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Soft elementary compact in soft elementary topology

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Abstract Very recently, the idea of soft topology was developed from soft elementary intersection and union. Following this work, we provided several soft topology complements: soft compactness concepts, including quasi-compact spaces, compact spaces, and sets, and locally compact spaces; and soft compactness properties. Additionally, we discussed the relationship between soft compactness and continuous functions. Finally, we concluded by proving a soft version of Baire's theorem.

1 Introduction

Molodtsov introduced the notion of soft sets in 1999 as a novel mathematical technique for dealing with uncertainty. For a list of this theory's numerous applications in the fields of economics, medicine, social sciences, and engineering, see [6, 15]. Maji et al. provided some operators for soft sets after Molodtsov [14]. Later, these ideas were refined by [1, 20], and [21].
Shabir and Naz gave a definition of soft topology in [18], which was based on the intersection and union of soft sets as in [20, 3, 5, 11, 12, 17, 19], and [22] use this notion of soft topology. In [18], soft conceptions of element, interior, closure, and separation were also defined. Two definitions of soft topology were presented by Hazra et al., who also deduced several conclusions and properties.
The definition of a soft element introduced by Das and Samanta in [10] differs from that provided in [18] and is compatible with the definition of soft subsets. Soft real numbers, soft complexes, and soft metric spaces are defined using it (see citations [8, 9, 10]). Das and Samanta develop various operations, such as elementary union, intersection, and complement, using this idea of a soft element. Aygunoglu and Aygun, as well as Zorlutuna et al., both of whom are based on the definition provided by Shabir in [18], examined the compactness.
Chiney and Samanta created a novel description of soft topology in [7] based on the elementary intersection, elementary union, and elementary complement. These soft topologies defined in the [18, 13] are distinct from this one. The soft inner element and set, limiting soft, soft closure set, soft neighborhood, soft base, soft continuous function, and soft separation axioms are defined as the necessary soft topological concept tools for continuity.
In this study, we embrace this new definition and, in order to avoid a possible paradox, we refer to the soft topology defined in [7] as soft elementary topology (or soft e-topology).
The paper was divided into five sections: In section two, we provide several well-known soft set theory results that are necessary for the paper. In section three, we define soft elementary topology, soft elementary sub-topology, as well as some additional features and findings. In the fourth part, we present soft elementary quasi-compact space, soft elementary compact space, soft elementary compact set, and their properties. The focus of the final section is the proof of soft Baire's Theorem.

2 Preliminaries

In this section, we provide definitions, characteristics, and some associated ideas related to soft sets. These terms will be used frequently throughout this paper. Let E represent the collection of parameters, and let X be the initial universe set. Consider a non-empty subset of E denoted as A.

Definition 2.1. [16] A pair (F, A) is called a soft set over X, if and only if F is a mapping of A into $\mathcal{P}(X)$.

Definition 2.2. [1, 14, 20] Let (F, A) and (G, A) be two soft sets over X.

i) (F, A) is called a soft subset of (G, A) (i.e., $(F, A)\widetilde{\subseteq}(G, A)$) if $F(\alpha)$ is a subset of $G(\alpha)$ for all $\alpha \in A$.

ii) (F, A) and (G, A) are called soft equal if (F, A) is a soft subset of (G, A) and (G, A) is a soft subset of (F, A).

iii) The complement or relative complement of a soft set (F, A) is denoted by $(F, A)^C$ and is defined as $(F, A)^C = (F^C, A)$, where $F^C(\alpha) = C_X^{F(\alpha)}$ for all $\alpha \in A$.

iv) The union of (F, A) and (G, A) is the soft set (H, A), defined by $H(\alpha) = F(\alpha) \cup G(\alpha)$ for all $\alpha \in A$ and denoted by $(F, A)\widetilde{\cup}(G, A)$.

v) The intersection of (F, A) and (G, A) is the soft set (H, A), defined by $H(\alpha) = F(\alpha) \cap G(\alpha)$ for all $\alpha \in A$ and denoted by $(F, A)\widetilde{\cap}(G, A)$.

Example 2.3. Putting $X = \{x, y, z\}, A = \{\alpha, \beta\}, (F, A)$ and (G, A) such that $F(\alpha) = \{x, y\}$, $F(\beta) = \{x, z\}, G(\alpha) = \{y, z\}, G(\beta) = \{x\}$. Then,

***)** $(F, A)\widetilde{\cup}(G, A) = (H, A)$ such that $H(\alpha) = X, H(\beta) = \{x, z\}$.

****)** $(F, A)\widetilde{\cap}(G, A) = (J, A)$ where $J(\alpha) = \{y\}, H(\beta) = \{x\}$.

*****)** $(F, A)^c = (F^C, A)$ with $F^c(\alpha) = \{z\}, F^C(\beta) = \{y\}$.

Definition 2.4. [18] Let Y be a subset of X. We denote $(\widetilde{Y}, A)$ as the soft set (F, A) such that $F(\alpha) = Y$ for all $\alpha \in A$.
If $Y = \emptyset$, the soft set (F, A) is called a null soft set, denoted by $(\widetilde{\Phi}, A)$.
If $Y = X$, the soft set (F, A) is called an absolute soft set, denoted by $(\widetilde{X}, A)$.

Definition 2.5. [9]

i) A soft element of $(\widetilde{X}, A)$ is a function $\widetilde{x}$ defined on A with values in the set X. A soft element $\widetilde{x}$ of $(\widetilde{X}, A)$ is said to belong to a soft set (F, A) over X, denoted by $\widetilde{x}\widetilde{\in}(F, A)$, if $\widetilde{x}(\alpha) \in F(\alpha)$ for all $\alpha \in A$. Therefore, if (F, A) is such that $F(\alpha) \neq \emptyset$ for all $\alpha \in A$, we have $F(\alpha) = \{\widetilde{x}(\alpha) : \widetilde{x}\widetilde{\in}(F, A)\}$ for all $\alpha \in A$.

ii) $S(\widetilde{X})$ is the collection of $(\widetilde{\Phi}, A)$ and the soft sets (F, A) such that $(F, A)(\alpha) \neq \emptyset$ for all $\alpha \in A$.

iii) The collection of all soft elements of a soft set (F, A) is denoted by $SE(F, A)$.

vi) For a collection $\mathcal{B}$ of soft elements of $(\widetilde{X}, A)$, we denote by $SS(\mathcal{B})$ the soft set (F, A) such that $F(\alpha) = \{\widetilde{x}(\alpha), \widetilde{x} \in \mathcal{B}\}$.

Remark 2.6. Let Y be a nonempty subset of X and $(F, A) \in S(\widetilde{X})$.

***)** It is evident that if $Y \neq \emptyset$, then $(\widetilde{Y}, A) \in S(\widetilde{X})$ and $(\widetilde{Y}, A) \neq (\widetilde{\Phi}, A)$.

****)** We designate by $S(\widetilde{Y})$ the collection of soft subsets (F, A) of $(\widetilde{Y}, A)$ such that $(F, A) = (\widetilde{\Phi}, A)$ or $(F, A)(\alpha) \neq \emptyset$ for all $\alpha \in A$.

*****)** If $(F, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{X})$ then $(F, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{Y})$.
Indeed, $(F, A)\widetilde{\cap}(\widetilde{Y}, A)\widetilde{\subset}(\widetilde{Y}, A)$, and $(F, A) = (\widetilde{\Phi}, A)$ or $(F, A)(\alpha) \neq \emptyset$ for all $\alpha \in A$.
Hence $(F, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{Y})$.

Definition 2.7. [9] For any two soft sets $(F, A), (G, A) \in S(\widetilde{X})$, we define the following operations:

i) The elementary union of (F, A) and (G, A) is denoted by $(F, A) \Cup (G, A)$ and defined as $(F, A) \Cup (G, A) = SS(SE(F, A) \cup SE(G, A))$.

ii) The elementary intersection of (F, A) and (G, A) is denoted by $(F, A) \Cap (G, A)$ and defined as $(F, A) \Cap (G, A) = SS(SE(F, A) \cap SE(G, A))$.

iii) The elementary complement of (F, A) is denoted by $(F, A)^{\mathbb{C}}$ and defined as $(F, A)^{\mathbb{C}} = SS(\mathcal{B})$, where $\mathcal{B} = \{\widetilde{x}\widetilde{\in}(\widetilde{X}, A) : \widetilde{x} \in (F, A)^C\}$.

Notation: Let Y be a nonempty subset of X and let $(Z, A) \in S(\widetilde{Y})$. We denote by $(Z, A)^C_Y$ the soft set (W, A) over Y, where $W(\alpha) = Y \setminus Z(\alpha)$ for all $\alpha \in A$. Also,we denote by $(Z, A)^{\mathbb{C}}_Y$ the soft set of the soft elements $\widetilde{x}$ such that $\widetilde{x}\widetilde{\in}(Z, A)^C_Y$.

Example 2.8. Putting $X = \{x, y, z\}$.
Then, we have: $(\widetilde{X}, A) = SS(\{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_6, \widetilde{x}_7, \widetilde{x}_8, \widetilde{x}_9\})$ such that:

$$\widetilde{x}_1(\alpha) = \widetilde{x}_1(\beta) = x, \quad \widetilde{x}_2(\alpha) = \widetilde{x}_2(\beta) = y, \quad \widetilde{x}_3(\alpha) = \widetilde{x}_3(\beta) = z,$$
$$\widetilde{x}_4(\alpha) = x,\ \widetilde{x}_4(\beta) = y, \quad \widetilde{x}_5(\alpha) = x,\ \widetilde{x}_5(\beta) = z, \quad \widetilde{x}_6(\alpha) = y,\ \widetilde{x}_6(\beta) = x,$$
$$\widetilde{x}_7(\alpha) = y,\ \widetilde{x}_7(\beta) = z, \quad \widetilde{x}_8(\alpha) = z,\ \widetilde{x}_8(\beta) = x, \quad \widetilde{x}_9(\alpha) = z,\ \widetilde{x}_9(\beta) = y.$$

Let $(F, A), (G, A) \in S(\widetilde{X})$ such that
$F(\alpha) = \{x, z\}, F(\beta) = \{y, z\}, G(\alpha) = \{x, y\}, G(\beta) = \{x\}$. Then, we have:
$(F, A) = SS(\{\widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_9\})$ and $(G, A) = SS(\{\widetilde{x}_1, \widetilde{x}_6\})$. Hence, we can compute:

***)** $(F, A) \Cup (G, A) = SS(\{\widetilde{x}_1, \widetilde{x}_3, \widetilde{x}_4, \widetilde{x}_5, \widetilde{x}_6, \widetilde{x}_9\})$ and $(F, A) \Cap (G, A) = (\widetilde{\Phi}, A)$.

****)** Since $(F, A)\widetilde{\cap}(G, A) = (H, A)$ where $H(\alpha) = \{x\}, H(\beta) = \emptyset$, we deduce that $(F, A) \Cup (G, A) = (F, A)\widetilde{\cup}(G, A) = (\widetilde{X}, A)$, but $(F, A) \Cap (G, A) \neq (F, A)\widetilde{\cap}(G, A)$.

*****)** $(F, A)^C = (F, A)^{\mathbb{C}} = SS(\{\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_6, \widetilde{x}_8\})$.

Definition 2.9. [7] Let τ be a collection of soft sets of $S(\widetilde{X})$. Then τ is called a soft e-topology on $(\widetilde{X}, A)$ if the following conditions hold:

i) $(\widetilde{\Phi}, A)$ and $(\widetilde{X}, A)$ belong to τ,

ii) the elementary union of any number of soft sets in τ belongs to τ,

iii) the elementary intersection of two soft sets in τ belongs to τ.

The triplet $(\widetilde{X}, \tau, A)$ is called a soft e-topological space. A member of τ is called a soft e-open set in $(\widetilde{X}, \tau, A)$.

Definition 2.10. [7] A soft set $(F, A) \in S(\widetilde{X})$ is called a soft e-closed set in $(\widetilde{X}, \tau, A)$ if its relative complement $(F, A)^C$ belongs to $S(\widetilde{X})$ and $(F, A)^{\mathbb{C}}$ belongs to τ.

Proposition 2.11. *[7]*

i) $(\widetilde{\Phi}, A)$ *and* $(\widetilde{X}, A)$ *are soft e-closed sets in* $(\widetilde{X}, \tau, A)$.

ii) *The arbitrary elementary intersection of soft e-closed sets is a soft e-closed set.*

Remark 2.12. [7] In general, the elementary union of two soft e-closed sets is not soft e-closed.

Definition 2.13. [7] Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space and $(F, A) \in S(\widetilde{X})$. Then, the soft e-closure of (F, A), denoted by $\overline{(F, A)}$ is defined as the elementary intersection of all soft e-closed super sets of (F, A).

Definition 2.14. [7] Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space. A soft element $\widetilde{x} \in (\widetilde{X}, A)$ is called a limiting soft element of a soft set $(F, A) \in S(\widetilde{X})$ if, for all $(G, A) \in \tau$ and for any $\alpha \in A, \widetilde{x}(\alpha) \in G(\alpha)$. This implies that $F(\alpha) \cap [G(\alpha) \setminus \widetilde{x}(\alpha)] \neq \emptyset$.

Definition 2.15. [7] Let $(\widetilde{X}, \tau, A)$ be a soft topological space, and $(F, A) \in S(\widetilde{X}$. A soft element $\widetilde{x}\widetilde{\in}(F, A)$ is called an interior soft element of (F, A) if there exists $(G, A) \in \tau$ such that $\widetilde{x}\widetilde{\in}(G, A)\widetilde{\subset}(F, A)$.
The interior of a soft set (F, A), denoted by $Int(F, A)$, is defined as:

$$Int(F, A) = \{\widetilde{x}\widetilde{\in}(F, A) : \widetilde{x}\widetilde{\in}(G, A)\widetilde{\subset}(F, A) \text{ for some } (G, A) \in \tau\}.$$

$SS[Int(F, A)]$ is called the soft interior of (F, A) and is denoted by $\overbrace{(F, A)}^{\circ}$.

Definition 2.16. [7] Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space. Then, $(\widetilde{\Phi}, A) \neq (F, A) \in S(\widetilde{X})$ is a soft neighbourhood (soft nbd, for short) of the soft element $\widetilde{x}$ if there exists a soft set $(G, A) \in \tau$ such that $\widetilde{x}\widetilde{\in}(G, A)\widetilde{\subset}(F, A)$.

Example 2.17. Let $X = \{a, b, c, d\}, A = \{\alpha, \beta\}$.
Set: $\tau = \{(\widetilde{\Phi}, A), (\widetilde{X}, A), (F_1, A), (F_2, A), (F_3, A), (F_4, A)\}$, such that
$F_1(\alpha) = \{a\}, F_1(\beta) = \{b\}, F_2(\alpha) = \{b, c\}, F_2(\beta) = \{c, d\}$,
$F_3(\alpha) = \{a, b, c\},\ F_3(\beta) = \{b, c, d\},\ F_4(\alpha) = X, F_4(\beta) = \{b, c, d\}$.
Then, $(\widetilde{X}, \tau, A)$ is a soft e-topological space.
The collection of e-closed sets is $\{(\widetilde{\Phi}, A), (\widetilde{X}, A), (F_1, A)^{\mathbb{C}}, (F_2, A)^{\mathbb{C}}, (F_3, A)^{\mathbb{C}}, \}$.
We remark that $(F_4, A)^C$ is not a soft e-closed set, since $(F_4, A)^C \in S(\widetilde{X})$.
Now, let $(F, A), (G, A)$ be such that $F(\alpha) = F(\beta) = \{c\}, G(\alpha) = \{a, b\}, G(\beta) = \{b\}$. Then,

$\overline{(F, A)} = (F_1, A)^{\mathbb{C}}, \overbrace{(G, A)}^{0} = (F_1, A)$, and (G, A) is a soft nbd of $(\widetilde{x})$ where $\widetilde{x}(\alpha) = a, \widetilde{x}(\beta) = b$.

Definition 2.18. [7] Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space. Let $\widetilde{x}, \widetilde{y} \in (\widetilde{X}, A)$ such that $\widetilde{x}(\alpha) \neq \widetilde{y}(\alpha)$, for all $\alpha \in A$. Then, if there exists $(F, A), (G, A) \in \tau$, such that $\widetilde{x} \in (F, A), \widetilde{y} \in (G, A)$ and $(F, A)\widetilde{\cap}(G, A) = (\widetilde{\Phi}, A)$, then $(\widetilde{X}, \tau, A)$ is called a soft $e - T_2$ space, or soft e-Hausdorff space.

Definition 2.19. [7] A soft e-topological space $(\widetilde{X}, \tau, A)$ is called a soft e-regular space if, for any soft closed set (F, A) and for any soft element $\widetilde{x}$ such that $\widetilde{x}(\alpha) \notin (F, A)(\alpha)$ for all $\alpha \in A$, there exist $(G, A), (H, A) \in \tau$ such that $(F, A)\widetilde{\subset}(G, A), \widetilde{x}\widetilde{\in}(H, A)$, and $(F, A) \Cap (G, A) = (\widetilde{\Phi}, A)$.

Definition 2.20. [7] A soft e-topological space $(\widetilde{X}, \tau, A)$ is called a soft e-normal space if, for any two soft closed sets (F, A) and (G, A) such that $(F, A)\widetilde{\cap}(G, A) = (\widetilde{\Phi}, A)$, there exist $(U, A), (V, A) \in \tau$ such that $(F, A)\widetilde{\subset}(U, A)$, $(G, A)\widetilde{\subset}(V, A)$, and $(U, A) \Cap (V, A) = (\widetilde{\Phi}, A)$.

Definition 2.21. [7] Let X, Y be two non-empty sets and $\{f_\alpha : X \to Y, \alpha \in A\}$ be a collection of functions. Then, a function $f : SE(\widetilde{X}) \to SE(\widetilde{Y})$ defined by $[f(\widetilde{x})](\alpha) = f_\alpha(x(\alpha))$, for all $\alpha \in A$ is called a soft function.

Definition 2.22. [7] Let $f : SE(\widetilde{X}) \to SE(\widetilde{Y})$ be a soft function. Then,

i) The image of a soft set (F, A) over X under the soft function f, denoted by $f[(F, A)]$, is defined as $f[(F, A)] = SS\{f(SE(F, A))\}$, i.e., $f[(F, A)](\alpha) = f_\alpha(F(\alpha))$ for all $\alpha \in A$.

ii) The inverse image of a soft set (G, A) over Y under the soft function f, denoted by $f^{-1}[(G, A)]$, is defined as $f^{-1}[(G, A)] = SS\{f^{-1}(SE(G, A))\}$, i.e., $f^{-1}[(G, A)](\alpha) = f_\alpha^{-1}(G(\alpha))$ for all $\alpha \in A$.

Definition 2.23. [7] Let $(\widetilde{X}, \tau, A)$ and $(\widetilde{Y}, \sigma, A)$ be two soft e-topological spaces, and let $f : SE(\widetilde{X}) \to SE(\widetilde{Y})$ be a soft function.
$f : (\widetilde{X}, \tau, A) \to (\widetilde{Y}, \sigma, A)$ is called soft e-continuous at $\widetilde{x}_0 \in (\widetilde{X}, A)$ if, for every $(V, A) \in \sigma$ such that $f(\widetilde{x}_0)\widetilde{\in}(V, A)$, there exists $(U, A) \in \tau$ such that $\widetilde{x}_0\widetilde{\in}(U, A)$ and $f(U, A)\widetilde{\subseteq}(V, A)$.
f is called soft e-continuous on $(\widetilde{X}, \tau, A)$ if it is soft e-continuous at each soft element $\widetilde{x}_0 \in (\widetilde{X}, A)$.

Proposition 2.24. *[7] Let $(\widetilde{X}, \tau, A), (\widetilde{Y}, \sigma, A)$ be two soft e-topological spaces and $f : SE(\widetilde{X}) \to SE(\widetilde{Y})$ be a soft function. Then, f is soft e-continuous on $(\widetilde{X}, \tau, A)$, if and only if for any soft e-open set $(U, A) \in S(\widetilde{Y})$ in $(\widetilde{Y}, \sigma, A)$, $f^{-1}(U, A)$ is a soft e-open set in $(\widetilde{X}, \tau, A)$.*

3 Soft elementary topology

Based on the definition 2.9, we will talk about the notion of soft sub-e-topological space, and we will explore their properties. First, we will examine the most important result.

Theorem 3.1. *Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space such that for all $(O_1, A), (O_2, A) \in \tau$, we have $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Let Y be a nonempty subset of X such that for all $(O, A) \in \tau$, we have $(O, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{X})$. Then, it holds that the collection $\tau_Y = \{(O_Y, A) = (O, A) \Cap (\widetilde{Y}, A), (O, A) \in \tau\}$ defines a soft e-topology for $(\widetilde{Y}, A)$.*

Proof. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space, and let Y be a nonempty subset of X. We will show that τ_Y satisfies the conditions of a soft e-topology.
On one hand, since $(\widetilde{\Phi}, A) = (\widetilde{\Phi}, A) \Cap (\widetilde{Y}, A)$ and $(\widetilde{Y}, A) = (\widetilde{X}, A) \Cap (\widetilde{Y}, A)$, then $(\widetilde{\Phi}, A)$ and $(\widetilde{Y}, A)$ belong to τ_Y.
On the other hand, suppose that $\{(O^i_Y, A) = (O^i, A) \Cap (\widetilde{Y}, A), i \in I\}$ is a family of soft sets in τ_Y. Then $\{(O^i, A), i \in I\}$ is a family of soft sets in τ, and $(O, A) = \underset{i\in I}{\Cup}(O^i, A) \in \tau$. It follows that:

$$\begin{aligned}
\underset{i\in I}{\Cup}(O^i_Y, A) &= \underset{i\in I}{\Cup}[(O^i, A) \Cap (\widetilde{Y}, A)] \\
&= \underset{i\in I}{\widetilde{\cup}}[(O^i, A) \Cap (\widetilde{Y}, A)] \\
&= \underset{i\in I}{\widetilde{\cup}}[(O^i, A)\widetilde{\cap}(\widetilde{Y}, A)] \\
&= [\underset{i\in I}{\widetilde{\cup}}(O^i, A)]\widetilde{\cap}(\widetilde{Y}, A) \\
&= [\underset{i\in I}{\Cup}(O^i, A)]\widetilde{\cap}(\widetilde{Y}, A) \\
&= (O, A)\widetilde{\cap}(\widetilde{Y}, A) \\
&= (O, A) \Cap (\widetilde{Y}, A) \in \tau_Y.
\end{aligned}$$

Finally, let $(O^1_Y, A) = (O^1, A) \Cap (\widetilde{Y}, A)$ and $(O^2_Y, A) = (O^2, A) \Cap (\widetilde{Y}, A)$ be two soft sets in τ_Y. Then, (O^1, A) and (O^2, A) are two soft sets in τ, and $(O, A) = (O^1, A) \Cap (O^2, A) \in \tau$. It follows that:

$$\begin{aligned}
(O^1_Y, A) \Cup (O^2_Y, A) &= [(O^1, A) \Cap (\widetilde{Y}, A)] \Cap [(O^2, A) \Cap (\widetilde{Y}, A)] \\
&= [(O^1, A)\widetilde{\cap}(\widetilde{Y}, A)] \Cap [(O^2, A)\widetilde{\cap}(\widetilde{Y}, A)].
\end{aligned}$$

If $[(O^1, A)\widetilde{\cap}(\widetilde{Y}, A)] \Cap [(O^2, A)\widetilde{\cap}(\widetilde{Y}, A)] = (\widetilde{\Phi}, A)$ then $(O^1_Y, A) \Cup (O^2_Y, A) \in \tau_Y$, else we obtain

$$\begin{aligned}
(O^1_Y, A) \Cup (O^2_Y, A) &= [(O^1, A)\widetilde{\cap}(\widetilde{Y}, A)]\widetilde{\cap}[(O^2, A)\widetilde{\cap}(\widetilde{Y}, A)] \\
&= [(O^1, A)\widetilde{\cap}(O^2, A)]\widetilde{\cap}(\widetilde{Y}, A) \\
&= (O, A) \Cap (\widetilde{Y}, A) \in \tau_Y.
\end{aligned}$$

Thus, the collection τ_Y define a soft topology for $(\widetilde{Y}, A)$. □

Based to the above theorem, we will now talk about sub-e-topological space.

Definition 3.2. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space such that for all $(O_1, A), (O_2, A) \in \tau$, we have $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Let Y be a nonempty subset of X such that for all $(O, A) \in \tau$, we have $(O, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{X})$. We define $\tau_Y = \{(O_Y, A) = (O, A) \Cap (\widetilde{Y}, A), (O, A) \in \tau\}$. The triplet $(\widetilde{Y}, \tau_Y, A)$ is called a soft sub-e-topological space of $(\widetilde{X}, \tau, A)$, and τ_Y is referred to as the soft sub-e-topology of τ. The members of τ_Y are called soft e_Y-open sets in $(\widetilde{Y}, \tau_Y, A)$.

Definition 3.3. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space such that for all $(O_1, A), (O_2, A) \in \tau$, we have $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Now, let $(\widetilde{Y}, \tau_Y, A)$ be a soft sub e-topological space of $(\widetilde{X}, \tau, A)$, and consider $(Z, A) \in S(\widetilde{Y})$.
The soft set (Z, A) is referred to as a soft e_Y-closed set in $(\widetilde{Y}, \tau_Y, A)$ if $(Z, A)^{\mathbb{C}}_Y \in S(\widetilde{Y})$ and $(Z, A)^{\mathbb{C}}_Y \in \tau_Y$.

Example 3.4. Let $X = \{a, b, c, d\}$, $A = \{\alpha, \beta\}$, and $\tau = \{(\widetilde{\Phi}, A), (\widetilde{X}, A), (F, A), (G, A), (H, A)\}$, where:
$F(\alpha) = \{a\}, F(\beta) = \{c, d\}, G(\alpha) = \{c, d\}, G(\beta) = \{a\}, H(\alpha) = \{a, c, d\}, H(\beta) = \{a, c, d\}$.

Then, $(\widetilde{X}, \tau, A)$ is a soft e-topological space.
Consider $Y = \{a, c\}$, and define $\tau_Y = \{(\widetilde{\Phi}, A), (\widetilde{Y}, A), (F_Y, A), (G_Y, A)\}$, where:
$F_Y(\alpha) = \{a\}$, $F_Y(\beta) = \{c\}$, $G_Y(\alpha) = \{c\}$, $G_Y(\beta) = \{a\}$.
Thus, τ_Y is a soft topology on $(\widetilde{Y}, A)$.

In the next proposition, we will describe the soft e_Y-closed sets in the sub-e-topological space $(\widetilde{Y}, \tau_Y, A)$.

Proposition 3.5. *Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space such that for all $(O_1, A), (O_2, A) \in \tau$, we have $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Consider $(\widetilde{Y}, \tau_Y, A)$ as a soft sub-e-topological space of $(\widetilde{X}, \tau, A)$.*
If the soft set (Z, A) is a soft e_Y-closed set in $(\widetilde{Y}, \tau_Y, A)$, then there exists a soft e-closed set (F, A) in $(\widetilde{X}, \tau, A)$ such that $(Z, A) = (F, A) \Cap (\widetilde{Y}, A)$.

Proof. Let (Z, A) be a soft e_Y-closed set in $(\widetilde{Y}, \tau_Y, A)$. There are two cases to be considered:

Case (1): If $(Z, A) = (\widetilde{\Phi}, A)$, then $(F, A) = (\widetilde{\Phi}, A)$.

Case (2): If $(Z, A) \neq (\widetilde{\Phi}, A)$, then there exists $(O, A) \in \tau$ such that
$(Z, A)^{\complement}_Y = (Z, A)^C_Y = (O, A) \Cap (\widetilde{Y}, A)$.
Hence, for all $\alpha \in A$, we have $Y \setminus Z(\alpha) = Y \cap O(\alpha)$.

Then, it follows that $Z(\alpha) = O(\alpha)^C \cap Y$ for all $\alpha \in A$. Since $(Z, A) \neq (\widetilde{\Phi}, A)$, we get that $(O, A)^C \in S(\widetilde{X})$, $(O, A)^C(\alpha) \neq \emptyset$ for all $\alpha \in A$, and $(Z, A)^C_Y(\alpha) = O(\alpha)^C \cap Y$.
We set $(F, A) = (O, A)^C$, and thus $(Z, A)^C_Y = (F, A)\widetilde{\cap}(\widetilde{Y}, A) = (F, A) \Cap (\widetilde{Y}, A)$.

□

Proposition 3.6. *Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space, (F, A) a soft subset of $(\widetilde{X}, A)$, and $\widetilde{x}\widetilde{\in}(\widetilde{X}, A)$. If $\widetilde{x}$ is a soft limiting element of (F, A) for all $(G, A) \in \tau$, where $\widetilde{x}\widetilde{\in}(G, A)$ implies that there exists $\widetilde{y}\widetilde{\in}SE(\widetilde{X})$ such that $\widetilde{y} \neq \widetilde{x}$ and $\widetilde{y}\widetilde{\in}(F, A) \Cap (G, A)$.*

Proof. Let $(\widetilde{X}, \tau, A)$ be a soft topological space, (F, A) a soft subset of $(\widetilde{X}, A)$, and $\widetilde{x}$ is a soft limiting element of (F, A). Then, for any $(G, A) \in \tau$ and for any $\alpha \in A$, $\widetilde{x}(\alpha) \in G(\alpha)$ implies that $F(\alpha) \cap [G(\alpha) \setminus \{\widetilde{x}(\alpha)\}] \neq \emptyset$. Hence, there exists $a_\alpha \in X$ such that, $\widetilde{x}(\alpha) \notin F(\alpha) \cap G(\alpha)$. Let $\widetilde{y} \in (\widetilde{X}, A)$ such that $\widetilde{y}(\alpha) = a_\alpha$ for all $\alpha \in A$, then $\widetilde{y} \neq \widetilde{x}$ and $\widetilde{y}\widetilde{\in}(F, A) \Cap (G, A)$. □

Now, we show that an e-Hausdorff space's sub-e-topological space is also an e-Hausdorff space.

Proposition 3.7. *Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space such that for all $(O_1, A), (O_2, A) \in \tau$ we have, $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$, and let $(\widetilde{Y}, \tau_Y, A)$ be a soft sub-e-topological space of $(\widetilde{X}, \tau, A)$. If $(\widetilde{X}, \tau, A)$ is a soft e-Hausdorff space then $(\widetilde{Y}, \tau_Y, A)$ is a soft e-Hausdorff space.*

Proof. Let $\widetilde{x}, \widetilde{y} \in (\widetilde{Y}, A)$ such that $\widetilde{x}(\alpha) \neq \widetilde{y}(\alpha)$ for all $\alpha \in A$, then $\widetilde{x}, \widetilde{y} \in (\widetilde{Y}, A)$ and since $(\widetilde{X}, \tau, A)$ is a soft e-Hausdorff space there exist $(F, A), (G, A) \in \tau$ such that $\widetilde{x} \in (F, A), \widetilde{y} \in (G, A)$ and $(F, A)\widetilde{\cap}(G, A) = (\widetilde{\Phi}, A)$.
Noting that $\widetilde{x} \in (F_Y, A) = (F, A) \Cap (\widetilde{Y}, A), \widetilde{y} \in (G_Y, A) = (G, A) \Cap (\widetilde{Y}, A)$
and $(F_Y, A)\widetilde{\cap}(G_Y, A) = (\widetilde{\Phi}, A)$. Then, $(\widetilde{Y}, \tau_Y, A)$ is a soft e-Hausdorff space. □

4 Soft e-compact space and soft e-compact set

In this section, we delve into the fundamental definitions and properties of soft e-quasi-compact space, e-compact space, and sets. We commence by introducing the concept of an "e-open cover."

Definition 4.1. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space, and $(F, A) \in S(\widetilde{X})$. Consider a family of soft e-open sets $\{(O_i, A)\}_{i \in I}$ in $(\widetilde{X}, \tau, A)$.

i) $\{(O_i, A)\}_{i\in I}$ is called a soft e-open cover of $(\widetilde{X}, A)$ if $(\widetilde{X}, A) = \underset{i\in I}{\Cup}(O_i, A)$.

ii) $\{(O_i, A)\}_{i\in I}$ is called a soft e-open cover of (F, A) if $(F, A)\widetilde{\subseteq} \underset{i\in I}{\Cup}(O_i, A)$.

In the following definition, we introduce the concept of a soft e-quasi-compact space.

Definition 4.2. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space. $(\widetilde{X}, \tau, A)$ is called a soft e-quasi-compact space if every soft e-open cover of $(\widetilde{X}, A)$ has a finite sub-e-cover of $(\widetilde{X}, A)$.

In the following theorem, we introduce the necessary condition for a soft e-topological space to have a soft e-quasi-compact space.

Theorem 4.3. *Let $(\widetilde{X}, \tau, A)$ be a soft e-quasi-compact space. For every family $\{(F_i, A)\}_{i\in I}$ of soft e-closed sets such that $\underset{i\in I}{\Cap}(F_i, A) = (\widetilde{\Phi}, A)$, we can select a finite subfamily $\{(F_i, A)\}_{i\in I_0\subset I}$ such that $\underset{i\in I_0}{\Cap}(F_i, A) = (\widetilde{\Phi}, A)$.*

Proof. Assume that $(\widetilde{X}, \tau, A)$ is a soft e-quasi-compact space, and let $\{(F_i, A)\}_{i\in I}$ be a family of soft e-closed sets such that $\underset{i\in I}{\Cap}(F_i, A) = (\widetilde{\Phi}, A)$. Consequently, $\{(F_i, A)^{\mathbb{C}}\}_{i\in I}$ forms a family of soft e-open sets, and we have $\underset{i\in I}{\Cup}(F_i, A)^{\mathbb{C}} = (\widetilde{X}, A)$.
Since $(\widetilde{X}, \tau, A)$ is quasi e-compact, there exists $I_0 \subset I$ such that $\underset{i\in I_0}{\Cup}(F_i, A)^{\mathbb{C}} = (\widetilde{X}, A)$.
Consequently, $\underset{i\in I_0}{\Cap}(F_i, A) = (\widetilde{\Phi}, A)$. □

Remark 4.4. Since the complement of a soft e-open set is not a soft e-closed set in general, the converse of Theorem 4.3 is not generally true. This is demonstrated in the following counterexample.
Consider $X =]1, +\infty[$, $A = [1, +\infty[$, $I = [1, +\infty[$, and $\tau = \{(\widetilde{\Phi}, A)\} \cup \{(O_i, A) \mid i \in I\}$, where:

$$(O_i, A)(\alpha) = \left]\frac{1 + i\alpha}{i + \alpha}, +\infty\right]$$

for all $\alpha \in A$. It is clear that $(\widetilde{\Phi}, A) \in \tau$ and $(\widetilde{X}, A) = (O_1, A) \in \tau$.
For any collection $I_0 \subset I$, we have $\underset{i\in I_0}{\Cup}(O_i, A) = (O_{i_0}, A)$, where $i_0 = \min\{i \mid i \in I_0\}$. However, for all $i, j \in I$ such that $i < j$, we have $(O_i, A) \Cap (O_j, A) = (O_j, A)$. Therefore, $(\widetilde{X}, \tau, A)$ is a soft e-topological space.
Now, for all $i \in I$, we have $(O_i, A)(1) = X$, which means that $(O_i, A)(1)^c = \widetilde{\Phi}$. In other words, $(O_i, A)^c \notin S(\widetilde{X})$ for all $i \in]1, +\infty[$. The collection of soft e-closed sets is only $\{(\widetilde{\Phi}, A), (\widetilde{X}, A)\}$, which is finite. However, the family $\{(O_i, A)\}_{i\in]1,+\infty[}$ is a soft e-open cover of $(\widetilde{X}, A)$, and we cannot extract a finite e-open subcover of $(\widetilde{X}, A)$.

The concept of a soft e-compact space is introduced in the following definition.

Definition 4.5. Let $(\widetilde{X}, \tau, A)$ be a soft e-topological space. $(\widetilde{X}, \tau, A)$ is called a soft e-compact space if it is a soft quasi e-compact space and a soft e-Hausdorff space.

To establish a necessary condition for a soft e-topological space to be a soft e-compact space, we introduce the following theorem.

Theorem 4.6. *Let $(\widetilde{X}, \tau, A)$ be a soft e-compact space, and $\{(F_i, A)\}_{i=1}^{\infty}$ be a family of decreasing soft e-closed sets. Then, $\Cap_{i=1}^{\infty}(F_i, A) \neq (\widetilde{\Phi}, A)$.*

Proof. Assume that $\Cap_{i=1}^{\infty}(F_i, A) = (\widetilde{\Phi}, A)$, then $\Cup_{i=1}^{\infty}(F_i, A)^{\mathbb{C}} = (\widetilde{X}, A)$. Since $(\widetilde{X}, A)$ is a soft e-compact space, and $\{(F_i, A)^{\mathbb{C}}\}_{i=1}^{\infty}$ is a family of soft e-open sets, we can extract a decreasing finite subfamily $\{(F_{i_k}, A)^{\mathbb{C}}\}_{k=1}^{n}$ such that $\Cup_{k=1}^{n}(F_{i_k}, A)^{\mathbb{C}} = (\widetilde{X}, A)$.
Then, $(F_{i_n}, A) = \Cap_{k=1}^{n}(F_{i_k}, A)^{\mathbb{C}} = (\widetilde{\Phi}, A)$, which is a contradiction. □

The concept of a soft e-compact set is introduced in the following definition.

Definition 4.7. Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff space, and $(F, A) \in S(\widetilde{X})$ such that $(F, A)^C \in S(\widetilde{X})$. (F, A) is called a soft e-compact set if all soft e-open cover of (F, A) has a finite sub-e-cover of (F, A).

Example 4.8. Let $X = \mathbb{R}$, $A = \{\alpha, \beta\}$, and τ be the collection of soft sets $(O, A) \in S(\widetilde{X})$ such that $(O, A) = (\widetilde{\Phi}, A)$ or for all $\widetilde{x}\widetilde{\in}(F, A)$, there exist $r_\alpha > 0$ and $r_\beta > 0$ such that $]\widetilde{x}(\alpha) - r_\alpha, \widetilde{x}(\alpha) + r_\alpha[\subset O(\alpha)$ and $]\widetilde{x}(\beta) - r_\beta, \widetilde{x}(\beta) + r_\beta[\subset O(\beta)$. It is obvious that $(\widetilde{X}, \tau, A)$ is a soft e-topological space, and $\tau_\alpha = \{O(\alpha), (O, A) \in \tau\}$, $\tau_\beta = \{O(\beta), (O, A) \in \tau\}$ are two crisp topologies of X, equivalent to the topology of the metric space $(\mathbb{R}, |.|)$.
Now, let $\widetilde{x}$ and $\widetilde{y}$ be two soft elements of $(\widetilde{X}, A)$ such that $\widetilde{x}(\alpha) \neq \widetilde{y}(\alpha)$, $\widetilde{x}(\beta) \neq \widetilde{y}(\beta)$,
$r_\alpha = |\widetilde{x}(\alpha) - \widetilde{y}(\alpha)|$ and $r_\beta = |\widetilde{x}(\beta) - \widetilde{y}(\beta)|$.
Then, $\left]\widetilde{x}(\alpha) - \frac{r_\alpha}{3}, \widetilde{x}(\alpha) - \frac{r_\alpha}{3}\right[\cap \left]\widetilde{y}(\alpha) - \frac{r_\alpha}{3}, \widetilde{y}(\alpha) - \frac{r_\alpha}{3}\right[= \emptyset$, and $\left]\widetilde{x}(\beta) - \frac{r_\beta}{3}, \widetilde{x}(\beta) - \frac{r_\beta}{3}\right[\cap \left]\widetilde{y}(\beta) - \frac{r_\beta}{3}, \widetilde{y}(\beta) - \frac{r_\beta}{3}\right[= \emptyset$. Hence, there exists (F, A), a soft e-neighborhood of $\widetilde{x}$, and (G, A), a soft e-neighborhood of $\widetilde{y}$, such that $(F, A)\widetilde{\cap}(G, A) = (\widetilde{\Phi}, A)$.
Therefore, it is a soft e-Hausdorff space. So, $(\widetilde{X}, \tau, A)$ is not a soft e-compact space, since we can't extract a soft e-subcover of the e-cover open $\{(F_n, A), n \in \mathbb{N})\}$ (where $F_n(\alpha) = F_n(\beta) =]-n, n[$), but the soft set $(\widetilde{[-1, 1]}, A)$ is a soft e-compact set.

We demonstrate that the sub-e-compact space is a soft e-compact set in the following theorem.

Theorem 4.9. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff space. Assume that for all $(O_1, A), (O_2, A) \in \tau$ we have: $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Let Y be a nonempty subset of X such that $Y \neq X$, and for all $(O, A) \in \tau$ we have: $(O, A)\widetilde{\cap}(\widetilde{Y}, A) \in S(\widetilde{X})$. If $(\widetilde{Y}, \tau_Y, A)$ is a soft e-compact space, then $(\widetilde{Y}, A)$ is a soft e-compact set in $(\widetilde{X}, \tau, A)$.*

Proof. Since $Y \neq \emptyset, Y \neq X$, we obtain $(\widetilde{Y}, A), (\widetilde{Y}, A)^C \in S(\widetilde{X})$. Now, let $\{(O_i, A), i \in I\} \subset \tau$ such that $(\widetilde{Y}, A)\widetilde{\subseteq} \Cup_{i\in I} (O_i, A)$. Then, $(\widetilde{Y}, A) = (\widetilde{Y}, A) \Cap [\Cup_{i\in I}(O_i, A)] = \Cup_{i\in I}[(\widetilde{Y}, A) \Cap (O_i, A)]$. Hence, the family $\{(\widetilde{Y}, A)\Cap(O_i, A), i \in I\}$ is a soft e_Y-open cover of $(\widetilde{Y}, A)$, and since $(\widetilde{Y}, \tau_Y, A)$ is a soft e-compact space we can extract a finite family $\{(\widetilde{Y}, A) \Cap (O_i, A), i \in I_0\}$ such that $(\widetilde{Y}, A) = \Cup_{i\in I_0}[(\widetilde{Y}, A)\Cap(O_i, A)] = (\widetilde{Y}, A)\Cap[\Cup_{i\in I_0}(O_i, A)]$. Then, $(\widetilde{Y}, A)\widetilde{\subseteq}\Cup_{i\in I_0}(O_i, A)$, hence $(\widetilde{Y}, A)$ is a soft e-compact set in $(\widetilde{X}, \tau, A)$. □

Remark 4.10. The converse of theorem 4.9 is not true in general. This is shown in the following counterexample. Consider the soft e-topological space introduced in example 3.4.
Let $X = \{a, b, c, d\}$, $A = \{\alpha, \beta\}$, and $\tau = \{(\widetilde{\Phi}, A), (F, A), (G, A), (H, A), (\widetilde{X}, A)\}$, where $F(\alpha) = G(\beta) = \{a\}, F(\beta) = G(\alpha) = \{c, d\}, H(\alpha) = H(\beta) = \{a, c, d\}$.
Let $Y = \{b, c, d\}$. $(\widetilde{Y}, A)$ is a soft compact set since it is a finite soft set, but we can't introduce a soft sub-e-topological space from $(\widetilde{Y}, A)$, as $(F, A)\widetilde{\cap}(\widetilde{Y}, A) \notin S(\widetilde{X})$.

We establish the relationship between a soft e-closed set and a soft e-compact set in the next two theorems.

Theorem 4.11. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff and for all $(O_1, A), (O_2, A) \in \tau$ we have: $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Let (F, A) be a soft e-compact set, then (F, A) is a soft e-closed set.*

Proof. Assume that $(F, A)^{\mathbb{C}} \neq (\widetilde{\Phi}, A)$, and let $\widetilde{y}\widetilde{\in}(F, A)^{\mathbb{C}}$, then for all $\widetilde{x}\widetilde{\in}(F, A)$ we have $\widetilde{x}(\alpha) \neq \widetilde{y}(\alpha)$ for all $\alpha \in A$.
Since $(\widetilde{X}, \tau, A)$ is a soft e-Hausdorff, there exist $(G_x, A), (H_x, A) \in \tau$ such that $\widetilde{x}\widetilde{\in}(G_x, A)$, $\widetilde{y}\widetilde{\in}(H_x, A)$ and $(G_x, A)\widetilde{\cap}(H_x, A) = (\widetilde{\Phi}, A)$. We have $(F, A)\widetilde{\subseteq} \Cup_{\widetilde{x}\widetilde{\in}(F,A)} (G_x, A)$, and since (F, A) be a soft e-compact set there exist $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\widetilde{\in}(F, A)$ such that $F, A)\widetilde{\subseteq} \Cup_{i=1}^n (G_{x_i}, A) = \Cup_{i=1}^n(G_i, A)$.
Putting $(H, A) = \Cup_{i=1}^n(H_{x_i}, A) \in \tau$, then $\widetilde{y} \in (H, A)$ and $(H, A) \Cap (G_i, A) = (\widetilde{\Phi}, A)$ for all

$i = 1 \dots n$. Since $(H, A), (G_i, A) \in \tau$, we have $(H, A) \Cap (G_i, A) = (H, A)\widetilde{\cap}(G_i, A)$ for all $i = 1 \dots n$. Then, for all $\alpha \in A$ we have $\Cup_{i=1}^{n}[(H, A) \Cap (G_i, A)](\alpha) = \widetilde{\cup}_{i=1}^{n}[(H, A)\widetilde{\cap}(G_i, A)](\alpha) = (H, A)\widetilde{\cap}[\widetilde{\cup}_{i=1}^{n}(G_i, A)](\alpha) = \emptyset$.
According to $(H, A), \widetilde{\cup}_{i=1}^{n}(G_i, A) \in \tau$, we have $(H, A)\widetilde{\cap}[\widetilde{\cup}_{i=1}^{n}(G_i, A)] \in S(\widetilde{X})$.
Hence, $(H, A)\widetilde{\cap}[\widetilde{\cup}_{i=1}^{n}(G_i, A)] = (H, A) \Cap [\Cup_{i=1}^{n}(G_i, A)] = (\widetilde{\Phi}, A)$.
Therefore, $(H, A)\widetilde{\subseteq}(\Cup_{i=1}^{n}(G_i, A))^C\widetilde{\subseteq}(F, A)^C$, so $(F, A)^{\mathbb{C}} \in \tau$, and (F, A) is soft e-closed set. □

Theorem 4.12. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff. Assume that there exists a soft e-compact set (K, A) such that $(F, A)\widetilde{\subseteq}(K, A)$, and let (F, A) be a soft e-closed set. Then, (F, A) is a soft e-compact set.*

Proof. Since (F, A) is a soft e-closed set, then $(F, A)^C \in S(\widetilde{X})$ and $(F, A)^{\mathbb{C}} \in \tau$. Let $\{(O_i, A), i \in I\}$ be a soft e-cover open of (F, A), then $(\widetilde{X}, A) = (F, A)^{\mathbb{C}} \Cup [\Cup_{i\in I}(O_i, A)]$, and $(K, A)\widetilde{\subseteq}(F, A)^{\mathbb{C}} \Cup [\Cup_{i\in I}(O_i, A)]$.
Since (K, A) is soft e-compact set, there exists a finite subfamily $\{(O_i, A), i \in I_0 \subset I\}$ such that $(K, A)\widetilde{\subseteq}(F, A)^{\mathbb{C}} \Cup [\Cup_{i\in I_0}(O_i, A)]$.
Then, $(F, A)\widetilde{\subseteq} \Cup_{i\in I_0}(O_i, A)$, so (F, A) is a soft e-compact set. □

The following proposition provide (in supplementary conditions) that the soft elementary union and intersection are compatible with the soft e-compactness.

Proposition 4.13. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff. Then,*

(i) The elementary union of two soft e-compact sets is a soft e-compact set.

(ii) If $(\widetilde{X}, \tau, A)$ is a soft e-compact, and for all $(O_1, A), (O_2, A) \in \tau$ we have: $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$, then the elementary intersection of any soft e-compact sets is a soft e-compact set.

Proof. Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff space.

(i) Let $(K_1, A), (K_2, A)$ be two soft e-compact sets, and let $\{(O_i, A), i \in I\}$ be an e-open cover of $(K_1, A) \Cup (K_2, A)$, Then $\{(O_i, A), i \in I\}$ be an e-open cover of (K_1, A) and (K_2, A). We can extract a finite sub-cover $\{(O_i, A), i \in I_1\}$ of (K_1, A) and a finite sub-cover $\{(O_i, A), i \in I_2\}$ of (K_2, A). Hence; $\{(O_i, A), i \in I_1 \cup I_2\}$ is a sub-cover of $(K_1, A) \Cup (K_2, A)$.

(ii) Assume that $(\widetilde{X}, \tau, A)$ is a soft e-compact and for all $(O_1, A), (O_2, A) \in \tau$ we have: $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. Let $\{(K_i, A), i \in I\}$ be a family of soft e-compact sets, then $\{(K_i, A), i \in I\}$ be a family of soft e-closed sets. $\Cap_{i\in I}(K_i, A)$ is a soft e-closed set and subset of any soft e-compact set (K_i, A), hence $\Cap_{i\in I}(K_i, A)$ is a soft e-compact set.

□

Remark 4.14. We can replace the condition $(\widetilde{X}, \tau, A)$ be a soft e-compact by the condition: there exists a soft e-compact set (K, A) such that for all $i \in I$ we have $(K_i, A)\widetilde{\subseteq}(K, A)$.

We now show some soft e-compact set and space properties.

Theorem 4.15. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff, (K, A) be a soft e-compact and $(F, A) \in S(\widetilde{X})$ be a soft subset not finite of (K, A). Then (F, A) has a limiting soft element.*

Proof. Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff, (K, A) be a soft e-compact, $(F, A) \in S(\widetilde{X})$.
Assume that $(F, A)\widetilde{\subseteq}(K, A)$ is not finite, and has not a soft limiting element, than by lemma 3.6 for all $\widetilde{x}\widetilde{\in}(\widetilde{X}, A)$ there exists $(G_x, A) \in \tau$ such that for all $\widetilde{y}\widetilde{\in}(\widetilde{X}, A), \widetilde{y}\widetilde{\in}(F, A) \Cap (G_x, A)$ implies that $\widetilde{y} = \widetilde{x}$. The family $\{(G_x, A), \widetilde{x}\widetilde{\in}(\widetilde{X}, A)\}$ us an e-open cover of (K, A), we can extract a finite cover $\{(G_{x_i}, A), \widetilde{x}_i\widetilde{\in}(\widetilde{X}, A), i = 1 \dots n\} = \{(G_i, A), \widetilde{x}_i\widetilde{\in}(\widetilde{X}, A), i = 1 \dots n\}$. In the family $\{\widetilde{x}_i\widetilde{\in}(\widetilde{X}, A), i = 1 \dots n\}$ there exists a subfamily $\{\widetilde{x}_i\widetilde{\in}(\widetilde{X}, A), i \in \{1, 2, \dots, n\}\}$. Since, $(K, A)\widetilde{\subseteq}[\Cup_{i=1}^{n}(G_i, A)]_{\widetilde{x}_i\widetilde{\in}(F,A)} \Cup [\Cup_{i=1}^{n}(G_i, A)]_{\widetilde{x}_i\widetilde{\notin}(F,A)}$. Then, (F, A) is a soft subset of $SS(\{\widetilde{x}_i\widetilde{\in}(F, A), i = 1 \dots n\})$, hence (F, A) is finite, which is a contradiction. □

Theorem 4.16. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff. If $(\widetilde{X}, \tau, A)$ is a soft e-compact, then $(\widetilde{X}, \tau, A)$ is an e-regular space.*

Proof. Let (F, A) be a soft e-closed and let $\widetilde{y}\widetilde{\in}(F, A)^{\mathbb{C}}$. According to the proof of theorem 4.11, there exist $(G, A), (H, A) \in \tau$ such that $(F, A)\widetilde{\subseteq}(G, A), \widetilde{y}\widetilde{\in}(H, A)$, and $(G, A) \Cap (H, A) = (\widetilde{\Phi}, A)$. Then, $(\widetilde{X}, \tau, A)$ is an e-regular space. □

Theorem 4.17. *Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff. If $(\widetilde{X}, \tau, A)$ is a soft e-compact, then $(\widetilde{X}, \tau, A)$ is an e-normal space.*

Proof. Let (F, A) be a soft e-closed and let $\widetilde{y}\widetilde{\in}(F, A)^{\mathbb{C}}$. According to the proof of theorem 4.15, if $(F_1, A), (F_2, A)$ are two soft e-closed sets such that $(F_1, A)\widetilde{\cap}(F_2, A) = (\widetilde{\Phi}, A)$, and for all $\widetilde{y}\widetilde{\in}(F_2, A)$ there exist $(G_1^y, A), (G_2^y, A) \in \tau$ such that $(F, A)\widetilde{\subseteq}(G_1^y, A), \widetilde{y}\widetilde{\in}(G_2^y, A)$, and $(G_1^y, A) \Cap (G_2^y, A) = (\widetilde{\Phi}, A)$, we can extract a sub-e-cover $\{(G_2^{y_i}, A)\}_{i=1}^n$ from the cover $\{(G_2^y, A), \widetilde{y}\widetilde{\in}(F_2, A)\}$ of (F_2, A).
Putting, $(G_1^i, A) = (G_1^{y_i}, A), (G_2^i, A) = (G_2^{y_i}, A)$ for all $i = 1 \ldots n, (G_1, A) = \Cap_{i=1}^n (G_1^i, A)$, $(G_2, A) = \Cap_{i=1}^n (G_2^i, A)$. Then, $(F_1, A)\widetilde{\subseteq}(G_1, A), (F_2, A)\widetilde{\subseteq}(G_2, A)$ and $(G_1, A) \Cap (G_2, A) = (\widetilde{\Phi}, A)$. Then $(\widetilde{X}, \tau, A)$ is an e-normal space. □

This proposition establishes that a soft e-continuous function in a soft e-Hausdorff space maps a soft e-compact set to an image that is also a soft e-compact set.

Proposition 4.18. *Let $(\widetilde{X}, \tau, A), (\widetilde{Y}, \sigma, A)$ be two soft e-Hausdorff spaces. Let $f : SE(\widetilde{X}) \to SE(\widetilde{Y})$ be a soft function and (K, A) be a soft e-compact set of $(\widetilde{X}, \tau, A)$. If f is a soft e-continuous then $f[(K, A)]$ is a soft e-compact set of $(\widetilde{Y}, \sigma, A)$.*

Proof. Let $\{(U_i, A) \in \sigma, i \in I\}$ be an open cover of $f[(K, A)]$, then $f[(K, A)]\widetilde{\subseteq} \Cup_{i\in I} (U_i, A)$. From Proposition 5.4 of [7], we have $(K, A)\widetilde{\subseteq} f^{-1}(f[(K, A)])\widetilde{\subseteq} f^{-1}[\underset{i\in I}{\Cup}(U_i, A)] = \underset{i\in I}{\Cup} f^{-1}[(U_i, A)]$.
Since f is a soft e-continuous function, then $\{f^{-1}[(U_i, A)], i \in I\}$ is a soft e-open cover of (K, A), and since (K, A) is a soft e-compact set we can extract a finite sub-cover $\{f^{-1}[(U_i, A)], i \in I_0\}$ of (K, A), i.e. $(K, A)\widetilde{\subseteq} \underset{i\in I_0}{\Cup} f^{-1}[(U_i, A)]$.
Then, $f[(K, A)]\widetilde{\subseteq} f(\underset{i\in I_0}{\Cup} f^{-1}[(U_i, A)]) = \underset{i\in I_0}{\Cup} f(f^{-1}[(U_i, A)])\widetilde{\subseteq} \underset{i\in I_0}{\Cup} (U_i, A)$. So, $f[(K, A)]$ is a soft e-compact set of $(\widetilde{Y}, \sigma, A)$. □

5 Soft e-locally compact space and soft e-Baire's theorem

In this section, we explain what a soft e-locally compact space is and how the Baire's theorem works in a soft elementary form. First, we will define some terms.

Definition 5.1. Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff space. $(\widetilde{X}, \tau, A)$ is called a soft e-locally compact space if, for all $\widetilde{x}\widetilde{\in}(\widetilde{X}, A)$ and for all soft neighborhoods (N, A) of $\widetilde{x}$, there exists a soft e-compact neighborhood (K, A) of $\widetilde{x}$, such that $(K, A)\widetilde{\subseteq}(N, A)$.

Definition 5.2. Let $(\widetilde{X}, \tau, A)$ be a soft e-Hausdorff, and $(F, A) \in S(\widetilde{X})$ such that $(F, A) \neq (\widetilde{\Phi}, A)$.

- **i)** (F, A) is called soft e-nowhere dense (or soft rare set) if $\overset{0}{\overline{(F, A)}} = (\widetilde{\Phi}, A)$
- **iii)** (F, A) is called of first e-category (or soft meager set) if it is a countable union of e-nowhere dense subsets of $(\widetilde{X}, A)$.
- **ii)** (F, A) is called of second e-category if it is not of first e-category.

Definition 5.3. Let $(\widetilde{X}, \tau, A)$ be a soft e- Hausdorff. We say that $(\widetilde{X}, \tau, A)$ is a soft e-Baire's space if for all countable family of soft e-closed sets $\{(F_i, A)\}_{i=1}^{\infty}$ such that $\overbrace{(F_i, A)}^{0} = (\widetilde{\Phi}, A)$ we have $\overbrace{\Cup_{i=1}^{\infty}(F_i, A)}^{0} = (\widetilde{\Phi}, A)$.

In this theorem, we use the soft elementary intersection to demonstrate the soft elementary version of Baire's theorem.

Theorem 5.4. *Let* $(\widetilde{X}, \tau, A)$ *be a soft locally compact space such that for all* $(O_1, A), (O_2, A) \in \tau$ *we have:* $(O_1, A)\widetilde{\cap}(O_2, A) \in S(\widetilde{X})$. *Then,* $(\widetilde{X}, \tau, A)$ *is a soft e-Baire's space.*

Proof. Let $(\widetilde{X}, \tau, A)$ be a soft e-locally compact space, and $\{(F_i, A)\}_{i=1}^{\infty}$ be a family of soft e-closed sets such that $\overbrace{(F_i, A)}^{0} = (\widetilde{\Phi}, A)$ for all $i = 1, 2, \ldots$. Set $\Cup_{i=1}^{\infty}(F_i, A) = (F, A)$.

To prove that $\overbrace{(F_i, A)}^{0} = (\widetilde{\Phi}, A)$ it is enough to prove that for all $(O, A) \in \tau$ we have $(O, A) \Cap (F, A)^C \neq (\widetilde{\Phi}, A)$.

Since $\overbrace{(F_1, A)}^{0} = (\widetilde{\Phi}, A)$ we have $(O, A)\widetilde{\nsubseteq}(F_1, A)$, hence $(O, A) \Cap (F, A)^C \neq (\widetilde{\Phi}, A)$.
Due to $(O, A), (F_1, A)^C \in \tau, (O, A)\widetilde{\cap}(F_1, A) \in S(\widetilde{X})$, there exists a soft element $\widetilde{x}_1\widetilde{\in}(O, A) \Cap (F_1, A)^C$, and taking into consideration that $(\widetilde{X}, \tau, A)$ is a soft e-locally compact space, we deduce that there exists a soft e-compact set $(K_1, A)\widetilde{\subseteq}(O, A) \Cap (F, A)^C$ such that $\widetilde{x}_1\widetilde{\in}\overbrace{(K_1, A)}^{0}$.
Therefore, $\overbrace{(F_2, A)}^{0} = (\widetilde{\Phi}, A)$ and $\overbrace{(K_1, A)}^{0} \neq (\widetilde{\Phi}, A)$, so there exists a soft element $\widetilde{x}_2$ and a soft e-compact set (K_2, A) such that $\widetilde{x}_2\widetilde{\in}(K_2, A)\widetilde{\subseteq}\overbrace{(K_1, A)}^{0} \Cap (F_2, A)^C$.
By recurrence, we construct a countable family of e-closed sets $\{(K_i, A)\}_{i=1}^{\infty}$ such that $(K_1, A)\widetilde{\supseteq}(K_2, A)\widetilde{\supseteq}\ldots$. The family $\{(K_i, A)\}_{i=1}^{\infty}$ is a family of decreasing soft subsets of the soft e-compact set (K_1, A). Then, from Theorem 4.6 we have $\Cap_{i=1}^{\infty}(K_i, A) \neq (\widetilde{\Phi}, A)$.
Let now $\widetilde{x}\widetilde{\in} \Cap_{i=1}^{\infty} (K_i, A) \neq (\widetilde{\Phi}, A)$, then $\widetilde{x}\widetilde{\in}(K_i, A) \Cap (F_i, A)^C$ for all $i = 1, 2, \ldots$.
Hence, $\widetilde{x}\widetilde{\notin}(F_i, A)$ for all $i = 1, 2, \ldots$. Then, $\widetilde{x}\widetilde{\notin}(F, A)$, which lead to $(O, A) \nsubseteq (F, A)$, so $\overbrace{(K_1, A)}^{0} = (\widetilde{\Phi}, A)$. □

6 Conclusion

In this paper, we have given a definition of a soft elementary compact set and space based on the work of Chiney and Samanta [7]. We have investigated some properties of the soft elementary compactness, and we have proved. The main result, which is the elementary version of Baire's theorem, is the soft elementary version.

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