6) is satisfied.

Let us next observe that from (2.5) we get

$$\begin{aligned}&\left\|\tfrac{f(a_1^k(a_{11}x_{11}+a_{12}x_{12}),\dots,a_n^k(a_{n1}x_{n1}+a_{n2}x_{n2}))}{A^k} -\right.\\ &\left.\textstyle\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \tfrac{f(a_1^kx_{1i_1},\dots,a_n^kx_{ni_n})}{A^k}, g(z)\right\| \leqslant \tfrac{\varepsilon+\theta\|z\|}{|A|^k}\end{aligned}$$

for $x_{11}, x_{12},\dots,x_{n1},x_{n2},z \in \mathcal{X}$ and $k \in \mathbb{N}_0$. Letting now $k \to \infty$ and applying definition (2.9) we deduce that

$$\begin{aligned}&\Big\|F(a_{11}x_{11}+a_{12}x_{12},\dots,a_{n1}x_{n1}+a_{n2}x_{n2}) -\\ &\textstyle\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F(x_{1i_1},\dots,x_{ni_n}), g(z)\Big\| = 0\end{aligned}$$

for $x_{11}, x_{12},\dots,x_{n1},x_{n2},z \in \mathcal{X}$. Since g is surjective, we infer that the function $F : \mathcal{X}^n \to Y$ is a solution of functional equation (1.1).

The uniqueness of F is easily follows from (2.6). □

Theorem 2.4. *Assume that* $m \in \mathbb{N}$, $\varepsilon \geqslant 0$ *and* $\mathcal{Y}$ *is an* $(m+1)$*-normed space. Let* $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ *and* $\{r_i\}_{i=1}^n$ *be nonnegative real numbers with* $\max_{1\leqslant i\leqslant n} r_i < 1$, *and* $f : \mathcal{Y}^n \to \mathcal{Y}$ *be a function such that*

$$\|Df(x_{11},x_{12},\cdots,x_{n1},x_{n2}), z\| \leqslant \varepsilon + \textstyle\sum_{i=1}^n [\alpha_i\|x_{i1},z\|^{r_i} + \beta_i\|x_{i2},z\|^{r_i}] \tag{2.10}$$

for $x_{11},x_{12},\dots,x_{n1},x_{n2} \in \mathcal{Y}$ *and* $z \in \mathcal{Y}^m$. *Then* f *satisfies* (1.1).

Proof. Replacing z by kz in (2.10) and dividing the resultant inequality by k^m, we obtain

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \frac{\varepsilon}{k^m} + \sum_{i=1}^{n} \left(\frac{k^{r_i}}{k}\right)^m [\alpha_i \|x_{i1}, z\|^{r_i} + \beta_i \|x_{i2}, z\|^{r_i}].$$

Letting now $k \to \infty$, we get

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| = 0, \quad x_{11}, \ldots, x_{n2} \in \mathcal{Y},\ z \in \mathcal{Y}^m.$$

Hence by Lemma 1.5, f satisfies (1.1). □

Theorem 2.5. *Assume that $m \in \mathbb{N}$ and $\mathcal{Y}$ is an $(m+1)$-normed space. Let $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ and $\{r_i\}_{i=1}^n$ be nonnegative real numbers with $\min_{1\leqslant i \leqslant n} r_i > 1$, and $f : \mathcal{Y}^n \to \mathcal{Y}$ be a function such that*

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \textstyle\sum_{i=1}^n [\alpha_i \|x_{i1}, z\|^{r_i} + \beta_i \|x_{i2}, z\|^{r_i}] \tag{2.11}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. Then f satisfies (1.1).

Proof. By replacing z by $\frac{z}{k}$ in (2.11) and applying a similar argument as in the proof of Theorem 2.4, the result is achieved. □

Theorem 2.6. *Assume that $\varepsilon \geqslant 0$ and $\mathcal{Y}$ is an $(m+1)$-Banach space. Let $\{\alpha_i\}_{i=1}^n and \{\beta_i\}_{i=1}^n$ be nonnegative real numbers and $f : \mathcal{Y}^n \to \mathcal{Y}$ be a function such that*

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \varepsilon + \textstyle\sum_{i=1}^n \Big[\alpha_i \|x_{i1}, z\| + \beta_i \|x_{i2}, z\|\Big], \tag{2.12}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. Suppose

$$\left| \sum_{i_1, \ldots, i_n \in \{1,2\}} A_{i_1, \ldots, i_n} \right| > \max_{1 \leqslant j \leqslant n} \{|a_{j1} + a_{j2}|\}.$$

Then there is a unique function $F : \mathcal{Y}^n \to \mathcal{Y}$ fulfilling equation (1.1) and

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n), z\| \leqslant \sum_{i=1}^n \frac{\alpha_i + \beta_i}{|A| - |a_{i1} + a_{i2}|} \|x_{i1}, z\| \tag{2.13}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$, where $A := \sum_{i_1, \ldots, i_n \in \{1,2\}} A_{i_1, \ldots, i_n}$.

Proof. Replacing z by kz in (2.12) and dividing the resultant inequality by k^m, we obtain

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \frac{\varepsilon}{k^m} + \sum_{i=1}^n \Big[\alpha_i \|x_{i1}, z\| + \beta_i \|x_{i2}, z\|\Big],$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. Letting $k \to \infty$, we get

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \textstyle\sum_{i=1}^n \Big[\alpha_i \|x_{i1}, z\| + \beta_i \|x_{i2}, z\|\Big], \tag{2.14}$$

$x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. Put $a_i := a_{i1} + a_{i2}$ for $i \in \{1, \ldots, n\}$. Letting $x_{i2} = x_{i1}$ for $i \in \{1, \ldots, n\}$ in (2.14), we get

$$\|f(a_1 x_{11}, \ldots, a_n x_{n1}) - Af(x_{11}, \ldots, x_{n1}), z\| \leqslant \sum_{i=1}^n (\alpha_i + \beta_i) \|x_{i1}, z\|,$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. Then

$$\left\| \frac{f(a_1^p x_{11}, \ldots, a_n^p x_{n1})}{A^p} - \frac{f(a_1^l x_{11}, \ldots, a_n^l x_{n1})}{A^l}, z \right\|$$
$$\leqslant \sum_{i=l}^n (\alpha_i + \beta_i) \|x_{i1}, z\| \sum_{j=l}^{p-1} \frac{|a_i|^j}{|A|^{j+1}},$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y}$ and $z \in \mathcal{Y}^m$. The rest of the proof is similar to the proof of Theorem 2.3. □

In 2-normed spaces, we have the following theorem.

Theorem 2.7. *Assume that* $\varepsilon \geqslant 0$ *and* $\mathcal{Y}$ *is a 2-normed space. Let* $\{\alpha_i\}_{i=1}^n, \{\beta_i\}_{i=1}^n$ *be nonnegative real numbers and* $\{r_i\}_{i=1}^n$ *be real numbers with* $\max_{1\leqslant i\leqslant n} r_i < 1$. *Suppose* $f : \mathcal{Y}^n \to \mathcal{Y}$ *satisfies*

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| \leqslant \varepsilon + \sum_{i=1}^n \Big[\alpha_i \|x_{i1}, z\|^{r_i} + \beta_i \|x_{i2}, z\|^{r_i}\Big] \tag{2.15}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y} \setminus \{0\}$ *and* $z \in \mathcal{Y}$ *with* $\|x_{ij}, z\| \neq 0$ *for* $1 \leqslant i \leqslant n$ *and* $j = 1, 2$. *Then* f *satisfies* (1.1) *for* $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y} \setminus \{0\}$.

Proof. Let $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathcal{Y} \setminus \{0\}$. By a similar argument as in the proof of Theorem 2.4, we get

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| = 0$$

for all $z \in \mathcal{Y}^m$ with $\|x_{ij}, z\| \neq 0$ for $1 \leqslant i \leqslant n$, $j = 1, 2$. Now, we can choose linearly independent elements $y, z \in \mathcal{Y}$ such that

$$\|x_{ij}, y\| \neq 0 \quad \text{and} \quad \|x_{ij}, z\| \neq 0, \quad 1 \leqslant i \leqslant n, \;\; j = 1, 2.$$

Thus

$$\|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), y\| = 0 \quad \text{and} \quad \|Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}), z\| = 0.$$

Then there exist scalars λ, μ such that

$$Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) = \lambda y \quad \text{and} \quad Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) = \mu z.$$

So, $\lambda y - \mu z = 0$, and we conclude that $\lambda = \mu = 0$. Therefore

$$Df(x_{11}, x_{12}, \cdots, x_{n1}, x_{n2}) = 0.$$

□

3 A special case of (1.1)

In this section, we deal with the following functional equation

$$f(ax + by, cz + dw) = A_1 f(x, z) + A_2 f(x, w) + A_3 f(y, z) + A_4 f(y, w), \tag{3.1}$$

which is a special case of (1.1). For convenience, we put

$$\Delta f(x, y, z, w) := f(ax + by, cz + dw) - A_1 f(x, z) - A_2 f(x, w) - A_3 f(y, z) - A_4 f(y, w).$$

In what follows $\mathcal{Y}$ is a 2-normed space.

Theorem 3.1. *Let* $\delta, \theta \geqslant 0$, $p, q, r, s \in \mathbb{R}^+$ *with* $p + q > 1, r + s > 1$ *and* $f : \mathcal{Y} \to \mathcal{Y}$ *satisfy*

$$\|\Delta f(x, y, z, w), t\| \leqslant \delta \|x, t\|^p \|y, t\|^q + \theta \|z, t\|^r \|w, t\|^s, \tag{3.2}$$

for all $x, y, z, w, t \in \mathcal{Y}$. *Then* f *satisfies* (3.1).

Proof. Replacing t by $\frac{t}{n}$ in (3.2), we infer that

$$\|\Delta f(x, y, z, w), t\| \leqslant \frac{n\delta}{n^{p+q}} \|x, t\|^p \|y, t\|^q + \frac{n\theta}{n^{r+s}} \|z, t\|^r \|w, t\|^s, \tag{3.3}$$

for all $x, y, z, w, t \in \mathcal{Y}$ and all positive integer n. Letting $n \to \infty$ in (3.3), we get

$$\|\Delta f(x, y, z, w), t\| = 0, \tag{3.4}$$

for all $x, y, z, w, t \in \mathcal{Y}$. Then f fulfills (3.1). □

Theorem 3.2. *Let $\varepsilon, \delta, \theta \geqslant 0$, $p, q, r, s \in \mathbb{R}$ with $p+q<1, r+s<1$ and $f : \mathcal{Y} \to \mathcal{Y}$ satisfy*

$$\|\Delta f(x,y,z,w),t\| \leqslant \varepsilon + \delta\|x,t\|^p\|y,t\|^q + \theta\|z,t\|^r\|w,t\|^s, \tag{3.5}$$

for all $x,y,z,w,t \in \mathcal{Y}$ with $\|x,t\|.\|y,t\|.\|z,t\|.\|w,t\| \neq 0$. Then f satisfies (3.1) for all $x,y,z,w \in \mathcal{Y}\setminus\{0\}$.

Proof. Replacing t by nt in (3.5), we infer that

$$\|\Delta f(x,y,z,w),t\| \leqslant \frac{\varepsilon}{n} + \frac{n^{p+q}}{n}\delta\|x,t\|^p\|y,t\|^q + \frac{n^{r+s}}{n}\theta\|z,t\|^r\|w,t\|^s, \tag{3.6}$$

for all $x,y,z,w,t \in \mathcal{Y}$ with $\|x,t\|.\|y,t\|.\|z,t\|.\|w,t\| \neq 0$ and all positive integer n. Letting $n \to \infty$ in (3.6), we get

$$\|\Delta f(x,y,z,w),t\| = 0, \tag{3.7}$$

for all $x,y,z,w,t \in \mathcal{Y}$ with $\|x,t\|.\|y,t\|.\|z,t\|.\|w,t\| \neq 0$. By a similar argument as in the proof of Theorem 3.1, it is concluded that f fulfills (3.1) for all $x,y,z,w \in \mathcal{Y}\setminus\{0\}$. □

The proof of the following theorem is almost similar to the proof of Theorem 3.2 and we leave its proof.

Theorem 3.3. *Let $\varepsilon, \delta \geqslant 0$, $p,q,r,s \in \mathbb{R}$ and $f : \mathcal{Y} \to \mathcal{Y}$ satisfy*

$$\|\Delta f(x,y,z,w),t\| \leqslant \begin{cases} \varepsilon + \delta\|x,t\|^p\|y,t\|^q\|z,t\|^r\|w,t\|^s, & p+q+r+s<1; \\ \delta\|x,t\|^p\|y,t\|^q\|z,t\|^r\|w,t\|^s, & p+q+r+s>1. \end{cases} \tag{3.8}$$

for all $x,y,z,w,t \in \mathcal{Y}$ with $\|x,t\|.\|y,t\|.\|z,t\|.\|w,t\| \neq 0$. Then f satisfies (3.1) for all $x,y,z,w \in \mathcal{Y}\setminus\{0\}$.

For the case $p+q=r+s=1$, we have the following stability theorem. It should be noted that in what follows $\mathcal{Y}$ is a 2-Banach space.

Theorem 3.4. *Let $\varepsilon, \delta, \theta \geqslant 0$, $p,q,r,s \in \mathbb{R}^+$ with $p+q=r+s=1$ and $f : \mathcal{Y} \to \mathcal{Y}$ satisfy (3.5) for all $x,y,z,w,t \in \mathcal{Y}$. Suppose that*

$$|A_1+A_2+A_3+A_4| > \max\{|a+b|, |c+d|\}.$$

Then there is a unique function $F : \mathcal{Y}^2 \to \mathcal{Y}$ fulfilling equation (3.1) and

$$\|F(x,z)-f(x,z),t\| \leqslant \frac{\delta\|x,t\|}{|A_1+A_2+A_3+A_4|-|a+b|} + \frac{\theta\|z,t\|}{|A_1+A_2+A_3+A_4|-|c+d|},$$

for all $x,z,t \in \mathcal{Y}$.

Proof. Replacing t by nt in (3.5), we infer that

$$\|\Delta f(x,y,z,w),t\| \leqslant \frac{\varepsilon}{n} + \delta\|x,t\|^p\|y,t\|^q + \theta\|z,t\|^r\|w,t\|^s,$$

for all $x,y,z,w,t \in \mathcal{Y}$. Then

$$\|\Delta f(x,y,z,w),t\| \leqslant \delta\|x,t\|^p\|y,t\|^q + \theta\|z,t\|^r\|w,t\|^s, \tag{3.9}$$

for all $x,y,z,w,t \in \mathcal{Y}$. Put $\alpha := a+b$, $\beta := c+d$ and $A := A_1+A_2+A_3+A_4$. Letting $y=x$ and $w=z$ in (3.9), we get

$$\|f(\alpha x, \beta z) - Af(x,z),t\| \leqslant \delta\|x,t\| + \theta\|z,t\|, \quad x,y,z,w,t \in \mathcal{Y}.$$

Then

$$\left\|\frac{f(\alpha^{n+1}x,\beta^{n+1}z)}{A^{n+1}} - \frac{f(\alpha^m x,\beta^m z)}{A^m},t\right\| \leqslant \delta\sum_{k=m}^{n}\left|\frac{\alpha^k}{A^{k+1}}\right|\|x,t\| + \theta\sum_{k=m}^{n}\left|\frac{\beta^k}{A^{k+1}}\right|\|z,t\|, \tag{3.10}$$

for all $x,y,z,w,t \in \mathcal{Y}$ and $n \geqslant m \geqslant 0$. Hence the sequence $\{\frac{f(\alpha^n x,\beta^n z)}{A^n}\}_n$ is Cauchy, and we can define

$$F : \mathcal{Y}^2 \to \mathcal{Y}, \quad F(x,z) := \lim_n \frac{f(\alpha^n x,\beta^n z)}{A^n}.$$

The rest of the proof is similar to the proof of Theorem 2.3. □

References

[1] S. AL-Ali, M. E. Hryrou, Y. Elkettani and M. A. Salman, *On a generalized p-radical functional equation related to additive mappings in 2-Banach spaces*, Palest. J. Math., **11**, 594–607, (2022).

[2] A. Bahyrycz, K. Ciepliński, And J. Olko, *On an equation characterizing multi-Cauchy-Jensen mappings and its Hyers-Ulam stability*, Acta Math. Sci. Ser. B (Engl. Ed.), **35**, 1349–1358, (2015).

[3] J. Brzdęk and K. Ciepliński, *A fixed point theorem in n-Banach spaces and Ulam stability*, J. Math. Anal. Appl., **470**, 632–646, (2019).

[4] X. Y. Chen and M. M. Song, *Characterizations on isometries in linear n-normed spaces*, Nonlinear Anal., **72**, 1895–1901, (2010).

[5] K. Ciepliński, *Generalized stability of multi-additive mappings*, Appl. Math. Lett., **23**, 1291–1294, (2010).

[6] K. Ciepliński, *On Ulam Stability of a Functional Equation*, Results Math., **75**, Paper No. 151, 11 pp., (2020).

[7] Y. Ma, *The Aleksandrov-Benz-Rassias problem on linear n-normed spaces*, Monatsh. Math., **180**, 305–316, (2016).

[8] A. Misiak, *n-inner product spaces*, Math. Nachr., **140**, 299–319, (1989).

[9] W. Prager and J. Schwaiger, *Stability of the multi-Jensen equation*, Bull. Korean Math. Soc., **45**, 133–142, (2008).

[10] T. Z. Xu, *Approximate multi-Jensen, multi-Euler-Lagrange additive and quadratic mappings in n-Banach spaces*, Abstr. Appl. Anal., Art. ID 648709, 12 pp., (2013).

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