Palestine Journal of Mathematics

Vol 13(3)(2024) , 773–778

ROUGH X-SUB-EXACT SEQUENCES OF ROUGH MODULES

Fitriani, Indah Emilia Wijayanti and Ahmad Faisol

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16D10; Secondary 16D70.

Keywords and phrases: Rough set, rough module, exact sequence, sub-exact sequence.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

The authors wish to thank the Directorate of Research, Technology and Community Services Directorate General of Higher Education, Research and Technology, Ministry of Education, Culture, Research and Technology for this research support and funding under the Research Contract No: 131/E5/PG.02.00.PL/2023.

Corresponding author: Fitriani

Abstract Discussion related to exact sequences is an essential tool in module theory. The exact sequences are generalized to U-exact sequences, and X-sub-exact sequences. The concept of X-sub-exact sequences can be used to generalize basis and free modules. Besides, the rough set theory is a mathematical tool for dealing with uncertainty and vagueness problems. The advancement of the algebraic concept into the rough set theory is expeditious. Several structures are developed in rough set theory, such as groups, semigroups, rings, and modules. In this research, we provide a rough X-sub-exact sequence of a rough module over a rough ring. Furthermore, we investigate the properties related to the intersection of a finite number of rough modules over the rough ring.

1 Introduction

The concept of exact sequences is an essential meaning in module theory. Exact sequences are used in defining projective and injective modules [17]. The quasi-exact sequence is a generalization of exact sequences ([4], [6]). Several concepts related to exact sequences are generalized into quasi-exact sequences. Anvariyeh and Davvaz [16] use the quasi-exact sequence to investigate the connection between U-split-sequence and the projective module. Moreover, in [15], Anvariyeh and Davvaz generalize Schanuel's Lemma. They obtain the connection between superfluous submodules and quasi-exact sequences [15]. In addition, Davvaz and Shabani-Solt [5] provide the concepts of U-exact sequences in homological algebra.

Based on the definition of a quasi-exact sequence, Fitriani et al. [9] define an X-sub-exact sequence as a generalization of the exact sequence. Moreover, in [10], they define an X-sub-linearly independent module, a generalization of a linearly independent module. In addition, to generalize the generator concept, in [11], Fitriani et al. provide the $\mathcal{U}_V$-generator using the V-coexact sequence. In the same year, they combine the concept of the $\mathcal{U}_V$-generator and the X-sub-linear independent module family to define (X, V)-basis and $\mathcal{U}$-free modules [12].

The Rough Set Theory is a mathematical tool for dealing with vagueness and uncertainty problems. Some researchers give some algebraic structures in rough set theory. In [13], Bagismaz and Ozcan introduce rough semigroups on approximation spaces. Moreover, Nelima and Isaac [14] give an anti-homomorphism in the rough group, Wang and Chen [8] obtain the properties of the rough group and its application to computers, and Sinha and Prakash [2] investigate the properties of the rough projective module. In [3], Sinha and Prakash introduce the injective module based on rough set theory. Furthermore, in [1], they give the rough exact sequence of the modules, and hence, there is an opportunity to develop the X-sub exact sequence into the rough set theory.

In this paper, we define a rough sub-exact sequence of a rough module over a rough ring. Then, we give examples of rough sub-exact sequences of a rough module over a rough ring. Moreover, we provide some properties of a rough sub-exact sequence of the rough module.

2 Preliminaries

In this section, we give basic notions of rough set, rough group, rough ring, rough module, rough module homomorphism, and rough exact sequence of a rough module over a rough ring. We use these concepts to define a rough sub exact sequence of rough modules over rough rings.

Definition 2.1. [18] Let S be a non-empty set and σ an equivalence relation on S. The pair (S, σ) is called an approximation space.

An example of an approximation space can be seen as follows:

Example 2.2. Let $S = \mathbb{Z}_6$, and $H = \{\bar{0}, \bar{2}, \bar{4}\}$ a subgroup of $\mathbb{Z}_6$ under addition modulo 6. We define $a\theta b$ if and only if $ab^{-1} \in H$, for all $a, b \in \mathbb{Z}_6$. We can show that θ is reflexive, simmetric, and transitive. Hence, θ is an equivalence relation on $\mathbb{Z}_6$. Therefore, a pair $(\mathbb{Z}_6, \theta)$ is an approximation space.

Definition 2.3. ([7], [13]) Let (S, σ) be an approximation space. A mapping:

$$Apr : P(S) \rightarrow P(S) \times P(S)$$

defined by $Apr(Y) = (\underline{Y}, \overline{Y})$, for every $Y \in P(S)$, where:

(i) $\underline{Y} = \{y | [y]_\sigma \subseteq Y\}$, $\underline{Y}$ is called under approximation of Y in (S, σ);

(ii) $\overline{X} = \{y | [y]_\sigma \cap X \neq \emptyset\}$, $\overline{Y}$ is called upper approximation of Y in approximation space (S, σ);

(iii) $RBN(Y) = \overline{Y} - \underline{Y}$, $RBN(Y)$ is called boundary region of Y in (S, σ).

Definition 2.4. [13] Let (S, σ) be an approximation space. A set $Y \subseteq S$ is called a rough set if and only if $RBN(Y) \neq \emptyset$.

For the illustration, we give an example of a rough set in an approximation space as follows.

Example 2.5. Let $S = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9\}$ and σ an equivalence relation on (S, σ) with the following equivalence classes:
$C_1 = \{u_1, u_3, u_5\}$,
$C_2 = \{u_4, u_6\}$,
$C_3 = \{u_2, u_7\}$,
$C_4 = \{u_8, u_9\}$.
Let $Y = \{u_1, u_2, u_8, u_9\}$. Then $\underline{Y} = \{u_8, u_9\}$, and $\overline{Y} = \{u_1, u_2, u_3, u_5, u_7, u_8, u_9\}$. We have $RBN(Y) = \{u_1, u_2, u_3, u_5, u_7\} \neq \emptyset$. Therefore Y is a rough set.

Now, we recall the definition of rough group as follows.

Definition 2.6. [1] Let (S, σ) be an approximation space and H a non-empty subset of S. H is called a rough group if $Apr(H) = (\underline{H}, \overline{H})$ the following conditions hold:

(i) $xy \in \overline{H}$, for every $x, y \in H$;

(ii) $(xy)z = x(yz)$, for all $x, y, z \in H$;

(iii) there exists e in $\overline{H}$ such that $ex = xe = x$, for every $x \in H$;

(iv) for every $x \in H$, there exists $y \in H$ such that $xy = yx = e$.

A rough group H is commutative if $xy = yx$, for every $x, y \in \overline{H}$. Furthermore, we give an example of a rough group as follows.

Example 2.7. Let $U = \mathbb{Z}_6$ and $H = \{\bar{0}, \bar{2}, \bar{4}\}$ a subset of U. We define an equivalence relation θ on U as follows. $a\theta b$ if and only if $ab^{-1} \in H$. Hence, we have two equivalence classes on U:

(i) $E_1 = \{\bar{0}, \bar{2}, \bar{4}\}$, and

(ii) $E_2 = \{\bar{1}, \bar{3}, \bar{5}\}$.

Choose $X = \{\bar{2}, \bar{4}\}$. We have $\underline{X} = \emptyset$ and $\overline{X} = \{\bar{0}, \bar{2}, \bar{4}\}$. Since $RBN(X) = \overline{X} - \underline{X} \neq \emptyset$, X is a rough set on approximation space (U, θ).

Then we define addition modulo 6 $(+_6)$ as a binary operation on U. Since $\bar{0}$ is not in X as an identity element, we get that X is not a group under this binary operation. But, $\bar{0}$ is in the upper approximation of X, i.e. $\bar{0} \in \overline{X}$. Furthermore X satisfies all conditions in Definition 2.6. So, X is a rough group even though X is not a group.

Then, we define a rough ring, a rough module, and a rough exact sequence of the rough module as follows.

Definition 2.8. [1] Let (S, σ) be an approximation space, and let $R \subseteq S$. R is called a rough ring if:

(i) $\langle R, + \rangle$ is a rough commutative group;

(ii) $\langle R, . \rangle$ is a rough semigroup;

(iii) $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$, for every $x, y, z \in \overline{R}$.

Definition 2.9. [1] Let R be a rough ring and M a rough commutative group. If there is a mapping:

$$\cdot : \overline{R} \times \overline{M} \to \overline{M},$$

where $(r, m) \mapsto rm$ such that:

(i) $r(m + n) = rm + rn$;

(ii) $(r + s)m = rm + sm$;

(iii) $(rs)m = r(sm)$;

(iv) $1m = m$,

for every $r, s \in \overline{R}, m, n \in \overline{M}$, then M is called rough left module over rough ring R.

We give the definition of a rough right module in a similar way. Now, we define a rough exact sequence of the rough module.

Definition 2.10. [1] A sequence

$$\overline{M'} \xrightarrow{f} \overline{M} \xrightarrow{g} \overline{M''}$$

of homomorphisms of the rough module over a rough ring R is called a rough exact if $\text{im}(f) = \text{ker}(g)$.

3 MAIN RESULT

Now, we give a definition of a rough X-sub-exact sequence of rough modules as follows:

Definition 3.1. Let $M^{'}, M, M^{''}$ be rough modules over rough ring R, and X a rough submodule of M. Tripel $(M^{'}, M, M^{''})$ is called a rough X-sub-exact if

$$\overline{M'} \xrightarrow{\alpha} \overline{X} \xrightarrow{\beta} \overline{M''}$$

is rough exact sequence.

We give an example of a rough module and a rough sub-exact sequence of a rough module over a rough ring.

Example 3.2. Consider the ring $\mathbb{Z}_n$ and subring H in $\mathbb{Z}_n$, where $H \neq \{0\}$. We collect all left cosets of H as follows:

$$\mathbb{Z}_n/H = \{a + H | a \in \mathbb{Z}_n\}.$$

The set $\mathbb{Z}_n/H$ form the equivalence classes and give a partition in $\mathbb{Z}_n$. We chooce $X = H - \{0\}$. Then, we have $\overline{X} = H$, and $\underline{X} = \emptyset$. Hence X is a rough module over itself. Besides that, for all $Y \subseteq X$ such that:

(i) If $a \in Y$, then $-a \in Y$,

(ii) $ab \in \overline{Y}$, for all $a, b \in Y$,

is a rough module over itself.

Example 3.3. Based on Example 3.2, we choose $K = \mathbb{Z}_n - \{0\}$. Then we have $\overline{K} = \mathbb{Z}_n$, and K is a rough ring. Hence, we have the following rough exact sequence of rough modules over rough ring K:

$$0 \to \overline{X} \xrightarrow{i} \overline{X},$$

where i is identity function. So, the triple $(0, K, X)$ is a rough X-sub-exact sequence of rough modules over a rough ring K.

Then, we give some properties of a rough X-sub-exact sequence of rough modules over a rough ring.

Proposition 3.4. *Let (S, σ) be an approximation space, K, L, M rough modules, and $Y_1, Y_2, ..., Y_n$ submodules of M, where $\overline{Y_1} = \overline{Y_2} = ... = \overline{Y_n}$. Triple (K, L, M) is a rough Y_i-sub-exact sequence if and only if the triple (K, L, M) is rough Y_j-sub-exact sequence, where $i, j = 1, 2, ..., n$, $i \neq j$.*

Proof. Let the triple (K, L, M) be a rough Y_i-sub-exact sequence, for $i = 1, 2, ..., n$. Then we have a rough exact sequence as follows:

$$\overline{K} \xrightarrow{f} \overline{Y_i} \xrightarrow{g} \overline{M}.$$

Choose $j \in \{1, 2, ..., n\}$, where $i \neq j$. By assumption, the upper approximation of Y_i is equal to the upper approximation of Y_j even though the set $Y_i \neq Y_j$. Hence, we have the following rough exact sequence:

$$\overline{K} \xrightarrow{f} \overline{Y_j} \xrightarrow{g} \overline{M}.$$

So, the triple (K, L, M) is a rough Y_j-sub-exact sequence at L. ■

Proposition 3.5. *Let (S, σ) be an approximation space, M a rough module over a rough ring R, and $Y_1, Y_2, ..., Y_n$ rough submodules of M. If $\overline{Y_1} \cap \overline{Y_2} \cap ... \cap \overline{Y_n} = \overline{Y_1 \cap Y_2 \cap ... \cap Y_n}$, then $Y_1 \cap Y_2 \cap ... \cap Y_n$ is a rough submodule of M over a rough ring R in approximation space (S, σ).*

Proof. Suppose $Y_1, Y_2, .., Y_n$ are rough submodules of M. Then $Y_1 \cap Y_2 \cap ... \cap Y_n \subset M$. Consider $a, b \in Y_1 \cap X_2 \cap ... \cap X_n$. This implies $a, b \in Y_i$, for all $i = 1, 2, ..., n$. We have $a + b \in \overline{Y_i}$, for all $i = 1, 2, ..., n$. Beside that $-x \in Y_i$, for all $i = 1, 2, ..., n$. Hence $-x \in \overline{Y_1} \cap \overline{Y_2} \cap ... \cap \overline{Y_n} = \overline{Y_1 \cap Y_2 \cap ... \cap Y_n}$. Therefore $Y_1 \cap Y_2 \cap ... \cap Y_n$ is a rough subgroup of M. Consider $a \in Y_1 \cap Y_2 \cap ... \cap Y_n$. So $a \in Y_i$, for all $i = 1, 2, ..., n$. Hence, for every $r \in R$, $ra \in \overline{Y_i}$, for all $i = 1, 2, ..., n$. It implies $ra \in \overline{Y_1} \cap \overline{Y_2} \cap ... \cap \overline{Y_n} = \overline{Y_1 \cap Y_2 \cap ... \cap Y_n}$. Since $Y_1 \cap Y_2 \cap ... \cap Y_n$ satisfies all conditions in Definition 2.9, $Y_1 \cap Y_2 \cap ... \cap Y_n$ is a rough submodule of a rough ring R. ■

Let (S, θ) be an approximation space, K, L, M rough modules over a rough ring R. We define the set

$$\overline{\sigma}(K, L, M) = \{Y \leq M | (K, L, M) \text{ rough } Y - \text{sub- exact at } M\}.$$

Proposition 3.6. *Let (S, θ) be an approximation space, L, M rough modules over a rough ring R. If $Y_i \in \overline{\sigma}(0, L, M)$, for all $i = 1, 2, ..., n$, then $\bigcap_{i=1}^{n} Y_i \in \overline{\sigma}(0, L, M)$.*

Proof. Consider $Y_i \in \overline{\sigma}(0, L, M)$, for all $i = 1, 2, ..., n$. Then we have a rough exact sequence as follows.

$$0 \to \overline{Y_i} \xrightarrow{f_i} \overline{M},$$

where f_i is a monomorphism, for all $i = 1, 2, ..., n$. Hence we can define $f = f_i|_{Y_1 \cap Y_2 \cap ... \cap Y_n}$. Based on Proposition 3.5, $\bigcap_{i=1}^{n} Y_i$ is a rough submodule of M. Since f_i a monomorphism, we have f is also a monomorphism. Therefore, we have a rough exact sequence:

$$0 \to \overline{\bigcap_{i=1}^{n} X_i} \xrightarrow{f} \overline{M}.$$

Therefore, $\bigcap_{i=1}^{n} Y_i \in \overline{\sigma}(0, L, M)$. ■

Proposition 3.7. *Let (S, θ) be an approximation space, K_i, L_i, M_i rough modules over a rough ring R and X_i a submodules of L_i, for all $i = 1, 2, ..., n$, in (U, θ). If $Y_i \in \overline{\sigma}(K_i, L_i, M_i)$, then $\prod_{i=1}^{n} Y_i \in \overline{\sigma}(\prod_{i=1}^{n} K_i, \prod_{i=1}^{n} L_i, \prod_{i=1}^{n} M_i)$.*

Proof. Consider $Y_i \in \overline{\sigma}(K_i, L_i, M_i)$, for all $i \in \{1, 2, .., n\}$. Then we have a rough exact sequence:

$$\overline{K_i} \xrightarrow{\mu_i} \overline{X_i} \xrightarrow{\phi_i} \overline{M_i}.$$

Therefore, we can define $\mu = \prod_{i=1}^{n} f_i$ and $\phi = \prod_{i=1}^{n} \phi_i$ so that the following sequence

$$\overline{\prod_{i=1}^{n} K_i} \xrightarrow{\mu} \overline{\prod_{i=1}^{n} Y_i} \xrightarrow{\phi} \overline{\prod_{i=1}^{n} M_i}.$$

is rough exact. Hence, $\prod_{i=1}^{n} Y_i \in \overline{\sigma}(\prod_{i=1}^{n} K_i, \prod_{i=1}^{n} L_i, \prod_{i=1}^{n} M_i)$. ■

Proposition 3.8. *Let (S, θ) be an approximation space, K, L, M rough modules over a rough ring R, Y_1, Y_2 submodules of M in approximation (S, θ). If $Y_1 \in \overline{\sigma}(K, L, M)$ and $Y_2 \subseteq Y_1$, where $\overline{Y_2}$ is a direct summand $\overline{Y_1}$, then $Y_2 \in \overline{\sigma}(K, L, M)$.*

Proof. Consider $Y_1 \in \overline{\sigma}(K, L, M)$. Then the following sequence:

$$\overline{K} \xrightarrow{f} \overline{Y_1} \xrightarrow{g} \overline{M}$$

is rough exact. By assumption, $Y_2 \subseteq Y_1$, where $\overline{Y_2}$ is a direct summand $\overline{Y_1}$. Based on [9], we have the following rough exact sequence:

$$\overline{K} \to \overline{Y_2} \to \overline{M}.$$

It implies $Y_2 \in \overline{\sigma}(K, L, M)$. ■

4 Conclusion remarks

A rough X-sub-exact sequence is a generalization of rough exact sequence of rough module over rough ring. If K, L, M rough modules over the rough ring R and $\overline{\sigma}(K, L, M)$ is the set of all submodules X of L, such that the triple (K, L, M) is rough sub-exact sequence, then $\overline{\sigma}(0, L, M)$ is closed under intersection, i.e. if $X_i \in \overline{\sigma}(0, L, M)$, for all $i = 1, 2, ..., n$, then $\bigcap_{i=1}^{n} X_i \in \overline{\sigma}(0, L, M)$. Moreover, if $X_i \in \overline{\sigma}(K_i, L_i, M_i)$, then $\prod_{i=1}^{n} X_i \in \overline{\sigma}(\prod_{i=1}^{n} K_i, \prod_{i=1}^{n} L_i, \prod_{i=1}^{n} M_i)$. Furthermore, if $X \in \overline{\sigma}(K, L, M)$ and $Y \subseteq X$, where $\overline{Y}$ is a direct summand $\overline{X}$, then $Y \in \overline{\sigma}(K, L, M)$.

References

[1] Sinha, A. K., and Prakash, A., *Rough Exact Sequences of Modules*, International Journal of Applied Engineering Research, **11(4)**, 2513–2517, (2016).

[2] Sinha, A. K. and Prakash, A., *Rough Projective Modules*, Proc. of the Second Intl Conference on Advances in Applied Science and Environmental Engineering-ASEE 2014, 35–38, (2014).

[3] Sinha, A. K. and Prakash, A., *Injective Module Based on Rough Set Theory*. Cogent Mathematics, **2(1)**, 1–7, (2015).

[4] Madanshekaf, A, *Quasi-Exact Sequence and Finitely Presented Modules*, Iranian Journal of Mathematical Sciences and Informatics, **3(2)** , 49–53, (2008).

[5] Davvaz, B. and Shabani-Solt, H., *A Generalization of Homological Algebra*, J. Korean Math. Soc., **6** , 881–898,(2002).

[6] Davvaz, B. and Parnian-Garamaleky, Y.A., *A Note on Exact Sequences*, Bull. Malays. Math. Sci. Soc., **22(1)**, 53–56,(1999).

[7] Davvaz, B. *Roughness in Rings*. Information Sciences, **164**, 147–163, (2004).

[8] Wang, C. and Chen, D., *A Short Note on Some Properties of Rough Groups*, Computers and Mathematics with Applications, **59(1)**, 431–436, (2010).

[9] Fitriani, Surodjo, B., and Wijayanti, I. E., *On Sub-Exact Sequences*, Far East J. Math. Sci.(FJMS), **7**, 1055–1065, (2016).

[10] Fitriani, Wijayanti, I. E. and Surodjo, B., *On X-sub-linearly independent modules*, J. Phys. Conf. Ser., **893(1)**, 012008, (2017).

[11] Fitriani, Wijayanti, I. E. and Surodjo, B., *Generalization of U-Generator and M-Subgenerator Related to Category $\sigma[M]$*, Journal of Mathematics Research, **10(4)**, 101–106,(2018).

[12] Fitriani, Wijayanti, I. E., and Surodjo, B., *Generalization of Basis and Free Modules Relatives to a Family $\mathcal{U}$ of R-Modules*, J. Phys. Conf. Series, **1097**, 012087,(2018).

[13] Bagismaz, N. and Ozcan, A., F., *SRough Semigroups on Approximation Spaces*, International Journal of Algebra, **7**, 339–350,(2015).

[14] Neelima, C. A. and Isaac, P., *Rough Anti-homomorphism on a Rough Group*, Annals of Fuzzy Mathematics and Informatics, **6(2)**, 7987,(2014).

[15] Anvariyeh, S. M. and Davvaz, B., *On Quasi-Exact Sequences*, Bull. Korean Math. Soc., **42(1)**, 149–155,(2005).

[16] Anvariyeh, S. M. and Davvaz, B., *U-Split-Exact Sequences*, Far East J. Math. Sci.(FJMS), **4(2)**, 209–219,(2002).

[17] Adkins, W.A. and Weintraub, S.H., *Algebra "An Approach via Module Theory"*, New York: Springer-Verlag, (1992).

[18] Pawlak, Z., *Rough Sets-Theoretical Aspects of Reasoning about Data*, Dordrecht: Kluwer, (1991).

Author information

Fitriani, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Indonesia.
E-mail: `fitriani.1984@fmipa.unila.ac.id`

Indah Emilia Wijayanti, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Indonesia.
E-mail: `ind_wijayanti@ugm.ac.id`

Ahmad Faisol, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Indonesia.
E-mail: `ahmadfaisol@fmipa.ugm.ac.id`

Received: 2023-07-18
Accepted: 2024-01-03