Palestine Journal of Mathematics

Vol 13(3)(2024) , 753–761

$\mathrm{L_p}\left(\mathcal{D}\right)$ Approximation by a Szasz Operator Type

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Communicated by Ayman Badawi

MSC 2010 Classifications: 41A10;41A25;41A36; 40A30.

Keywords and phrases: Exact order of approximation, simultaneous approximation, Voronovskajas theorem, overconvergence and complex Szsz-Stancu type operator.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

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Abstract *Previously we introduced a modification of szasz operator, which is a complex valued operator and study the approximation of functions in* $\mathrm{L_p}\left(\mathcal{D}\right)$.
Here we introduce another type of complex valued operator which is a modification for our previous operator. Then we prove some of its convergence properties and estimates its rate of approximation. Then we prove a type of voronvskaja theorem for functions in $\mathrm{L_p}\left(\mathcal{D}\right)$ *quasi normed spaces for* $0 < p < 1$.

1 Introduction

$|f(u)| \leq Ce^{Ku}, u \in [0, \infty)$ *for a function* $f \in \mathrm{L_p}[0, \infty)$ *satisfying an exponential growth condition, with some constants* $C > 0$ *and* $K > 0$, *Phillips [24] initially, denoted the following operators*

$$\mathcal{N}_{\delta}\left(f;u\right)=\delta\sum_{\mu=1}^{\infty}\mathcal{W}_{\delta,\mu}\left(u\right)\int_{0}^{\infty}\mathcal{W}_{\delta,\mu-1}\left(y\right)f(y)\,dy+\mathcal{W}_{\delta,0}\left(u\right)f(0)\,,\delta>K,$$

where

$$\mathcal{W}_{\delta,\mu}\left(u\right)=e^{\delta u}\frac{(\delta u)^{\mu}}{\mu!},\ \mu\in\mathbb{N}_{0,}\ u\in\left[0,\infty\right)\,,$$

which are frequently referred to as Phillips operators in the literature. Phillips operators can also be referred to as authentic Szsz-Durrmeyer operators because these operators preserve constants and linear functions. After that, these operators and their various generalizations have been thoroughly studied by numerous researchers in the case of real variables; see, for example, [1, 7, 16, 22, 23] et al.
One of the interesting research areas in recent years has been the issue of complex operator approximation. Some complex Bernstein approximation properties Wright presented polynomials in several domains in the complex plane [27], Tonne [25], Kantorovich [18], Bernstein [2], and Lorentz [21]. Gergen et al. [11] proved the initial conclusion regarding the convergence of complex Szasz operators, which is a generalization of the Bernstein polynomials. Then, in complex domains, Szsz operators were studied by Jakimovski et al. [17], Wood [26], and Deeba [6]. However, all of the aforementioned outcomes were attained.
without a measurement or calculation. In addition to the findings from [21] and [11], Gal also obtained quantitative estimates for the convergence and Voronovskaja's theorem in [8]. In his book [8], Gal also compiled the findings regarding the overconvergence characteristics of popular complex operators. Later, a large number of authors (see, for example,[3, 9, 10, 13, 14, 15]) established approximation properties with quantitative estimates for various operators in complex domain.The complex modified genuine Szasz-Durrmeyer operators that were introduced and studied in [4] served as the inspiration for the current work. For more related study see for examples [12, 19, 20].
A modification of the complex genuine Szasz-Durrmeyer operators is defined below:

$$\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(\mathrm{f};z\right)=\frac{h_\delta}{g_\delta}\sum_{\mu=1}^{\infty}\mathcal{W}_{h_\delta,g_\delta,\mu}\left(z\right)\int_{o}^{\infty}\mathcal{W}_{h_\delta,g_\delta,\mu-1}\left(y\right)\mathrm{f}\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)dy$$
$$+\mathcal{W}_{h_\delta,g_\delta,0}\left(z\right)\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right), \qquad (1.1)$$

where

$$\mathcal{W}_{h_\delta,g_\delta,\mu}(z) = e^{-\frac{h_\delta}{g_\delta}z}\frac{(h_\delta z)^\mu}{\mu! g_\delta^\mu}.$$

Where the strictly positive number sequences $\{h_\delta\}$and $\{g_\delta\}$are provided in such a way that $\lim_{\delta\longrightarrow\infty}\frac{g_\delta}{h_\delta}$ is convergent sequence and $\frac{g_\delta}{h_\delta}\le M$ and also γ,ϑ are two real parameters that have been given and satisfy the condition $0\le\gamma\le\vartheta$. Define $\mathrm{L_p}(\mathcal{D}) = \left\{ f:\mathcal{D}\longrightarrow\mathbb{C}:\|f\|_{\mathrm{L_p}(\mathcal{D})} = \left(\int_{\mathcal{D}}|f|^{\mathrm{p}}\right)^{\frac{1}{\mathrm{p}}}<\infty\right\}$ where $0<p<1$ and $\mathcal{D}=\{z\in\mathcal{D}_r : |z|\le r,\ 1<r<\infty\}$.
The objective of this paper is to show approximation results for functions in $\mathrm{L_p}(\mathcal{D})$ using the operators given by (1.1).
For this operator, we obtain the rate of convergence, the Voronovskaja-type result with a quantitative estimate, and the exact degree of approximation. In the outcomes, using $G(\mathcal{D}_\mathcal{R})$ with $\mathcal{D}_\mathcal{R}=\{z\in\mathbb{C}:|z|<\mathcal{R},\ 1<\mathcal{R}<\infty\}$.we consider the class of functions satisfying f : $[\mathcal{R},\infty)\cup\overline{\mathcal{D}_\mathcal{R}}\longrightarrow\mathbb{C}$ is integrable on $[0,\infty)$ i.e. $f(z)=\sum_{\theta=0}^\infty a_\theta z^\theta$ for any $z\in\mathcal{D}_\mathcal{R}$.

2 AUXILIARY LEMMAS

The next auxiliary result will be very helpful to prove our results.

Lemma 2.1. *For* $0\le\gamma\le\vartheta$. *Then, we get*

$$\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}\left(z^\theta;z\right) = \sum_{\beta=0}^{\theta}\binom{\theta}{\beta}\frac{h_\delta^\beta(\gamma g_\delta)^{\theta-\beta}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\beta;z\right),$$

where $\beta\in\mathbb{N}\cup\{0\}$ *and* $\mathcal{N}_{h_\delta,g_\delta}$ *denotes* $\mathcal{N}^{(0,0)}_{h_\delta,g_\delta}$.

Proof. From the equation (1.3), we get

$$\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}\left(z^\theta;z\right) = \frac{h_\delta}{g_\delta}\sum_{\mu=1}^{\infty}\mathcal{W}_{h_\delta,g_\delta,\mu}(z)\int_o^\infty \mathcal{W}_{h_\delta,g_\delta,\mu-1}(y)\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)^\theta dy$$
$$+\mathcal{W}_{h_\delta,g_\delta,0}(z)\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)^\theta.$$

Using the binomial theorem, we have

$$\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}\left(z^\theta;z\right) = \frac{h_\delta}{g_\delta}\sum_{\mu=1}^{\infty}\mathcal{W}_{h_\delta,g_\delta,\mu}(z)\int_o^\infty \mathcal{W}_{h_\delta,g_\delta,\mu-1}(y)\sum_{\beta=0}^{\theta}\binom{\theta}{\beta}\frac{h_\delta^\beta(\gamma g_\delta)^{\theta-\beta}}{(h_\delta+\vartheta g_\delta)^\theta}y^\beta dy$$
$$+\mathcal{W}_{h_\delta,g_\delta,0}(z)\sum_{\beta=0}^{\theta}\binom{\theta}{\beta}\frac{h_\delta^\beta(\gamma g_\delta)^{\theta-\beta}}{(h_\delta+\vartheta g_\delta)^\theta}0^\beta$$
$$=\sum_{\beta=0}^{\theta}\binom{\theta}{\beta}\frac{h_\delta^\beta(\gamma g_\delta)^{\theta-\beta}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\beta;z\right).$$

□

MAIN RESULTS:

Theorem 2.2. *Let* γ,ϑ *satisfy* $0\le\gamma\le\vartheta$, *and assume that* $f\in G(\mathcal{D}_\mathcal{R})\subset \mathrm{L_p}(\mathcal{D}_\mathcal{R})$ *and* $\exists\ K,\ C>0\ni\ |f(u)|\le Ce^{Ku},\ \forall\ u\in[\mathcal{R},+\infty)$. *Also let* $\delta_0\in\mathbb{N}$ *be such that* $\frac{h_\delta}{g_\delta}+\vartheta-K>0\ \forall\ \delta>\delta_0$. *Denoting* $f(z)=\sum_{\theta=0}^\infty a_\theta z^\theta,\ z\in L_p(\mathcal{D})$, *we have* $\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}(f;z)=\sum_{\theta=0}^\infty a_\theta\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}\left(z^\theta;z\right)$, $\forall\ z\in\mathcal{D}_\mathcal{R}$ *and* $\delta>\delta_0$ *with* $\delta,\delta_0\in\mathbb{N}$.

Proof. Let $0<r<\mathcal{R}$ and $\forall\ i\in\mathbb{N}$, we define

$$\mathrm{f}_i(z)=\sum_{\mu=0}^{i}a_\mu z^\mu \text{ if } |z|\le r \text{ and } \mathrm{f}_i(u)=\mathrm{f}(u) \text{ if } u\in(r,\ +\infty).$$

Since $|\mathrm{f}_i(z)|\le\sum_{\mu=0}^\infty|a_\mu|\ r^\mu:=C_r,\ \forall\ |z|\le r$ and $i\in\mathbb{N}$. from the hypothesis on f, $\exists$ a constant $C_{r,\mathcal{R}}>0$ s.t $|\mathrm{f}_i(u)|\le C_{r,\mathcal{R}}e^{Ku},\ \forall i\in\mathbb{N}$ and $u\in[0,\ +\infty)$.
This implies that using the ratio criterion, $\forall$ fixed $z\in\mathbb{C},\ i,\ \delta\in\mathbb{N}$ and $\forall\ \delta>\delta_0$ with $\frac{h_\delta}{g_\delta}+\vartheta-K>0$, and using definition $L_p(\mathcal{D})$, we get

$$\left\|\mathcal{N}^{(\gamma,\vartheta)}_{h_\delta,g_\delta}(\mathrm{f}_i;z)\right\|_{\mathrm{L_p}(\mathcal{D})} = \left(\int_{\mathcal{D}}\left|\frac{h_\delta}{g_\delta}\sum_{\mu=1}^{\infty}e^{-\frac{h_\delta}{g_\delta}z}\frac{(h_\delta z)^\mu}{\mu! g_\delta^\mu}\int_o^\infty e^{-\frac{h_\delta}{g_\delta}y}\frac{(h_\delta y)^{\mu-1}}{(\mu-1)! g_\delta^{\mu-1}}\mathrm{f}_i\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)dy\right.\right.$$
$$\left.\left.+e^{-\frac{h_\delta}{g_\delta}z}\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right|^p dz\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathcal{D}}\left(\frac{h_\delta}{g_\delta}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\frac{h_\delta^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\int_{\mathcal{D}}^{\infty}e^{-\frac{h_\delta}{g_\delta}y}y^{\mu-1}C_{r,\mathcal{R}}e^{K\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)}dy\right.\right.$$
$$\left.\left.+\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\left|c_0+c_1\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}+\cdots+c_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)^i\right|\right)^p dz\right)^{\frac{1}{p}}$$
$$\leq \left(\int_{\mathcal{D}}\left(\frac{h_\delta}{g_\delta}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\frac{h_\delta^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}C_{r,\mathcal{R}}e^{K\frac{yg_\delta}{h_\delta+\vartheta g_\delta}}\int_{\mathcal{D}}^{\infty}e^{-y\left(\frac{h_\delta}{g_\delta}-\frac{Kh_\delta}{h_\delta+\vartheta g_\delta}\right)}y^{\mu-1}dy\right.\right.$$
$$\left.\left.+\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\{|c_0|+|c_1|+\cdots+|c_i|\}\right)^p dz\right)^{\frac{1}{p}}$$
$$\leq \left(\int_{\mathcal{D}}\left(\sum_{\mu=1}^{\infty}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\frac{h_\delta^\mu}{g_\delta^\mu}C_{r,\mathcal{R}}e^{K\frac{yg_\delta}{h_\delta+\vartheta g_\delta}}\frac{1}{\left(\frac{h_\delta}{g_\delta}-\frac{Kh_\delta}{h_\delta+\vartheta g_\delta}\right)^\mu}\right.\right.$$
$$\left.\left.+\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\{|c_0|+|c_1|+\cdots+|c_i|\}\right)^p dz\right)^{\frac{1}{p}}$$
$$=\left(\int_{\mathcal{D}}\left(C_{r,\mathcal{R}}e^{K\frac{yg_\delta}{h_\delta+\vartheta g_\delta}}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{\left[\left(\frac{h_\delta}{g_\delta}\right)^2|z|\,/\left(\frac{h_\delta}{g_\delta}-\frac{Kh_\delta}{h_\delta+\vartheta g_\delta}\right)\right]^\mu}{\mu!}\right.\right.$$
$$\left.\left.+\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\{|c_0|+|c_1|+\cdots+|c_i|\}\right)^p dz\right)^{\frac{1}{p}}$$
$$\leq\left[C_{r,\mathcal{R}}+|c_0|+|c_1|+\cdots+|c_i|\right]\left|e^{-\frac{h_\delta}{g_\delta}z}\right|e^{K\frac{yg_\delta}{h_\delta+\vartheta g_\delta}}e^{\left(\frac{h_\delta}{g_\delta}\right)^2|z|/\left(\frac{h_\delta}{g_\delta}-\frac{Kh_\delta}{h_\delta+\vartheta g_\delta}\right)}(\pi r^2)^{\frac{1}{p}}$$
$$<\infty.$$

Hence, $\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f}_i;z)$ is well- defined. Denoting

$$\mathrm{f}_{i,\theta}(u)=\frac{\mathrm{f}(u)}{i+1}\ \text{if}\ u\in(r,\ \infty)\ \text{and}\ \mathrm{f}_{i,\theta}(z)=a_\theta z^\theta\ if\ |z|\leq r,$$

It is evident that each $\mathrm{f}_{i,\theta}$ has an exponential growth rate relative to $[0,\infty)$ and $\mathrm{f}_i(z)=\sum_{\theta=0}^{i}\mathrm{f}_{i,\theta}(z)$.
Since $\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}$ is linear, so we have

$$\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f}_i;z)=\sum_{\theta=0}^{i}a_\theta\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(z^\theta;z\right),\ \forall\ |z|\leq r,$$

demonstrating that $\lim_{i\longrightarrow\infty}\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f}_i;z)=\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)$ for any fixed $|z|\leq r$ and $\delta\in\mathbb{N}$ is sufficient. But this is immediate from $\lim_{i\longrightarrow\infty}\|\mathrm{f}_i-\mathrm{f}\|_{\mathrm{L_p}(\mathcal{D})}=0$, and from the definition $\mathrm{L_p}(\mathcal{D})$, we obtain

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f}_i;z)-\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$=\left\|\frac{h_\delta}{g_\delta}e^{-\frac{h_\delta}{g_\delta}z}\sum_{\mu=1}^{\infty}\frac{(h_\delta z)^\mu}{\mu!g_\delta^\mu}\int_{\mathcal{D}}^{\infty}e^{-\frac{h_\delta}{g_\delta}y}\frac{(h_\delta y)^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\left[\mathrm{f}_i\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]dy\right.$$
$$\left.+e^{-\frac{h_\delta}{g_\delta}z}\left[\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\leq 2^{\frac{1}{p}-1}\left\|\frac{h_\delta}{g_\delta}e^{-\frac{h_\delta}{g_\delta}z}\sum_{\mu=1}^{\infty}\frac{(h_\delta z)^\mu}{\mu!g_\delta^\mu}\int_{\mathcal{D}}^{\infty}e^{-\frac{h_\delta}{g_\delta}y}\frac{(h_\delta y)^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\left[\mathrm{f}_i\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]dy\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$+2^{\frac{1}{p}-1}\left\|e^{-\frac{h_\delta}{g_\delta}z}\left[\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\leq 2^{\frac{1}{p}-1}\left(\int_{\mathcal{D}}\left|\frac{h_\delta}{g_\delta}e^{-\frac{h_\delta}{g_\delta}z}\sum_{\mu=1}^{\infty}\frac{(h_\delta z)^\mu}{\mu!g_\delta^\mu}\int_{\mathcal{D}}^{\infty}e^{-\frac{h_\delta}{g_\delta}y}\frac{(h_\delta y)^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\left[\mathrm{f}_i\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]dy\right|^p dz\right)^{\frac{1}{p}}$$
$$+2^{\frac{1}{p}-1}\left(\int_{\mathcal{D}}\left|e^{-\frac{h_\delta}{g_\delta}z}\left[\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right]\right|^p dz\right)^{\frac{1}{p}}$$
$$\leq 2^{\frac{1}{p}-1}\left(\int_{\mathcal{D}}\left(\frac{h_\delta}{g_\delta}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\int_{\mathcal{D}}^{\infty}e^{-\frac{h_\delta}{g_\delta}y}\frac{(h_\delta y)^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\left|\mathrm{f}_i\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{h_\delta y+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right|dy\right)^p dz\right)^{\frac{1}{p}}$$
$$+2^{\frac{1}{p}-1}\left(\int_{\mathcal{D}}\left(\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\left|\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right|\right)^p dz\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p}-1}\frac{h_\delta}{g_\delta}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\frac{h_\delta^{\mu-1}}{(\mu-1)!g_\delta^{\mu-1}}\left(\int_{\mathcal{D}}\left|\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right|^p dz\right)^{\frac{1}{p}}\int_0^\infty e^{-\frac{h_\delta}{g_\delta}y}y^{\mu-1}dy$$

$$+2^{\frac{1}{p}-1}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\left(\int_{\mathcal{D}}\left|\mathrm{f}_i\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)-\mathrm{f}\left(\frac{\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)\right|^p dz\right)^{\frac{1}{p}}$$

$$\leq 2^{\frac{1}{p}-1}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\sum_{\mu=1}^{\infty}\frac{(h_\delta\,|z|)^\mu}{\mu!g_\delta^\mu}\frac{h_\delta^\mu}{g_\delta^\mu}\|\mathrm{f}_i-\mathrm{f}\|_{\mathrm{L_p}(\mathcal{D})}\frac{g_\delta^\mu}{h_\delta^\mu}+2^{\frac{1}{p}-1}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|\|\mathrm{f}_i-\mathrm{f}\|_{\mathrm{L_p}(\mathcal{D})}$$

$$=2^{\frac{1}{p}-1}\left|e^{-\frac{h_\delta}{g_\delta}z}\right|e^{\frac{h_\delta}{g_\delta}|z|}\|\mathrm{f}_i-\mathrm{f}\|_{\mathrm{L_p}(\mathcal{D})}.$$

□

Theorem 2.3. *If we indicate* $\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right):=\mathcal{N}_{h_\delta,g_\delta}^{(0,0)}\left(z^\theta;z\right)$*, then for any* $|z|\leq r$ *with* $r\geq 1$*,* $\theta\in\mathbb{N}\cup\{0\}$ *and* $\delta\in\mathbb{N}$*, we have*

$$\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\right\|_{L_p(\mathcal{D})}\leq\prod_{\mu=1}^{\theta}\left(r+2\left(\theta-1\right)M\right)\left(\pi r^2\right)^{\frac{1}{p}}.$$

Proof. By the definition $\mathrm{L_p}(\mathcal{D})$, we obtain

$$\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\right\|_{\mathrm{L_p}(\mathcal{D})}=\left(\int_{\mathcal{D}}\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\right|^{\mathrm{p}}dz\right)^{\frac{1}{\mathrm{p}}}. \tag{2}$$

Now, we will be using the recurrence formula shown below, which was discovered during the proof of Theorem 1(i) in [5]

$$\mathcal{N}_{h_\delta,g_\delta}\left(z^{\theta+1};z\right)=\frac{g_\delta}{h_\delta}z\Big(\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\Big)'+\left(z+\frac{g_\delta}{h_\delta}\,\theta\right)\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)$$

for any $z\in\mathbb{C}$, $\delta\in\mathbb{N}$ and $\theta\in\mathbb{N}\cup\{0\}$. Since $\mathcal{N}_{h_\delta,g_\delta}\left(z^0;z\right)=1$, for $\theta=0$ we have

$$\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\right|\leq r$$

for any $|z|\leq r$. Then, for $\theta=1$ we obtain

$$\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^2;z\right)\right|=\left|\frac{g_\delta}{h_\delta}z\Big(\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\Big)'+\left(z+\frac{g_\delta}{h_\delta}\right)\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\right|,$$

since $\frac{g_\delta}{h_\delta}\leq M$, we get

$$\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^2;z\right)\right|\leq Mr\left|\Big(\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\Big)'\right|+\left(r+M\right)\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\right|.$$

The following from the Bernstein's inequality for $\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)$ polynomial of the degree θ we get:

$$\left|\Big(\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\Big)'\right|\leq\frac{\theta}{r}r=\theta.$$

Using the final inequality, we discover

$$\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^2;z\right)\right|\leq Mr+\left(r+M\right)\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^1;z\right)\right|\leq r\left(r+2M\right).$$

We can easily obtain by writing for $\theta=2,3,\ \ldots,$ step by step

$$\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\right|\leq r\left(r+2M\right)\ldots\left(r+2\left(\theta-1\right)M\right)=\prod_{\mu=1}^{\theta}\left(r+2\left(\theta-1\right)M\right). \tag{3}$$

By substituting inequality (3) in equation (2), we get

$$\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)\right\|_{\mathrm{L_p}(\mathcal{D})}\leq\left(\int_{\mathcal{D}}\left|\prod_{\mu=1}^{\theta}\left(r+2\left(\theta-1\right)M\right)\right|^{\mathrm{p}}dz\right)^{\frac{1}{\mathrm{p}}}$$

$$=\prod_{\mu=1}^{\theta}\left(r+2\left(\theta-1\right)M\right)\left(\int_{\mathcal{D}}dz\right)^{\frac{1}{p}}$$

$$=\prod_{\mu=1}^{\theta}\left(r+2\left(\theta-1\right)M\right)\left(\pi r^2\right)^{\frac{1}{p}}.$$

.for any $|z|\leq r$, $\delta\in\mathbb{N}$ and $\theta\in\mathbb{N}\cup\{0\}$.■ □

Now, In the following theorem, we get upper quantitative estimates for the operator (1.3)

Theorem 2.4. *Let* $0 \le \gamma \le \vartheta$, $f \in G(\mathcal{D_R}) \subset \mathrm{L_p}(\mathcal{D_R})$ *and assume that there exist* $E,\ C,\ K > 0$ *and* $X \in \left(\frac{1}{\mathcal{R}}, 1\right)$, *with the property* $|a_\theta| \le E\frac{X^\theta}{\theta!}$, *for any* $\theta = 0,\ 1,\ \ldots,$ *(which suggests* $|f(z)| \le Ee^{X|z|}$ *for any* $z \in \mathcal{D_R}$*) and* $|f(u)| \le Ce^{Ku}$, *for any* $u \in [\mathcal{R}, \infty)$ *. Also let* $\delta_0 \in \mathbb{N}$ *be such that* $\frac{h_\delta}{g_\delta} + \vartheta - K > 0$ *for any* $\delta > \delta_0$.

(a) Let $1 \le r < \frac{1}{X}$ *be arbitrarily fixed. For any* $\delta > \delta_0$ *and* $|z| \le r$ *with* $\delta,\ \delta_0 \in \mathbb{N}$, *we have*

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(f;z) - f(z)\right\|_{\mathrm{L_p}(\mathcal{D})} \le \frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\mathcal{G}_{r,\ X},$$

where

$$\mathcal{G}_{r,\ X} = EM\left(2\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=1}^{\infty}(\theta+1)(rX)^\theta < \infty.$$

(b) For the simultaneous approximation, we have: if $1 \le r < r_1 < \frac{1}{X}$ *are arbitrarily fixed, then for any* $|z| \le r$, $\delta > \delta_0$ *and* $\delta,\ q \in \mathbb{N}$,

$$\left\|\left(\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(f;z)\right)^q - f^q(z)\right\|_{\mathrm{L_p}(\mathcal{D})} \le \frac{q!r_1}{(r_1-r)^{q+1}}\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\mathcal{G}_{r_1,\ X},$$

where $\mathcal{G}_{r_1,\ X}$ *is specified in (a).*

Proof of (a) Using Lemma 1, we can prove

$$\begin{aligned}&\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z) - z^\theta\\ &= \sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\left(\mathcal{N}_{h_\delta,g_\delta}(z^\mu;z) - z^\mu\right) + \sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}z^\mu\\ &\quad + \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\left(\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right) - z^\theta\right) - \left(1 - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)z^\theta,\end{aligned}$$

Then

$$\begin{aligned}&\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z) - z^\theta\right\|_{\mathrm{L_p}(\mathcal{D})}\\ &\le 2^{\frac{1}{p}-1}\left[\sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\left\|\mathcal{N}_{h_\delta,g_\delta}(z^\mu) - z^\mu\right\|_{\mathrm{L_p}(\mathcal{D})} + \sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}r^\mu\right.\\ &\quad\left. + \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta\right) - z^\theta\right\|_{\mathrm{L_p}(\mathcal{D})} + \left(1 - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)r^\theta\right].\end{aligned}$$

By the following inequality

$$\begin{aligned}\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta\right) - z^\theta\right\|_{\mathrm{L_p}(\mathcal{D})} &= \left(\int_{\mathcal{D}}\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta\right) - z^\theta\right|^p dz\right)^{\frac{1}{p}}\\ \le \left(\int_{\mathcal{D}}\left(M(\theta+1)!r^{\theta-1}\right)^p dz\right)^{\frac{1}{p}} &= \left(M(\theta+1)!r^{\theta-1}\right)\left(\pi r^2\right)^{\frac{1}{p}}.\end{aligned} \tag{4}$$

Using inequality (2.1) in the last inequality, we get

$$\begin{aligned}&\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z) - z^\theta\right\|_{\mathrm{L_p}(\mathcal{D})}\\ &\le 2^{\frac{1}{p}-1}\left[\left(M(\theta+1)!r^{\theta-1}\right)\left(\pi r^2\right)^{\frac{1}{p}}\sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta} + r^\theta\sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\right.\\ &\quad\left. + \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\left(M(\theta+1)!r^{\theta-1}\right)\left(\pi r^2\right)^{\frac{1}{p}} + \left(1 - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)r^\theta\right]\\ &\le 2^{\frac{1}{p}-1}\left[\left(\frac{h_\delta+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)^\theta\left(M(\theta+1)!r^{\theta-1}\right)\left(\pi r^2\right)^{\frac{1}{p}} + \left[\left(\frac{h_\delta+\gamma g_\delta}{h_\delta+\vartheta g_\delta}\right)^\theta - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right]r^\theta\right.\\ &\quad\left. + \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\left(M(\theta+1)!r^{\theta-1}\right)\left(\pi r^2\right)^{\frac{1}{p}} + \left(1 - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)r^\theta\right]\\ &\le 2^{\frac{1}{p}-1}\left[2M(\theta+1)!r^{\theta-1}\left(\pi r^2\right)^{\frac{1}{p}} + 2\left(1 - \frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)r^\theta\right]\\ &\le 2^{\frac{1}{p}-1}\left[2M(\theta+1)!r^{\theta-1}\left(\pi r^2\right)^{\frac{1}{p}} + 2\frac{\theta\vartheta g_\delta}{h_\delta+\vartheta g_\delta}r^\theta\right]\\ &\le 2^{\frac{1}{p}-1}\left[2M(\theta+1)!r^\theta\left(\pi r^2\right)^{\frac{1}{p}}\left(1 + \frac{\vartheta h_\delta}{h_\delta+\vartheta g_\delta}\right)\right]\\ &= \left(2\pi r^2\right)^{\frac{1}{p}}M(\theta+1)!r^\theta\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}.\end{aligned} \tag{5}$$

Thus, from Theorem 1, (2.2), and the hypothesis on a_θ, we obtain

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)-\mathrm{f}(z)\right\|_{\mathrm{L_p}(\mathcal{D})}\le\sum_{\theta=1}^{\infty}|a_\theta|\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)-z^\theta\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\le\sum_{\theta=1}^{\infty}E\frac{X^\theta}{\theta!}\left(2\pi r^2\right)^{\frac{1}{p}}M(\theta+1)!r^\theta\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}$$
$$=\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}EM\left(2\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=1}^{\infty}(\theta+1)(rX)^\theta$$
$$=\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\mathcal{G}_{r,\,X},$$

where

$$\mathcal{G}_{r,\,X}=EM\left(2\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=1}^{\infty}(\theta+1)(rX)^\theta<\infty,$$
$$\forall\,1\le r<\frac{1}{X}.$$

Note that $\mathrm{f}(z)=\sum_{\theta=1}^{\infty}z^{\theta+1}$ and its derivative $\mathrm{f}'(z)=\sum_{\theta=1}^{\infty}(\theta+1)z^\theta$ are absolutely and uniformly convergent in $|z|\le r,\forall\,1\le r<\frac{1}{X}$.

Proof of (b) In order to approximate simultaneously, denote by L the circle with radius $r_1>r$ and center 0; since we have $|\tau-z|\ge r_1-r\;\forall\,\tau\in L$ and $|z|\le r$; it follows from Cauchy's formulas that $\forall\,|z|\le r$ and $\delta>\delta_0$with an that $\frac{h_\delta}{g_\delta}+\vartheta-K>0$, we have

$$\left\|\left(\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)\right)^q-\mathrm{f}^q(z)\right\|_{\mathrm{L_p}(\mathcal{D})}=\left\|\frac{q!}{2\pi}\int_L\frac{\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};\tau)-\mathrm{f}(\tau)}{(\tau-z)^{q+1}}d\tau\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\le\frac{q!}{2\pi}\int_L\frac{\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};\tau)-\mathrm{f}(\tau)\right\|_{\mathrm{L_p}(\mathcal{D})}}{(r_1-r)^{q+1}}d\tau\le\frac{q!}{2\pi}\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\mathcal{G}_{r_1,\,X}\frac{2\pi r_1}{(r_1-r)^{q+1}}$$
$$=\frac{q!r_1}{(r_1-r)^{q+1}}\frac{h_\delta(1+\vartheta)+\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\mathcal{G}_{r_1,\,X}.$$

We present the following for the Voronovskaja-type formula with a quantitative estimate.
Theorem 4:

Theorem 2.5. *Assume that the hypotheses on the function $f\in\mathrm{L_p}(\mathcal{D})$ and on the constants $\delta_0,\ \mathcal{R},\ E,\ C,\ K,\ X$ in the statement of the Theorem 3 hold. Also, let $0\le\gamma\le\vartheta$ and $1\le r<\frac{1}{X}$. Then $\forall\ |z|\le r$ and $\delta>\delta_0$, we have*

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(f;z)-f(z)-\frac{(\gamma-\vartheta z)g_\delta}{h_\delta+\vartheta g_\delta}f'(z)-\frac{g_\delta z}{h_\delta}f''(z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\le M^2\mathcal{T}_{r,\,X}+\frac{g_\delta^2}{(h_\delta+\vartheta g_\delta)^2}\mathcal{G}_{r,\,1}^{(\gamma,\,\vartheta)}+\frac{g_\delta^2}{h_\delta(h_\delta+\vartheta g_\delta)}\mathcal{G}_{r,\,2}^{(\gamma,\,\vartheta)},$$

where

$$\mathcal{T}_{r,\,X}=\frac{4E}{r^2}2^{\frac{1}{p}-1}\left(\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}(\theta+2)(\theta+1)(rX)^\theta<\infty,$$
$$\mathcal{G}_{r,\,1}^{(\gamma,\,\vartheta)}=\left(\gamma^2+\gamma\vartheta+2\vartheta^2\right)2^{\frac{1}{p}-1}E\left(\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}\theta(\theta-1)(rX)^\theta$$
$$\mathcal{G}_{r,\,2}^{(\gamma,\,\vartheta)}=(\gamma+\vartheta)2^{\frac{1}{p}-1}EX\left(\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}\theta(\theta+1)(rX)^{\theta-1}.$$

Proof. $\forall\,z\in\mathcal{D}_{\mathcal{R}}$, we can write

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)-\mathrm{f}(z)-\frac{(\gamma-\vartheta z)g_\delta}{h_\delta+\vartheta g_\delta}\mathrm{f}'(z)-\frac{g_\delta z}{h_\delta}\mathrm{f}''(z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$=\left\|\mathcal{N}_{h_\delta,g_\delta}(\mathrm{f};z)-\mathrm{f}(z)-Mz\mathrm{f}''(z)+\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)-\mathcal{N}_{h_\delta,g_\delta}(\mathrm{f};z)-\frac{(\gamma-\vartheta z)g_\delta}{h_\delta+\vartheta g_\delta}\mathrm{f}'(z)\right\|_{\mathrm{L_p}(\mathcal{D})}.$$

Taking $\mathrm{f}(z)=\sum_{\theta=0}^{\infty}a_\theta z^\theta$

$$\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}(\mathrm{f};z)-\mathrm{f}(z)-\frac{(\gamma-\vartheta z)g_\delta}{h_\delta+\vartheta g_\delta}\mathrm{f}'(z)-\frac{g_\delta z}{h_\delta}\mathrm{f}''(z)\right\|_{\mathrm{L_p}(\mathcal{D})}$$
$$\le 2^{\frac{1}{p}-1}\left[\sum_{\theta=0}^{\infty}|a_\theta|\left\|\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-z^\theta-Mz\theta(\theta-1)z^{\theta-2}\right\|_{\mathrm{L_p}(\mathcal{D})}\right.$$

$$+\sum_{\theta=0}^{\infty}|a_\theta|\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(z^\theta;z\right)-\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\right\|_{L_p(\mathcal{D})}\Bigg].$$

Using [5], $\forall\ |z|\le r$, $\delta>\delta_0$ and by the definition $L_p(\mathcal{D})$, we get

$$\left\|\mathcal{N}_{h_\delta,g_\delta}(\mathrm{f};z)-\mathrm{f}(z)-Mz\mathrm{f}''(z)\right\|_{L_p(\mathcal{D})}\le M^2\mathcal{T}_{r,\,X},$$

where $\mathcal{T}_{r,\,X}=\frac{4E}{r^2}2^{\frac{1}{p}-1}\left(\pi r^2\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}(\theta+2)(\theta+1)(rX)^\theta<\infty$.
Using Lemma 1 and some rearrangements, we can easily obtain an estimate for the second sum.

$$\begin{aligned}
&\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(z^\theta;z\right)-\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\right\|_{L_p(\mathcal{D})}\\
&=\left(\int_{\mathcal{D}}\left|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(z^\theta;z\right)-\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\right|^p dz\right)^{\frac{1}{p}}\\
&=\Bigg(\int_{\mathcal{D}}\Bigg|\sum_{\mu=0}^{\theta-1}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)\\
&\quad-\left(1-\frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\right)\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\Bigg|^p dz\Bigg)^{\frac{1}{p}}\\
&=\Bigg(\int_{\mathcal{D}}\Bigg|\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)+\frac{\theta h_\delta^{\theta-1}\gamma g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^{\theta-1};z\right)\\
&\quad-\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{\theta h_\delta^{\theta-1}\gamma g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\Bigg|^p dz\Bigg)^{\frac{1}{p}}\\
&=\Bigg(\int_{\mathcal{D}}\Bigg|\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)+\frac{\theta h_\delta^{\theta-1}\gamma g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\left[\mathcal{N}_{h_\delta,g_\delta}\left(z^{\theta-1};z\right)-z^{\theta-1}\right]\\
&\quad-\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{\theta h_\delta^{\theta-1}\gamma g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\left[\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-z^\theta\right]\\
&\quad-\frac{\theta\gamma g_\delta}{h_\delta+\vartheta g_\delta}z^{\theta-1}\left(1-\frac{h_\delta^{\theta-1}}{(h_\delta+\vartheta g_\delta)^{\theta-1}}\right)+\frac{\theta\vartheta g_\delta}{h_\delta+\vartheta g_\delta}z^\theta\left(1-\frac{h_\delta^{\theta-1}}{(h_\delta+\vartheta g_\delta)^{\theta-1}}\right)\Bigg|^p dz\Bigg)^{\frac{1}{p}}.
\end{aligned}\tag{2.3}$$

From Theorem 2, using inequality (4) and the following inequalities

$$1-\frac{h_\delta^\theta}{(h_\delta+\vartheta g_\delta)^\theta}\le\sum_{\mu=0}^{\theta}\left(1-\frac{h_\delta}{h_\delta+\vartheta g_\delta}\right)=\frac{\theta\vartheta g_\delta}{h_\delta+\vartheta g_\delta}$$

and

$$\begin{aligned}
\left|\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)\right|&\le\sum_{\mu=0}^{\theta-2}\binom{\theta}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\left|\mathcal{N}_{h_\delta,g_\delta}\left(z^\mu;z\right)\right|\\
&\le\sum_{\mu=0}^{\theta-2}\frac{\theta(\theta-1)}{(\theta-\mu)(\theta-\mu-1)}\binom{\theta-2}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu}}{(h_\delta+\vartheta g_\delta)^\theta}\mu!r^\mu\\
&\le\frac{\theta(\theta-1)}{2}\frac{(\gamma g_\delta)^2}{(h_\delta+\vartheta g_\delta)^2}(\theta-2)!r^{\theta-2}\sum_{\mu=0}^{\theta-2}\binom{\theta-2}{\mu}\frac{h_\delta^\mu(\gamma g_\delta)^{\theta-\mu-2}}{(h_\delta+\vartheta g_\delta)^{\theta-2}}\\
&\le\frac{\theta(\theta-1)}{2}\frac{(\gamma g_\delta)^2}{(h_\delta+\vartheta g_\delta)^2}(\theta-2)!r^{\theta-2},
\end{aligned}$$

In (6), we get

$$\begin{aligned}
&\left\|\mathcal{N}_{h_\delta,g_\delta}^{(\gamma,\vartheta)}\left(z^\theta;z\right)-\mathcal{N}_{h_\delta,g_\delta}\left(z^\theta;z\right)-\frac{(\gamma-\vartheta z)\,g_\delta}{h_\delta+\vartheta g_\delta}\theta z^{\theta-1}\right\|_{L_p(\mathcal{D})}\\
&\le\Bigg(\int_{\mathcal{D}}\Bigg(\frac{\theta(\theta-1)}{2}\frac{(\gamma g_\delta)^2}{(h_\delta+\vartheta g_\delta)^2}(\theta-2)!r^{\theta-2}+\frac{\theta\gamma h_\delta^{\theta-1}g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\frac{g_\delta}{h_\delta}\theta!\ r^{\theta-2}\\
&\quad+\frac{\theta(\theta-1)}{2}\frac{(\vartheta g_\delta)^2}{(h_\delta+\vartheta g_\delta)^2}\theta!r^\theta+\frac{\theta h_\delta^{\theta-1}\vartheta g_\delta}{(h_\delta+\vartheta g_\delta)^\theta}\frac{g_\delta}{h_\delta}(\theta+1)!r^{\theta-1}\\
&\quad+\frac{\theta\gamma g_\delta}{h_\delta+\vartheta g_\delta}r^{\theta-1}\frac{(\theta-1)\vartheta g_\delta}{h_\delta+\vartheta g_\delta}+\frac{\theta\vartheta g_\delta}{h_\delta+\vartheta g_\delta}r^\theta\frac{(\theta-1)\vartheta g_\delta}{h_\delta+\vartheta g_\delta}\Bigg)^p dz\Bigg)^{\frac{1}{p}}\\
&\le\Bigg(\int_{\mathcal{D}}\Bigg(\frac{\theta(\theta-1)(\gamma g_\delta)^2}{2(h_\delta+\vartheta g_\delta)^2}(\theta-2)!r^{\theta-2}+\frac{\theta\gamma g_\delta^2}{h_\delta(h_\delta+\vartheta g_\delta)}\theta!\ r^{\theta-2}
\end{aligned}$$

$$+\frac{\theta\left(\theta-1\right)\left(\vartheta g_{\delta}\right)^{2}}{2(h_{\delta}+\vartheta g_{\delta})^{2}}\theta!r^{\theta}+\frac{\theta\vartheta g_{\delta}{}^{2}}{h_{\delta}\left(h_{\delta}+\vartheta g_{\delta}\right)}\left(\theta+1\right)!r^{\theta-1}$$

$$+\frac{\theta\left(\theta-1\right)\gamma\vartheta g_{\delta}^{2}}{\left(h_{\delta}+\vartheta g_{\delta}\right)^{2}}r^{\theta-1}+\frac{\theta\left(\theta-1\right)\left(\vartheta g_{\delta}\right)^{2}}{\left(h_{\delta}+\vartheta g_{\delta}\right)^{2}}r^{\theta}\Bigg)^{p}dz\Bigg)^{\frac{1}{p}}$$

$$\leq\left(\int_{\mathcal{D}}\left(\frac{g_{\delta}^{2}}{\left(h_{\delta}+\vartheta g_{\delta}\right)^{2}}r^{\theta}\theta\left(\theta-1\right)\theta!\left[\gamma^{2}+\gamma\gamma+2\vartheta^{2}\right]\right.\right.$$

$$\left.\left.+\frac{g_{\delta}^{2}}{h_{\delta}\left(h_{\delta}+\vartheta g_{\delta}\right)}r^{\theta}\theta\left(\theta+1\right)!\left[\gamma+\vartheta\right]\right)^{p}dz\right)^{\frac{1}{p}}$$

$$=\left(\frac{g_{\delta}^{2}}{\left(h_{\delta}+\vartheta g_{\delta}\right)^{2}}r^{\theta}\theta\left(\theta-1\right)\theta!\left[\gamma^{2}+\gamma\vartheta+2\vartheta^{2}\right]+\frac{g_{\delta}^{2}}{h_{\delta}\left(h_{\delta}+\vartheta g_{\delta}\right)}r^{\theta-1}\theta\left(\theta+1\right)!\left[\gamma+\vartheta\right]\right)\left(\pi r^{2}\right)^{\frac{1}{p}}.$$

Therefore, we have

$$\sum_{\theta=0}^{\infty}\left|a_{\theta}\right|\left\|\mathcal{N}_{h_{\delta},g_{\delta}}^{(\gamma,\vartheta)}\left(z^{\theta};z\right)-\mathcal{N}_{h_{\delta},g_{\delta}}\left(z^{\theta};z\right)-\frac{\left(\gamma-\vartheta z\right)g_{\delta}}{h_{\delta}+\vartheta g_{\delta}}\theta z^{\theta-1}\right\|_{\mathrm{L_p}(\mathcal{D})}$$

$$\leq\frac{g_{\delta}^{2}}{\left(h_{\delta}+\vartheta g_{\delta}\right)^{2}}\left(\gamma^{2}+\gamma\vartheta+2\vartheta^{2}\right)2^{\frac{1}{p}-1}E\left(\pi r^{2}\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}\theta\left(\theta-1\right)\left(rX\right)^{\theta}$$

$$+\frac{g_{\delta}^{2}}{h_{\delta}\left(h_{\delta}+\vartheta g_{\delta}\right)}\left(\gamma+\vartheta\right)2^{\frac{1}{p}-1}EX\left(\pi r^{2}\right)^{\frac{1}{p}}\sum_{\theta=0}^{\infty}\theta\left(\theta+1\right)\left(rX\right)^{\theta-1},$$

thus, this the series are convergent for $1\leq r<\frac{1}{X}$.■ □

References

[1] P.N. Agrawal and V. Gupta, *On convergence of derivatives of Phillips operators*, Demonstratio Math, **27 (2)**, 501- 510, (1994).

[2] S. N. Bernstein, *Sur la convergence de certaines suites des polynomes*, J. Math. Pures Appl., **15 (9)**, 345-358, (1935).

[3] N. Cetin, *Approximation by complex modiÖed Sz·sz-Mirakjan-Stancu operators in compact disks*, Filomat, **29 (5)**, 1007- 1019. (2015).

[4] N. Cetin, *On complex modified genuine szasz- durrmeyed- stancu operators*, Commun. Fac. Sci. Univ. Ank. Ser. Al Math. Stat., **68 (1)**, 283- 298, (2019).

[5] N. Cetin and N. Ispir, *Approximation properties of complex modiÖed genuine Sz·sz- · Durrmeyer operators*, Comput. Methods Funct. Theory, **14**, 623-638, (2014).

[6] E. Y. Deeba, *On the convergence of generalized Szasz operator on complex plane*, Tamkang J. Math., **13(1)**, 79- 86, (1982).

[7] Z. Finta and V. Gupta, *Direct and inverse estimates for Phillips type operators*, J. Math. Anal. Appl., **303 (2)**, 627-642, (2005).

[8] S. G. Gal, *Approximation by Complex Bernstein and Convolution Type Operators*, Series on Concrete and Applicable Mathematics, 8. World ScientiÖc Publishing Co. Pte. Ltd., Hackensack, NJ, xii+337 pp. ISBN: 978- 981- 4282-42-0, (2009).

[9] S. G. Gal, *Approximation by complex genuine Durrmeyer type polynomials in compact disks*, Appl. Math. Comput., **217**, 1913-1920, (2010).

[10] S. G. Gal and V. Gupta, *Approximation by Complex Durrmeyer Type Operators in Compact Disks*, Mathematics Without Boundaries, in:P. Pardalos, T.M. Rassias (Eds.), Surveys in Interdisciplinary Research, Springer, (2014).

[11] J. J. Gergen, F. G. Dressel and W.H. Purcell, *Convergence of extended Bernstein polynomials in the complex domain*, PaciÖc J. Math., **13 (4)**, 1171- 1180 1171- 1180 ,(1963).

[12] S. Y. Golbol and U. Deger, *Approximation in the weighted generalized Lipchitz class by deferred cesaro-matrix product sub methods*, Palestine Journal of Mathematics, **10(2)**, 740—750, (2021).

[13] V. Gupta and G. Tachev , *Approximation by linear combinations of complex Phillips operators in compact disks*, Results. Math., DOI 10.1007/s0025-014– 0377-3.

[14] V. Gupta and D. K. Verma, *Approximation by complex Favard-Sz·sz-Mirakjan-Stancu operators in compact disks*, Mathematical Sciences, , **6:25**, (2012).

[15] V. Gupta, *Complex Baskakov-Sz·sz operators in compact semi-disks*, Lobachevskii J. Math., **35(2)**, 65–73, (2014).

[16] M. Heilmann and G. Tachev, , *Commutativity, direct and strong converse results for Phillips operators*, East J. Approx., **17(3)**, 299– 317, (2011).

[17] A. Jakimovski and D. Leviatan, *Generalized Sz·sz operators for the approximation in the inÖnite interval*, Mathematica (Cluj), **34**,97– 103, (1969).

[18] L. V. Kantorovich, *Sur la convergence de la suite de polynomes de S. Bernstein en dehors de líinterval fundamental*. Bull. Acad. Sci. URSS, 1103– 1115, (1931).

[19] T. Khan and S. A. Khan, *Approximation by Stancu type Lupa¸s operators*, Palestine Journal of Mathematics, **11(3)**, 700– 707,(2022).

[20] A. Kumar and Dakshita, *Approximation of functions by Stancu variant of genuine Gupta-Srivastava operators*, Palestine Journal of Mathematics, **10(2)**, 863– 873,(2021).

[21] G. G. Lorentz, *Bernstein Polynomials*, 2nd ed., Chelsea Publ., New York (1986).

[22] C.P. May, *On Phillips operator*, J. Approx. Theory, **20**, 315– 332, (1977).

[23] S.M. Mazhar and V. Totik, *Approximation by modioed Sz·sz operators*, Acta Sci. Math., **49**, 257–269,(1985).

[24] R. S. Phillips, *An inversion formula for semi-groups of linear operators*, Ann. Math., **59**, 352– 356,(1954).

[25] P.C. Tonne, *On the convergence of Bernstein polynomials for some unbounded analytic functions*, Proc. Am. Math. Soc., **22**,1-6.

[26] B. Wood, *Generalized Sz·sz operators for the approximation in the complex domain*, SIAM J. Appl. Math., **17(4)**, 790– 801,(1969).

[27] E. M. Wright, *The Bernstein approximation polynomials in the complex plane*, J. Lond. Math. Soc., **5**, 265– 269, (1930).

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Received: 2023-06-02
Accepted: 2024-01-31