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CENTRAL ELEMENTS AND ARMENDARIZ BIMODULES

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Abstract Let M be an (R,S)-bimodule and $\alpha : R \to S$ a ring homomorphism. In this paper, we defined central elements for a bimodule M relative to α and studied its various properties. We proved that the set $C_\alpha(M)$ of all central elements of bimodule M over a commutative ring R forms a submodule of M. We also introduced the notion of α-Central Armendariz bimodule as a generalization of Central Armendariz rings and investigated their properties. Various examples which illustrate and delimit the results of this paper are also provided.

1 Introduction

Throughout this article, all rings are associative with identity. Recall that if R and S are rings and M is a left module over R and right module over S, then M is called an (R,S)-bimodule if for all $r \in R\,, s \in S$ and $m \in M$ we have $(rm)s = r(ms)$. We write ${}_RM_S$ to mean M is a (R,S)-bimodule. Every left R-module M is a $(R,\mathbb{Z})$-bimodule. Any ring S is an (S,R)-bimodule for any subring R of S with $1_R = 1_S$. More generally, if $f : R \to S$ is any ring homomorphism with $f(1_R) = 1_S$, then S can be considered as a right R-module with the action $s.r = s.f(r)$ and with respect to this action S becomes an (S,R)-bimodule. Clearly, every ring R is an (R,R)-bimodule. Suppose that R is a commutative ring then a left (respectively right) R-module M can always be given the structure of a right (respectively left) R-module by defining $mr = rm(rm = mr)\ \forall\ m \in M,\ r \in R$ and this makes M into an (R,R)-bimodule. Thus every module(right or left) over a commutative ring R has at least one natural (R,R)-bimodule structure. An element a of a ring R is called central element if $ar = ra\,\forall\, r \in R$. Let $C(R)$ denote the set of all central elements of R. For a ring R, the set $C(R)$ forms a subring. An element r of a ring R is called nilpotent element if there exists some $n \in \mathbb{N}$ such that $r^n = 0$. Let $N(R)$ denote the set of all nilpotent elements of ring R. Suppose that M is a left R-module and S is a subring contained in the center of R, then M can be given a right S-module structure. Thus every left R-module M is a $(R, C(R))$-bimodule. We write $I \trianglelefteq R$ to mean I is an ideal of R, $N \leq M$ to mean N is a submodule of M and 'id' to represent the identity homomorphism of a ring R.

2 Central elements

In this section we define central elements for a bimodule M.

Definition 2.1. Let M be a (R,S)-bimodule and $\alpha : R \longrightarrow S$ a ring homomorphism. We say that $m \in M$ is a central element relative to α if $rm = m\alpha(r)\,\forall\, r \in R$. Thus we define center of M relative to α as

$$C_\alpha(M) = \{m \in M | rm = m\alpha(r) \ \forall\, r \in R\}$$

Remark 2.2. By the Definition 2.1, the following can be easily obtained:

(1) $r \in C(R)$ if and only if $r \in C_{id}({}_RR_R)$, where id represent identity homomorphism.

(2) For every abelian group M as a $(\mathbb{Z},\mathbb{Z})$-bimodule, we have $C_{id}(M) = M$.

(3) 0 is an central element of every (R,S)-bimodule M relative to any ring homomorphism $\alpha : R \to S$.

(4) For a commutative ring R, every left module ${}_RM$ can be made right module by defining as $mr = rm$. Thus M is (R,R)-bimodule and $C_{id}(M) = M$.

Example 2.3. Consider $R = \mathbb{C}$ and $S = M_2(\mathbb{R})$. Then $M = M_2(\mathbb{C})$ is a (R,S)-bimodule. Let us consider a homomorphism $\alpha : \mathbb{C} \to M_2(\mathbb{R})$ as $\alpha(r_1 + ir_2) = \begin{pmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{pmatrix}$. Then under this homomorphism we have

$$\begin{aligned}
C_\alpha(M) &= \Big\{ m \in M_2(\mathbb{C}) \mid rm = m\alpha(r) \ \forall\ r \in \mathbb{C} \Big\} \\
&= \left\{ \begin{pmatrix} U & V \\ W & X \end{pmatrix} \in M_2(\mathbb{C}) \mid (r_1 + ir_2) \begin{pmatrix} U & V \\ W & X \end{pmatrix} = \begin{pmatrix} U & V \\ W & X \end{pmatrix} \begin{pmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} U & V \\ W & X \end{pmatrix} \in M_2(\mathbb{C}) \mid V = iU \ , X = iW \right\} \\
&= \left\{ \begin{pmatrix} U & iU \\ W & iW \end{pmatrix} \mid U, W \in \mathbb{C} \right\}
\end{aligned}$$

Similarly let $R := \{A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \ \in \mathbb{R}\}$ and $\alpha : R \to \mathbb{C}$ a ring homomorphism defined by $\alpha(A) = a + ib$. Then center of $M = M_2(\mathbb{C})$ as a $(R,\mathbb{C})$-bimodule is

$$\begin{aligned}
C_\alpha(M) &= \Big\{ m \in M_2(\mathbb{C}) \mid Am = m\alpha(A) \ \forall\ A \in R \Big\} \\
&= \left\{ \begin{pmatrix} U & V \\ W & X \end{pmatrix} \in M_2(\mathbb{C}) \mid \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} U & V \\ W & X \end{pmatrix} = \begin{pmatrix} U & V \\ W & X \end{pmatrix} (a + ib) \right\} \\
&= \left\{ \begin{pmatrix} U & V \\ W & X \end{pmatrix} \in M_2(\mathbb{C}) \mid W = iU \ , X = iV \right\} \\
&= \left\{ \begin{pmatrix} U & V \\ iU & iV \end{pmatrix} \mid U, V \in \mathbb{C} \right\}
\end{aligned}$$

From the Definition (2.1), it is obvious that a finite sum of central elements relative to $\alpha \in Hom(R,S)$ of a bimodule ${}_RM_S$ is central in ${}_RM_S$. However for $r \in R$ and $m \in C_\alpha(M)$, rm is not necessarily central in ${}_RM_S$. For this consider $R = S = M_n(\mathbb{R})$, $M = M_n(\mathbb{R})$ and $\alpha = id$. Then we have $C_\alpha(M) = kI_n$ for all $k \in \mathbb{R}$. Let us take $A = \sum_{i=1}^{n-1} r_i E_{i,i+1} \in M_n(\mathbb{R})$, where $E_{i,i+1}$ are the matrix having 1 in $(i, i+1)^{th}$ position and zero elsewhere. Then it is clear that $A.rI_n \notin C_\alpha(M)$. Here in the next proposition we have given a sufficient condition for the set of central elements of a bimodule to be closed under left(or right) multiplication.

Recall that for a left R-module M, Torsion of M is defined as $Tor(M) = \{m \in M \ : rm = 0$ for some non-zero $r \in M\}$. A module M is said to be torsion free if $Tor(M) = \{0\}$.

Proposition 2.4. *Let M be a $(R,S)-$bimodule and $\alpha : R \to S$ is a ring homomorphism. Let $0 \neq m \in C_\alpha(M)$, then*

(1) If $r \in C(R)$ then $rm \in C_\alpha(M)$. Converse part hold if $m \notin Tor(_RM)$.

(2) If $s \in C(S)$ then $ms \in C_\alpha(M)$. Converse part hold if $m \notin Tor(M_S)$.

Proof. (1)$\Rightarrow$ Consider $m \in C_\alpha(M)$ and $r \in C(R)$. Then for any $k \in R$, we have $k(rm) = k(m\alpha(r)) = (km)\alpha(r) = (m\alpha(k))\alpha(r) = m\alpha(kr) = m\alpha(rk) = m\alpha(r)\alpha(k) = (rm)\alpha(k)$. Conversely suppose that $rm \in C_\alpha(M)$ for some $r \in R$ and $m \in M$. We need to proof that $r \in C(R)$. We have for any $k \in R$, $k(rm) = rm\alpha(k) = m\alpha(r)\alpha(k)$. Also we can write $k(rm) = k(m\alpha(r)) = m\alpha(k)\alpha(r)$. Thus we see that $m\alpha(rk) = m\alpha(kr)$. Again since $m \in C_{\alpha(M)}$, implies $(rk)m = (kr)m \Rightarrow (rk - kr).m = 0$. AS $m \notin Tor(M)$, implies that $rk - kr = 0$ for any $k \in R$. Thus $rk = kr$ for all $k \in R \Rightarrow r \in C(R)$.

(2)$\Rightarrow$ Follows straightforward. □

Note that the converse part of Proposition (2.4) may not hold if $m \in Tor(M)$. For this consider $R = S = M = M_2(\mathbb{Z}_4)$. let $m = \begin{pmatrix} \bar{2} & 0 \\ 0 & \bar{2} \end{pmatrix}$, then it is clear that $m \in C_{id}(M) \cap Tor(M)$ as we have a non-zero $r = \begin{pmatrix} 0 & \bar{2} \\ 0 & 0 \end{pmatrix}$ in R such that $rm = 0$. This also implies $rm \in C_{id}(M)$. Now let us consider $k = \begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix} \in R$. Then we have $rk = \begin{pmatrix} 0 & \bar{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix} = \begin{pmatrix} \bar{2} & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & \bar{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix}\begin{pmatrix} \bar{2} & 0 \\ 0 & 0 \end{pmatrix} = kr$. This implies $r \notin C(R)$.

Recall that for an (R, S)-bimodule M, a subset N of M is said to be sub-bimodule of M if N is itself a (R, S)-bimodule. In the next Proposition we have given sufficient condition for the center of (R, S)-bimodule M to become sub-bimodule.

Proposition 2.5. *Let M be a $(R, S)-$bimodule and $\alpha : R \to S$ is a ring homomorphism. Then the following condition holds.*

(1) If R is commutative then $C_\alpha(M)$ is a submodule of $_RM$.

(2) If S is commutative then $C_\alpha(M)$ is a submodule of M_S.

(3) If R and S are both commutative then $C_\alpha(M)$ is a sub-bimodule of $_RM_S$.

Proof. The proof follows from Proposition (2.4). □

Recall that if M and N are both (R, S)-bimodules, then a map $f : M \longrightarrow N$ which is simultaneously R-linear and S-linear is called a homomorphism of bimodules.

Proposition 2.6. *Central elements of modules are preserved by module homomorphism.*

Proof. Suppose m is a central element in $_RM_S$ and $f : M \longrightarrow N$ is a bimodule homomorphism. Thus we have $rm = m\alpha(r)$ for all $r \in R$. Now, $rf(m) = f(rm) = f(m\alpha(r)) = f(m)\alpha(r)$. □

3 α-Central Armendariz bimodule

In [9] Lee and Zhou introduced Reduced module as a left R-module M in which for a given $a \in R$ and $m \in M$ satisfying $a^2m = 0$ implies $aRm = 0$. Similarly we can say an (R, S)-bimodule M is reduced if both $_RM$ and M_S is reduced. Rege and Chhawchharia in [14] introduced the notion of Armendariz ring. A ring R is said to be Armendariz if $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$, $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j = 0$ for all i, j. The term Armendariz ring was chosen because E. Armendariz in [6] had shown that a reduced ring(ring without nonzero nilpotent elements) satisfies this condition. For more details on this topic, we refer the reader to [3], [4], [5], [10], [11] and [12]. In [1] Agayev et. al introduced the concept of Central Armendariz ring as an extension of Armendariz ring. A ring R is said to be Central Armendariz if $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m$, $g(x) =$

$b_0 + b_1x + b_2x^2 + \ldots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$ then $a_ib_j \in C(R)$ for all i, j. In [9] Lee and Zhou introduced the notion of an Armendariz module. They defined a nodule $_RM$ to be an Armendariz module if whenever polynomials $r(x) = r_0 + r_1x + r_2x^2 + \ldots + r_mx^m \in R[x]$ and $m(x) = m_0 + m_1x + m_2x^2 + \ldots + m_nx^n \in M[x]$ satisfy $r(x)m(x) = 0$ then $r_im_j = 0$ for each i, j. Thus the concept of Armendariz property can be extended for bimodule as (R, S)-bimodule M is said to satisfies Armendariz property if both $_RM$ and M_S satisfies Armendariz properties.

In this section the notion of an α-central Armendariz bimodule is introduced as a generalization of central Armendariz rings to bimodules. We prove that many results of central Armendariz rings can be extended to bimodules for this general setting.

The ring R is called central Armendariz if whenever $f(x)g(x) = 0$ for some $f(x) = \sum_{i=1}^{n} a_ix^i, g(x) = \sum_{j=1}^{m} b_jx^j \in R[x]$, then $a_ib_j \in C(R)$.

Definition 3.1. Let *M* be a (*R*, *S*)-bimodule and $\alpha : R \to S$ a ring homomorphism. We say *M* is α-central Armendariz bimodule if whenever elements $f(x) = \sum_{i=1}^{m} a_ix^i \in R[x]$, $s(x) = \sum_{k=1}^{p} s_kx^k \in S[x]$ and $m(x) = \sum_{j=1}^{n} m_jx^j$, $n(x) = \sum_{l=1}^{t} n_lx^l \in M[x]$ satisfy $f(x)m(x) = 0$(or $n(x)s(x) = 0$), then a_im_j(or n_ls_k) $\in C_\alpha(M)$ for each $1 \leq i \leq m$, $1 \leq k \leq p$, $1 \leq l \leq t$ and $1 \leq j \leq n$.

Remark 3.2. By Definition 3.1, the following remark can be easily obtained.

(1) R is an central Armendariz ring if and only if $_RR_R$ is a 1-central Armendariz bimodule.

(2) If $_RM_S$ is Armendariz bimodule then $_RM_S$ is α-central Armendariz bimodule for some $\alpha \in Hom(R, S)$. But the converse part is not true as shown in the following example.

Example 3.3. Recall that if R is a ring and M is an (R, R)-bimodule, then the trivial extension of R by M is defined to be the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. Let $R = S = M = T(\mathbb{Z}_8, \mathbb{Z}_8)$ and $\alpha = id$. Consider $f(x) = (\overline{4}, \overline{0}) + (\overline{4}, \overline{1})x$, then the square of this polynomial is zero but the product $(\overline{4}, \overline{0})(\overline{4}, \overline{1}) = (\overline{0}, \overline{4})$ is not zero. On the other hand being commutative, $_RM_S$ is α-central Armendariz bimodule.

Proposition 3.4. *Let $\alpha \in Hom(R, S)$ and $_RM_S$ be an α-central Armendariz bimodule. If $a \in R$ and $m \in M$ satisfy $am = 0$, then $acm \in C_\alpha(M)$ for any $c \in N(R)$.*

Proof. As $c \in N(R)$, there exists a positive integer n such that $c^n = 0$. Thus, we have $(a - acx)(m + cmx + \ldots + c^{n-1}mx^{n-1}) = a(1 - cx)(1 + cx + \ldots + c^{n-1}x^{n-1})m = am = 0$ in $_{R[x]}M[x]$, and so $acm \in C_\alpha(M)$ by the central Armendariz property of M. □

Proposition 3.5. *Let $\alpha \in Hom(R, S)$. The class of α-central Armendariz bimodule is closed under direct sums, direct products and sub-bimodules.*

An R-module M is torsionless if it is a submodule of a direct product of copies of R. If M is faithful R-module, then R is a submodule of a direct product of copies of M. The following corollary is easy to be obtained by Proposition 3.5.

Corollary 3.6. *Let R be ring and M be a (R,R)-bimodule. The following conditions are equivalent.*

(1) R is central Armendariz ring.

(2) Every torsionless R-module is id-Central Armendariz.

(3) Every submodule of a free R-module is id-central Armendariz.

(4) There exists a faithful R-module which is id-central Armendariz.

Proposition 3.7. *Let R, S are any two rings and $\alpha \in Hom(R, S)$ and M be a (R, S)-bimodule. Then M is α-central Armendariz if and only if every finitely generated(cyclic) sub-bimodule of M is α-Central Armendariz.*

Proposition 3.8. *Let $\alpha \in Hom(D, K)$, where D and K are commutative domain. Then $_DM_K$ is α-central Armendariz if and only if its torsion sub-bimodule $T(M)$ is α-central Armendariz.*

Proof. Consider $f(x) = \sum_{i=0}^{n} a_i x^i \in D[x]$ and $m(x) = \sum_{j=0}^{k} m_j x^j \in M[x]$ satisfy $f(x)m(x) = 0$, we have

$$\begin{cases} a_0 m_0 = 0 \\ a_0 m_1 + a_1 m_0 = 0 \\ a_0 m_2 + a_1 m_1 + a_2 m_0 = 0 \\ \dots \\ a_n m_k = 0 \end{cases} \tag{3.1}$$

we can assume that $a_0 \neq 0$, then $m_0 \in T(M)$. Now multiplying the second equation of (3.1) by a_0 from left side, we get $a_0^2 m_1 = 0$. Since D is domain, this implies $m_1 \in T(M)$. Multiplying the third equation of (3.1) by a_0^2 from left side, again we get $a_0^3 m_2 = 0$, this implies $m_2 \in T(M)$. Continuing this process, we get $m(x) \in T(M)[x]$. Since $T(M)$ is α-Central Armendariz, we conclude that $a_i m_j \in C_\alpha(T(M))$ for all i, j. Thus ${}_D M_K$ is α-Central Armendariz. The converse part is trivial. □

Let $\alpha : R \to S$ be a ring homomorphism, the map $\overline{\alpha} : R[x] \to S[x]$ defined by $\overline{\alpha}(a_0 + a_1 x^1 + \ldots + a_m x^m) = \alpha(a_0) + \alpha(a_1)x^1 + \ldots + \alpha(a_m)x^m$ is a ring homomorphism. In ([9], Theorem 1.12), it is proved that a module ${}_R M$ is Armendariz if and only if ${}_{R[x]} M[x]$ is Armendariz. In Next Proposition we have extended the same result for α-central Armendariz.

Proposition 3.9. *Let M be a (R,S)-bimodule and $\alpha : R \to S$ be a ring homomorphism. Then the following are equivalent:*

(1) ${}_R M_S$ is α-central Armendariz.

(2) ${}_{R[x]} M[x]_{S[x]}$ is $\overline{\alpha}$-central Armendariz.

Proof. Suppose that ${}_R M_S$ is α-Central Armendariz. let $r(y) = r_0 + r_1 y + r_2 y^2 + \ldots + r_m y^m \in R[x][y]$ and $m(y) = m_0 + m_1 y + m_2 y^2 + \ldots + m_n y^n \in M[x][y]$ are such that $r(y)m(y) = 0$, where $r_i = r_{i0} + r_{i1} x^1 + \ldots + r_{im_i} x^{m_i} \in R[x]$ and $m_j = m_{j0} + m_{j1} x^1 + \ldots + m_{jn_j} x^{n_j} \in M[x]$ for each $0 \leq i \leq m$ and $0 \leq j \leq n$. Let us consider $t = deg\, r_0 + deg\, r_1 + deg\, r_2 + \ldots + deg\, r_m + deg\, m_0 + \ldots + deg\, m_n$. Then $r(x^t) = r_0 + r_1 x^t + r_2 x^{2t} + \ldots + r_m x^{mt} \in R[x]$ and $m(x^t) = m_0 + m_1 x^t + m_2 x^{2t} + \ldots + m_n x^{nt} \in M[x]$ and the set of coefficients of the $r_i(m_j)$ equals the set of coefficients of the $(r(x^t))(resp.\ m(x^t))$ for each i, j. Since $r(y)m(y) = 0$, this implies $r(x^t)m(x^t) = 0$ in ${}_{R[x]} M[x]$. Since ${}_R M_S$ is α-central Armendariz, thus $r_{ip_i} m_{jq_j} \in C_\alpha(M)$, where $0 \leq p_i \leq m_i$, $0 \leq q_j \leq n_j$. Also from Proposition 2.8 it is clear that $C_\alpha(M)$ is closed under addition, thus $r_i mj \in C_\alpha(M[x])$. Converse part is obvious as any submodule of α-central Armendariz is α-central Armendariz. □

Next we study localizations. Let M be an (R, S)-bimodule. Let K and L be a multiplicative closed subset consisting of central regular elements of R and S respectively, then $K^{-1}ML^{-1}$ has a $(K^{-1}R, SL^{-1})$-bimodule structure.

Proposition 3.10. *Let M be an (R,S)-bimodule and $\alpha : R \to S$ a ring homomorphism. let K and L be as defined above. Then M is α-central Armendariz if and only if $K^{-1}ML^{-1}$ is $\overline{\alpha}$-central Armendariz.*

Proof. ($\Rightarrow$) Let $f(x) = \sum_{i=0}^{m} \xi_i x^i \in K^{-1}R[x]$ and $m(x) = \sum_{j=0}^{n} \eta_j x^j \in K^{-1}M[x]$ satisfy $f(x)m(x) = 0$. Here $\xi_i = s_i^{-1} a_i$ and $\eta_j = t_j^{-1} m_j$ where $s_i, t_j \in K$, $a_i \in R$ and $m_j \in M$. Let us fix $s = (s_0 s_1 \ldots s_m)$ and $t = (t_0 t_1 \ldots t_n)$, then define $\widehat{f(x)} = \sum_{i=0}^{m} s\xi_i x^i$ and $\widehat{m(x)} = \sum_{j=0}^{n} t\eta_j x^j$. Thus clearly $\widehat{f(x)} \in R[x]$ and $\widehat{m(x)} \in M[x]$ and also we have $\widehat{f(x)}\widehat{m(x)} = 0$ in $M[x]$ which implies $\sum_{i=0}^{m} s\xi_i x^i \sum_{j=0}^{n} t\eta_j x^j = 0$, since M is α- Central Armendariz bimodule thus we have $s\xi_i t\eta_j \in C_\alpha(M)$. This implies $st\xi_i\eta_j \in C_\alpha(M)$. Again we know that s_i and t_j are central regular elements. Thus it follows that $\xi_i\eta_j \in C_{\overline{\alpha}}(K^{-1}M)$. Conversely, assume that $k^{-1}M$ is a $\overline{\alpha}$-Central Armendariz bimodule. Since we know that sub-bimodule of α-Central Armendariz bimodule are α-Central Armendariz bimodule and M is a sub-bimodule of $K^{-1}M$. Thus M is α-Central Armendariz (R, S)-bimodule. □

Corollary 3.11. *Let M be a (R,S)-bimodule and $\alpha : R \to S$ a ring homomorphism. Then the following are equivalent:*

(1). ${}_RM_S$ *is α-Central Armendariz bimodule.*

(2). ${}_{R[x]}M[x]_{S[x]}$ *is $\overline{\alpha}$-Central Armendariz bimodule.*

(3). ${}_{R[x,x^{-1}]}M[x,x^{-1}]_{S[x,x^{-1}]}$ *is $\overline{\alpha}$-Central Armendariz bimodule.*

Proof. Consider $K = \{1, x, x^2, \ldots, x^4, \ldots\}$ and $L = \{1, x, x^2, \ldots, x^4, \ldots\}$. Then S and L are the multiplicative closed subsets of $R[x]$ and $S[x]$ respectively consisting of central regular elements. Then the proof follows from Proposition 3.10 □

A module M is called $p.p$ *-module*, if for any $m \in M$ $r_R(m) = eR$ where $e^2 = e \in R$([9], Definition 2.1). Recall from [2], A module ${}_RM$ is called abelian if, for any $m \in M$ and any $a \in R$, any idempotent $e \in R$, $aem = eam$. We say an (R,S)-bimodule M is abelian if it abelian from both the sides. In [2], N.Agayev et.al proved that Armendariz modules are abelian and converse hold if the module M is $p.p$ *-module*. But in case of α-Central Armendariz bimodule, we pose the following question: Let M is a (R,S)-bimodule and $\alpha : R \to S$, a ring homomorphism. If M is α-Central Armendariz then M is abelian. In fact we do not know any example of a α-Central Armendariz bimodule that is not abelian. However we have an example of bimodule M, which is not α-Central Armendariz.

Example 3.12. There exist an abelian bimodule which is not α-Central Armendariz bimodule. For this let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^{2\times 2}$ the 2×2 full matrix ring over $\mathbb{Z}$.

$$R = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} : a \equiv d(mod\ 2), b \equiv c \equiv 0(mod\ 2)\}$$

and consider M to be the (R,R)-bimodule ${}_RR_R$. Since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are only idempotent in R. Thus ${}_RM_R$ is an abelian bimodule. Now let $f(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \in R[x]$ and $m(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x \in M[x]$. Then we have $f(x)m(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \notin C_{id}(M)$.

4 Examples of Armendariz submodules

We write $M_n(R)$ and $T_n(R)$ for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over R respectively. The $n \times n$ identity matrix is denoted by I_n. For a left R-module M and $A = (a_{ij}) \in M_n(R)$, let $AM = \{(a_{ij}m : m \in M\}$ for $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ where $\{E_{ij} : 0 \leq i,j \leq n\}$ are the matrix units and set $V_n(R) = I_nR + VR + \ldots + V^{n-1}R$, $V_n(M) = I_nM + VM + \ldots + V^{n-1}M$. Then $V_n(R)$ is a ring and $V_n(M)$ becomes a left module over $V_n(R)$ under the usual addition and multiplication of matrices. There is a ring isomorphism $\theta : V_n(R) \to \dfrac{R[x]}{(x^n)}$ given by $\theta(I_nr_0 + \ldots + V^{n-1}r_{n-1}) = (r_0 + r_1x + \ldots + r_{n-1}x^{n-1}) + (x^n)$ and an abelian group isomorphism $\phi : V_n(M) \to \dfrac{M[x]}{(x^n)M[x]}$ given by given by $\phi(I_nm_0 + \ldots + V^{n-1}m_{n-1}) = (m_0 + m_1x + \ldots + m_{n-1}x^{n-1}) + (x^n)M[x]$ such that $\phi(AW) = \theta(A)\phi(W)$ for all $W \in V_n(m)$ and $A \in V_n(R)$. In ([15], Corollary 3.7), Zhang and Chen proved that ${}_RM$ is reduced module if and only if $V_n(M)$ is Armendariz over $V_n(R)$. So for a reduced module ${}_RM$, we find some bigger class Armendariz submodules of $T_n(M)$ over $T_n(R)$ which contain all these known Armendariz submodule of $T_n(M)$. For this purpose recall from [10], the following notations

For an even number $n = 2k \geq 2$, let

$$A_n^e(M) = \sum_{i=1}^{k} \sum_{j=k+i}^{n} E_{i,j}M$$

and for an odd number $n = 2k + 1 \geq 3$

$$A_n^o(M) = \sum_{i=1}^{k+1} \sum_{j=k+i}^{n} E_{i,j}M$$

Let

$$A_n(M) = I_nM + VM + \ldots + V^{k-1} + A_n^e(M) \text{ for } n = 2k \geq 2$$

and

$$A_n(M) = I_nM + VM + \ldots + V^{k-1} + A_n^o(M) \text{ for } n = 2k + 1 \geq 3$$

thus we have $A_n(M) = \begin{pmatrix} x_1 & x_2 & \ldots & x_k & a_{1(k+1)} & a_{1(k+2)} & \ldots & a_{1n} \\ 0 & x_1 & \ldots & x_{k-1} & x_k & a_{1(k+2)} & \ldots & a_{2n} \\ 0 & 0 & x_1 & \ldots & & & & a3n \\ & & & \ldots & & & & \\ & & & & & & & x_1 \end{pmatrix}$ for any $n \geq 2$.

where $x_i, a_{js} \in M, 1 \leq i \leq k, 1 \leq j \leq n-k$ and $k+1 \leq s \leq n$. For $A = (a_{ij})$, $B = (b_{ij})$, we write $[A.B]_{ij} = 0$ to mean that $a_{il}b_{lj} = 0$ for $l = 0, , \ldots, n$.

Lemma 4.1. *([10], Lemma 1.2)For $r(x) = A_0+A_1x+\ldots+A_px^p \in M_n(R)[x]$ and $m(x) = B_0+B_1x+\ldots+B_qx^q \in M_n(M)[x]$, let $f_{ij} = a_{ij}^0+a_{ij}^1x+\ldots+a_{ij}^px^p$ and $g_{ij} = b_{ij}^0+b_{ij}^1x+\ldots+b_{ij}^qx^q$ where a_{ij}^l are the (i,j)-entries of A_l for $l = 0, 1, \ldots, p$ and b_{ij}^s are the (i,j)-entries of B_s for $s = 0, 1, \ldots, q$. Then $r(x) = (f_{ij}(x)) \in M_n(R[x])$ and $m(x) = (g_{ij}(x)) \in M_n(M[x])$. If ${}_RM$ is Armendariz and $[r(x).m(x)]_{ij} = 0$ for all i, j, then $A_iB_j = 0$ for all i, j*

Proposition 4.2. *Let $n = 2k + 1 \geq 3$ be a natural number. Then ${}_RM$ is a reduced module if and only if ${}_{A_n(R)}A_n(M)$ is Armendariz module.*

Proof. $(\Rightarrow)$ Let $r(x) = A_0+A_1x+\ldots+A_px^p \in A_n(R)[x]$ and $m(x) = B_0+B_1x+\ldots+B_qx^q \in A_n(M)[x]$ are such that $r(x).m(x) = 0$. We need to proof that $A_iB_j = 0$ for all $0 \leq i \leq p$ and $0 \leq j \leq q$. Here we identify $A_n(R)[x]$ with $A_n(R[x])$ and $A_n(M)[x]$ with $A_n(M[x])$ canonically. Let $f_{ij} = a_{ij}^{(0)}+a_{ij}^{(1)}x+\ldots+a_{ij}^{(p)}x^p$ and $g_{ij} = b_{ij}^{(0)}+b_{ij}^{(1)}x+\ldots+b_{ij}^{(q)}x^q$ where $a_{ij}^{(l)}$ are the (i,j)-entries of A_l for $l = 0, 1, \ldots, p$ and $b_{ij}^{(s)}$ are the (i,j)-entries of B_s for $s = 0, 1, \ldots, q$. Then $r(x) = (f_{ij}(x)) \in A_n(R[x])$ and $m(x) = (g_{ij}(x)) \in A_n(M[x])$. By Lemma 4.1, it is suffices to show that $[r(x).m(x)]_{ij} = 0$ for all i, j. clearly $[r(x).m(x)]_{ij} = 0$ for $i > j$ and for $t = 1, 2, \ldots, k$, we have

$$f_t := f_{1,t} = f_{2,t+1} = \ldots = f_{n-t+1,n} \text{ and } g_t := g_{1,t} = g_{2,t+1} = \ldots = g_{n-t+1,n}.$$

It follows from $r(x).m(x) = 0$,

$$\begin{cases} f_1g_1 = 0 \\ f_1g_2 + f_2g_1 = 0 \\ f_1g_3 + f_2g_2 + f_3g_1 = 0 \\ \ldots \\ f_1g_k + f_2g_{k-1} + \ldots + f_kg_1 = 0 \end{cases} \tag{4.1}$$

we know that ${}_RM$ is reduced module if and only if ${}_{R[x]}M[x]$ is reduced([9], Theorem 1.6). Thus from $f_1g_1 = 0$ we get $f_1^2g_1 = 0$ and hence $f_1R[x]g_1 = 0$. Multiplying by f_1 from left side to $f_1g_2 + f_2g_1 = 0$, we get $f_1^2g_2 = 0$ which implies $f_1g_2 = 0$, thus $f_2g_1 = 0$. similarly multiplying by f_1 from left to $f_1g_3 + f_2g_2 + f_3g_1 = 0$, we get $f_1^2g_3 + f_1f_2g_2 + f_1f_3g_1 = 0$, hence $f_1^2g_3$, which implies $f_1g_3 = 0$. Again multiplying f_2 to the same equation we get $f_2^2g_2 + f_2f_3g_1 = 0$, this implies $f_2^2g_2 = 0$ and hence $f_3g_1 = 0$. Similarly Continuing this process, we get $f_ig_j = 0$ for all $i + j \leq k + 1$. This implies $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i, j) \notin \Gamma$ where $\Gamma = \{(u, k+u) : u = 1, \ldots, k+1\} \cup \{(u, k+u+1) : u = 1, \ldots, k\} \cup \ldots \cup \{(u, u+n-2) : u = 1, 2\} \cup \{(u, n-1+u) : u = 1\}$. Now we need to prove that $[r(x).m(x)]_{ij} = 0$ for all $(i, j) \in \Gamma$.

Again from $r(x).m(x) = 0$, we have

$$\begin{cases} f_1 m_{1,k+1} + f_2 g_k + f_k g_2 + f_{1,k+1} g_1 = 0 \\ f_1 g_{2,k+2} + f_2 g_k + f_k g_2 + f_{2,k+2} g_1 = 0 \\ \dots \\ f_1 g_{k+1,2k+1} + f_2 g_k + \dots + f_{k-1} g_3 + f_k g_2 + f_{k+1,2k+1} g_1 = 0 \end{cases} \tag{4.2}$$

Now using left multiplications with 4.2 and using the result obtained previously in 4.1, we have, for $u = 1, 2, \dots, k+1$

$$\begin{cases} f_1 g_{u,k+u} = f_{u,k+u} g_1 = 0, \text{ for } u = 1, 2, \dots, k+1 \\ f_i g_j = 0, \text{ for all } i, j \text{ with } i + j = k + 2. \end{cases} \tag{4.3}$$

Thus it follows from 4.3 that $[r(x).m(x)]_{u,u+k} = 0$ for $u = 1, 2, \dots, k+1$.

Now let us assume that, for some $0 < l \leq k$, $[r(x).m(x)]_{u,k+u+t} = 0$ for $t = 0, 1, \dots, l-1$ and $u = 1, \dots, k-t+1$.. We now prove that $[r(x).m(x)]_{u,k+u+l} = 0$ for $u = 1, \dots, k-l+1$. Since $r(x).m(x) = 0$, we have

$$\textstyle\sum_{j=1}^{n} f_{u,j} g_{j,k+u+l} = 0 \text{ for } u = 1, \dots, k-l+1$$

Thus

$$f_1 g_{u,k+u+l} + \dots + f_{l+1} g_{u+l,k+u+l} + f_{l+2} g_k + \dots + f_k g_{l+2} + f_{u,k+u} g_{l+1} + \dots + f_{u,k+u+l-1} g_2 + f_{u,k+u+l} g_1 = 0$$

Again using results obtained in 4.1, 4.2, 4.3 and the induction hypothesis, the following obtained:

(1) (a) $f_1 g_{u,k+u+t} = f_{u,k+u+t} g_1 = 0$, for $t = 0, 1, \dots, l-1$; $s = 1, 2, \dots, k-t+1$

(b) $f_2 g_{u+1,k+u+t} = f_{u,k+u+t-1} g_2 = 0$, for $t = 1, \dots, l-1$; $s = 1, 2, \dots, k-t+1$

$\vdots$

(c) $f_{t+1} g_{u+t,k+u+t} = f_{u,k+u} g_{l+1} = 0$, for $t = l-1$; $s = 1, 2, \dots, k-t+1$

(2) $f_i g_j = 0$ for all $i, j \geq u$ with $i + j = u + k$ for $u = 1, 2, \dots, l+1$

using the method of left multiplications with the help of (1) and (2)(note that $fg = 0$ in $M[x]$, then $fR[x]g = 0$), we obtain that every term in left side of (4) is zero. hence $[r(x).m(x)]_{u,k+u+t} = 0$ for $u = 1, \dots, k-l+1$. Hence by mathematical induction, we get $[r(x).m(x)] = 0$ for all $(i, j) \in \Gamma$.

($\Leftarrow$) Conversely suppose that ${}_{A_n(R)}A_n(M)$ is Armendariz for $n = 2k+1 \geq 3$, then being submodule, ${}_{V_n(R)}V_n(M)$ is Armendariz. Hence by ([9], Theorem 1.9), ${}_RM$ is reduced. □

Corollary 4.3. *For $n = 2k+1 \geq 3$, a ring R is reduced if and only if ${}_{A_n(R)}A_n(R)$ is Armendariz.*

Proposition 4.4. *Let $n = 2k \geq 2$ be a natural number. Then ${}_RM$ is a reduced if and only if $A_n(M) + E_{1,k}M$ is Armendariz module over $A_n(R) + E_{1,k}R$.*

Proof. ($\Rightarrow$) Let $S = A_n(R) + E_{1,k}R$ and $T = A_n(R) + E_{1,k}R$. Similarly by Lemma 4.1, it is suffices to show that $[r(x).m(x)]_{ij} = 0$ for all i, j. We have $r(x) = (f_{ij})$ and $m(x) = (g_{ij})$ with $f_{ij} = g_{ij} = 0$ for all $i > j$ and for $t = 1, 2, \dots, k-1$, we have

$f_t := f_{1,t} = f_{2,t+1} = \dots = f_{n-t+1,n}$ and $g_t := g_{1,t} = g_{2,t+1} = \dots = g_{n-t+1,n}$,

$f_k := f_{2,k+1} = f_{3,k+2} = \dots = f_{k+1,n}$,

$g_k := g_{2,k+1} = g_{3,k+2} = \dots = g_{k+1,n}$,

$f_0 := f_{1,k}$ and $g_0 := g_{1,k}$

Now from $r(x).m(x) = 0$, we have

$$\begin{cases} f_1 g_1 = 0 \\ f_1 g_2 + f_2 g_1 = 0 \\ f_1 g_3 + f_2 g_2 + f_3 g_1 = 0 \\ \dots \\ f_1 g_k + f_2 g_{k-1} + \dots + f_k g_1 = 0 \end{cases} \tag{4.4}$$

and

$$f_1 g_0 + f_2 g_{k-1} + \dots + f_{k-1} g_2 + f_0 g_1 = 0 \tag{4.5}$$

By the method of left multiplication with (4.4) and (4.5), one obtains

$$f_i g_j = 0 \; \forall \; i + j \leq k + 1 \tag{4.6}$$

and

$$f_1 g_0 = f_0 g_1 = 0 \tag{4.7}$$

Thus it follows that $[r(x).m(x)]_{ij} = 0$ for all i, j with $(i, j) \notin \Gamma$ where $\Gamma = \{(u, k + u) : u = 1, \dots, k\} \cup \{(u, k+u+1) : u = 1, \dots, k-1\} \cup \dots \cup \{(u, u+n-2) : u = 1, 2\} \cup \{(u, n-1+u) : u = 1\}$. Now we need to prove that $[r(x).m(x)]_{ij} = 0$ for all $(i, j) \in \Gamma$. Now we need to prove that $[r(x).m(x)]_{ij} = 0$ for all $(i, j) \in \Gamma$.

Again from $r(x).m(x) = 0$, we have

$$f_1 g_{1,k+1} + f_2 g_k + f_3 g_{k-1} + \dots + f_{k-1} g_3 + f_0 g_2 + f_{1,k+1} g_1 = 0 \tag{4.8}$$

and

$$\begin{cases} f_1 g_{2,k+2} + f_2 g_k + f_k g_2 + f_{2,k+2} g_1 = 0 \\ \dots \\ f_1 g_{k,2k} + f_2 g_k + \dots + f_{k-1} g_3 + f_k g_2 + f_{k,2k} g_1 = 0 \end{cases} \tag{4.9}$$

Now using left multiplications with (4.8) and (4.9) and using the result obtained previously in (4.6) and (4.7), we have, for $u = 1, 2, \dots, k$

$$\begin{cases} f_1 g_{u,k+u} = f_{u,k+u} g_1 = 0 \; \forall \; u = 1, 2, \dots, k \\ f_i g_j = 0 \; \forall \; i, j \geq 2, i + j = k + 2 \\ f_0 g_2 = f_2 g_0 = 0 \end{cases} \tag{4.10}$$

Thus it follows from (4.10) that $[r(x).m(x)]_{u,u+k} = 0$ for $u = 1, 2, \dots, k$.

Now let us assume that, for some $0 < l \leq k$, $[r(x).m(x)]_{u,k+u+t} = 0$ for $t = 0, 1, \dots, l$ and $u = 1, \dots, k - t$.. We now prove that $[r(x).m(x)]_{u,k+u+l} = 0$ for $u = 1, \dots, k - l$. Since $r(x).m(x) = 0$, we have

$$\begin{cases} f_1 g_{u,k+u+l} \dots + f_{l+1} g_{u+l,k+u+l} + f_{l+2} g_k + \dots + f_k g_{l+2} + f_{u,k+u} g_{l+1} + \\ \dots + f_{u,k+u+l-1} g_2 + f_{u,k+u} g_{l+1} + \dots + f_{u,k+u+l-1} g_2 + f_{u,k+u+l} g_1 = 0. \end{cases} \tag{4.11}$$

and

$$\begin{cases} f_1 g_{1,k+l+l} \dots + f_{l+1} g_{l+l,k+l+l} + f_{l+2} g_k + \dots + f_{k-1} g_{l+3} + f_0 g_{l+2} + f_{1,k+1} g_{l+1} + \\ \dots + f_{1,k+1} g_2 + f_{1,k+l+1} g_1 = 0. \end{cases} \tag{4.12}$$

Again using results obtained in (4.10), (4.6), (4.7) and the induction hypothesis, we obtained the following:

(1) (a) $f_1 g_{u,k+u+t} = f_{u,k+u+t} g_1 = 0$, for $t = 0, 1, \dots, l - 1$; $u = 1, 2, \dots, k - t$

(b) $f_2g_{u+1,k+u+t} = f_{u,k+u+t-1}g_2 = 0$, for $t = 1, \ldots, l-1$; $u = 1, 2, \ldots, k-t+1$

$\vdots$

(c) $f_{t+1}g_{u+t,k+u+t} = f_{u,k+u}g_{t+1} = 0$, for $t = l-1$; $u = 1, 2, \ldots, k-t$

(2) $f_ig_j = 0$ for all $i, j \geq u$ with $i+j = u+k$ for $u = 1, 2, \ldots, l+1$

(3) $f_0g_u = 0$ for $u = 1, 2, \ldots, l+1$

using the method of left multiplications with the help of (1)-(3)(note that $fg = 0$ in $M[x]$, then $fR[x]g = 0$), we obtain that every term in left side of (4.11) and (4.12) is zero. hence $[r(x).m(x)]_{u,k+u+t} = 0$ for $u = 1, \ldots, k-l$. Hence by mathematical induction, we get $[r(x).m(x)] = 0$ for all $(i, j) \in \Gamma$.

($\Leftarrow$) Conversely suppose that ${}_{A_n(R)}A_n(M)$ is Armendariz for $n = 2k+1 \geq 3$, then being submodule, ${}_{V_n(R)}V_n(M)$ is Armendariz. Hence by ([9], Theorem 1.9), ${}_RM$ is reduced. □

Corollary 4.5. *Let $n \geq 2$ be a natural number. Then ${}_RM$ is a reduced module if and only if $A_n(M)$ is left Armendariz module over $A_n(R)$.*

Corollary 4.6. *For $n = 2k \geq 2$, a ring R is reduced if and only if ${}_{A_n(R)}A_n(R)$ is Armendariz.*

Lemma 4.7. *Let M be a (R,S)-bimodule and $\alpha \in hom(R, S)$. Suppose that there exist $a, b \in R$ and $m \in M$ such that $a^2m = b^2m = 0$ and $abm = bam$ is not central. Then ${}_RM_S$ is not α-Central Amendariz bimodule.*

Proof. Let $a + bx \in R[x]$ and $am - bmx \in M[x]$. Then we have $(a+bx)(am-bmx) = 0$ in $M[x]$, but $abm = bam$ is not central. So, ${}_RM_S$ is not Central Armendariz bimodule. □

Proposition 4.8. *Let $n \geq 3$ be a natural number. Then ${}_RM_S$ is reduced bimodule if and only if $A_n(M)$ is a $(A_n(R), A_n(R))$ id-Central Armendariz bimodule.*

Proof. Let ${}_RM$ be reduced module. Thus from Proposition 4.2 and 4.4, it is obvious that $A_n(M)$ is an Armendariz module and so it is central Armendariz bimodule. Conversely, suppose that ${}_RM$ is not a reduced module. Choose a non-zero element $a \in R$ and $m \in M$ such that $a^2m = 0$ but $aRm \neq 0$. Then for elements $A = a(E_{11}+E_{22}+\ldots+E_{nn})$, $B = (E_{1(k+1)}+E_{1(k+2)}+\ldots+E_{1n}) \in A_n(R)$ and $N = m(E_{11}+E_{22}+\ldots+E_{nn}) \in A_n(M)$. Then we have $A^2N = B^2N = 0$ and $ABN = BAN$ is not central. Therefore by Lemma 4.7, $A_n(M)$ is not Central Armendariz module. This complete the proof. □

Corollary 4.9. *([1], Theorem 2.4)Let $n \geq 3$ be a natural number. R is reduced ring if and only $A_n(R)$ is central Armendariz.*

Proposition 4.10. *let M be (R, R)-bimodule. then for $n \geq 3$, the following are equivalent.*

(1). (R, R)-bimodule M is reduced.

(2). $(A_n(R), A_n(R))$-bimodule $A_n(M)$ is Armendariz

(3). $(A_n(R), A_n(R))$-bimodule $A_n(M)$ is id-Central Armendariz.

Proof. The proof is straightforward □

5 Conclusion remarks

Since every ring is a module over itself, thus generalization of the concept of ring theory to the modules is one of the key interests for many algebraists. Motivated by this, presently in this note a concept of central elements for bimodules is been introduced and their relations with other subclasses of modules are investigated. Furthermore, some new classes of Bimodules concerning ring homomorphism have been discussed and various examples have been constructed. Therefore, the results of this article are significant and so it is interesting and capable of developing further study in the future.

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