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CERTAIN NEW RESULTS FOR SOME HORNS HYPERGEOMETRIC FUNCTIONS IN TWO VARIABLES

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Abstract In this paper, we introduce new transformations for Horn's double hypergeometric functions G_1, G_2 and G_3. Also deduce new integral representations of Euler-type involving these functions in terms of hypergeometric functions of two variables.

1 Introduction

The theory of hypergeometric functions has evolved in many directions and has been useful in the study of a variety of useful properties in diverse areas of mathematics, physics, and engineering. Numerous works have been published on this topic in the literature (for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). The enormous success of the theory of hypergeometric functions in a single variable motivated the development of the theory of hypergeometric functions in multiple variables, as the solutions of partial differential equations arising in various problems of mathematical physics are expressed in terms of such hypergeometric functions [11, 12, 13]. The study of multivariable hypergeometric functions has grown in popularity as a result of advances in number theory, Lie algebras, group theory, representation theory, algebraic geometry, and combinatorics, etc. have led to increasing interest in the study of multivariable hypergeometric functions [14, 15, 16].

First of all, we recall Horn's hypergeometric functions of two variables $G_1, G_2, G_3, H_1, H_2, H_3$, H_4, H_5, H_6 and H_7 defined by (see[17, 18]):

$$G_1\left(a, b, c; x, y\right) = \sum_{m,n=0}^{\infty} (a)_{m+n}(b)_{n-m}(c)_{m-n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.1}$$

with $|x| + |y| < 1$;

$$G_2\left(a, b, c, d; x, y\right) = \sum_{m,n=0}^{\infty} (a)_m(b)_n(c)_{n-m}(d)_{m-n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.2}$$

with $|x| < 1$ and $|y| < 1$;

$$G_3\left(a, b; x, y\right) = \sum_{m,n=0}^{\infty} (a)_{2n-m}(b)_{2m-n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.3}$$

with $27|x|^2|y|^2 + 18|x||y| \pm 4(|x| - |y|) < 1$.

$$H_1\left(a, b, c; d; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.4}$$

with $4|x||y| + 2|y| - |y|^2 < 1$.

$$H_2(a,b,c,d;e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.5}$$

with $\frac{1}{|y|-|x|} < 1$.

$$H_3(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.6}$$

with $|x| + |y|^2 - |y| < 0$.

$$H_4(a,b;c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(c)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.7}$$

with $4|x| + 2|y|^2 - |y| < 1$.

$$H_5(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_{n-m}}{(c)_n} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.8}$$

with $16|x|^2 - 36|x||y| \pm (8|x| - |y| + 27|x||y|^2) < 1$.

$$H_6(a,b,c;x,y) = \sum_{m,n=0}^{\infty} (a)_{2m-n}(b)_{n-m}(c)_n \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.9}$$

with $|x||y|^2 + |y| < 1$.

$$H_7(a,b,c;d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{1.10}$$

with $4|x| + \frac{2}{|s|} - \frac{1}{|s|^2} < 1$.

Here $(\alpha)_\beta$ denotes the Pochhammer symbol defined (for $\alpha, \beta \in \mathbb{C}$) by

$$(\alpha)_\beta := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} = \begin{cases} 1 & (\beta = 0;\ \alpha \neq 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (\beta = n \in \mathbb{N}), \end{cases} \tag{1.11}$$

Γ being the familiar Gamma function and it being read traditionally that $(0)_0 := 1$.

Appell [19] developed four hypergeometric functions of two variables and designated them by F_1, F_2, F_3 and F_4. The following is a de
nition of one of these functions:

$$F_3(a,b,c,d;e;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{1.12}$$

for $|x| < 1, \quad |y| < 1$, and $e \neq 0, -1, -2, \ldots$.

2 Transformation formulas

In this section, we find some new transformations of Horn functions G_1, G_2 and G_3 in series of some generalized hypergeometric functions. We recall the generalized hypergeometric series ${}_pF_q$ $(p, q \in \mathbb{N}_0)$ given by (see cite14):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\lambda_1, \ldots, \lambda_p; \\ \mu_1, \ldots, \mu_q;\end{matrix}\ z\right] &= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_p)_n}{(\mu_1)_n \cdots (\mu_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\lambda_1, \ldots, \lambda_p; \mu_1, \ldots, \mu_q; z). \end{aligned} \tag{2.1}$$

Theorem 2.1. *If* $|x| \neq 0$ *and* $\left|\frac{x}{y}\right| < 1$, *then the following transformations holds*

$$G_1\,(a,b,c;x,y) = \sum_{m=0}^{\infty} \frac{(a)_m (c)_m}{(b-1)_m} \frac{x^m}{m!} \times$$
$${}_5F_4\left[-m, \frac{b}{2}, \frac{b+1}{2}, \frac{b-1}{2}, \frac{b}{2}; \frac{b-1+m}{2}, \frac{b+m}{2}, \frac{1-c-m}{2}, \frac{2-c-m}{2}; \frac{y}{x}\right]. \tag{2.2}$$

$$G_2\,(a,b,c,d;x,y) = \sum_{m=0}^{\infty} \frac{(a)_m (d)_m}{(c-1)_m} \frac{x^m}{m!} \times$$
$${}_5F_5\left[-m, \frac{c}{2}, \frac{c+1}{2}, \frac{c-1}{2}, \frac{c}{2}; 1-a-m, \frac{c-1+m}{2}, \frac{c+m}{2}, \frac{1-d-m}{2}, \frac{2-d-m}{2}; \frac{y}{x}\right]. \tag{2.3}$$

$$G_3\,(a,b;x,y) = \sum_{m=0}^{\infty} \frac{(b)_m}{(a-1)_m} \frac{x^m}{m!} \times$$
$${}_7F_6\left[\begin{matrix} -m, \frac{a}{3}, \frac{a+1}{3}, \frac{a+3}{3}, \frac{a-1}{3}, \frac{a}{3}, \frac{a+1}{3}; \\ \frac{a-1+m}{a}, \frac{a+m}{3}, \frac{a+m+1}{3} \frac{1-b-2m}{3}, \frac{2-b-2m}{3} \frac{3-b-2m}{3}; \end{matrix} \frac{y}{x}\right]. \tag{2.4}$$

Proof. To prove the result in equality (2.2), let Ω denote the left side of the equality (2.2). And with the help of following identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{k,n-k} \tag{2.5}$$

we get

$$\Omega = \sum_{m=0}^{\infty}\sum_{n=0}^{m} (a)_m (b)_{2n-m} (c)_{m-2n} \frac{x^{m-n}}{(m-n)!} \frac{y^n}{n!}. \tag{2.6}$$

By using the following identities (see [19])

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n}, \tag{2.7}$$

$$(b)_{m-n} = \frac{(-1)^n (b)_m}{(1-b-m)_n}, \quad \text{and} \quad (b)_{2n-m} = \frac{(b)_{2n}(b-1)_{2n}}{(b-1)_m (b-1+m)_{2n}}. \tag{2.8}$$

and

$$(c)_{m-2n} = \frac{(-1)^{2n}(c)_m}{(1-c-m)_{2n}}, \tag{2.9}$$

Ω becomes

$$\Omega = \sum_{m=0}^{\infty} \frac{(a)_m (c)_m}{(b-1)_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (b)_{2n} (b-1)_{2n}}{(b-1-m)_{2n}(1-c-m)_{2n}}. \tag{2.10}$$

From above equation (2.10) second summation over n is

$${}_5F_4\left[-m, \frac{b}{2}, \frac{b+1}{2}, \frac{b-1}{2}, \frac{b}{2}; \frac{b-1+m}{2}, \frac{b+m}{2}, \frac{1-c-m}{2}, \frac{2-c-m}{2}; \frac{y}{x}\right].$$

in view of which, complete the proof of (2.2). Similarly, we can prove (2.3) and (2.4) by using identities (2.5), (2.7), (2.8) and (2.9). □

3 Integral representations

In this section, we give certain integral representations for Horn's double hypergeometric functions G_1, G_2 and G_3.

Theorem 3.1. *The integral representations are satisfied for G_1:*

$$G_1\left(a,b,c;x,y\right)=\frac{\Gamma(a+a')(V-T)^{a}(W-T)^{a'}}{\Gamma(a)\Gamma(a')(V-W)^{a+a'-1}}\int_W^V\frac{(\alpha-W)^{a-1}(V-\alpha)^{a'-1}}{(\alpha-T)^{a+a'}}$$
$$\times G_2\left(a+a',1-a',b,c;\frac{(V-T)(\alpha-W)x}{(V-W)(\alpha-T)},\frac{-y}{(W-T)(V-\alpha)}\right)d\alpha,$$
$$\left(\Re(a)>0,\Re(a')>0,T<W<V\right),\quad (3.1)$$

$$G_1\left(a,b,c;x,y\right)=\frac{4M_1^{a'}M_2^{b}\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}$$
$$\times\int_0^\infty\int_0^\infty\frac{\cosh\alpha\left(\sinh^2\alpha\right)^{a'-\frac{1}{2}}}{\left(1+M_1\sinh^2\alpha\right)^{a+a'}}\frac{\cosh\beta\left(\sinh^2\beta\right)^{b-\frac{1}{2}}}{\left(1+M_2\sinh^2\beta\right)^{b+b'}}$$
$$\times H_2\left(c,1-a',a+a',b+b';b';-\frac{x\,csch^2\alpha\,csch^2\beta}{M_1M_2},\frac{M_2y}{\left(1+M_1\sinh^2\alpha\right)\left(1+M_2\sinh^2\beta\right)}\right)d\alpha d\beta,$$
$$\left(\Re(a)>0,\Re(a')>0,\Re(b)>0,\Re(b')>0,M_1>0,M_2>0\right),\quad (3.2)$$

$$G_1\left(a,b,c;x,y\right)=\frac{\Gamma(a+b)\Gamma(c+c')}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')}\int_0^1\int_0^1\alpha^{c-1}\left(1-\alpha\right)^{c'-1}\beta^{a-1}\left(1-\beta\right)^{b-1}$$
$$\times H_7\left(1-c',\frac{a+b}{2},\frac{a+b+1}{2};1-c-c';\frac{\alpha\beta x}{\left(1-\alpha\right)^2\left(\beta-1\right)},\frac{4\left(\alpha-1\right)\beta\left(1-\beta\right)y}{\alpha}\right)d\alpha d\beta,$$
$$\left(\Re(a)>0,\Re(b)>0,\Re(c)>0,\Re(c')>0\right),\quad (3.3)$$

$$G_1\left(a,b,c;x,y\right)=\frac{4\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a+b-\frac{1}{2}}(\cos^2\alpha)^{c-\frac{1}{2}}(\sin^2\beta)^{b-\frac{1}{2}}(\cos^2\beta)^{a-\frac{1}{2}}$$
$$\times\left[1-x\cos^2\alpha\cot^2\beta-\frac{1}{4}y\sin^2\alpha\tan^2\alpha\sin^2 2\beta\right]^{-(a+b+c)}d\alpha,$$
$$\left(\Re(a)>0,\Re(b)>0,\Re(c)>0\right).\quad (3.4)$$

Proof. To prove the result in equality (3.1), let $\mho$ denote the right side of the equality (3.1). Then from the definition (2), we get

$$\mho=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\sum_{m,n=0}^{\infty}\frac{(a+a')_m(b)_{n-m}(c)_{m-n}}{(a)_{-n}}$$
$$\times\frac{(V-T)^{a+m}(W-T)^{a'-n}}{(V-W)^{a+a'+m-1}}\int_W^V\frac{(\alpha-W)^{a+m-1}(V-\alpha)^{a'-n-1}}{(\alpha-T)^{a+a'+m}}\times\frac{x^m}{m!}\frac{y^n}{n!}d\alpha.\quad (3.5)$$

Applying the following integral representation (see, e.g., [17, p. 10,(14)])

$$B(a,b)=\frac{(V-T)^{a}(W-T)^{b}}{(V-W)^{a+b-1}}\int_W^V\frac{(\alpha-W)^{a-1}(V-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha,$$

$$(T < W < V, \Re(a) > 0, \Re(b) > 0).$$

in (27), we have

$$\mho = \frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')} \sum_{m,n=0}^{\infty} \frac{(a+a')_m (b)_{n-m} (c)_{m-n} B(a+m+n, a'-n)}{(a)_{-n}}. \quad (3.6)$$

Now applying well known beta function (see, e.g., [20, 21])

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (28), we get the required result. On the same way, we find the results (24)-(26). □

As in the proof of Theorem 3.1, one can easily prove the following Theorems. So, details are omitted.

Theorem 3.2. *The integral representations are satisfied for G_2:*

$$G_2(a,b,c,d;x,y) = \frac{\Gamma(b+d)\Gamma(c+c')}{\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)} \int_0^\infty \int_0^\infty \frac{\alpha^{c-1}}{(1+\alpha)^{c+c'}} \frac{\beta^{d-1}}{(1+\beta)^{b+d}}$$
$$\times F_3\left(a, \frac{c+c'}{2}, b+d, \frac{c+c'+1}{2}; c'; \frac{\beta x}{\alpha(1+\beta)}, \frac{4\alpha y}{(1+\alpha)^2 \beta}\right) d\alpha d\beta,$$
$$(\Re(b) > 0, \Re(c) > 0, \Re(c') > 0, \Re(d) > 0), \quad (3.7)$$

$$G_2(a,b,c,d;x,y) = \frac{(1+M_1)^a (1+M_2)^b \Gamma(a+d)\Gamma(b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}$$
$$\times \int_0^1 \int_0^1 \frac{\alpha^{a-1}(1-\alpha)^{d-1}}{(1+M_1\alpha)^{a+d}} \frac{\beta^{b-1}(1-\beta)^{c-1}}{(1+M_2\beta)^{b+c}}$$
$$\times G_3\left(b+c, a+d; \frac{(1+M_1)\alpha(1-\alpha)(1+M_2\beta)x}{(1+M_1\alpha)^2(1-\alpha)}, \right.$$
$$\left. \frac{(1+M_2)\beta(1-\beta)(1+M_1\alpha)y}{(1+M_2\beta)^2(1-\alpha)}\right) d\alpha d\beta,$$
$$(\Re(a) > 0, \Re(b) > 0, \Re(c) > 0, \Re(d) > 0, M_1 > -1, M_2 > -1), \quad (3.8)$$

$$G_2(a,b,c,d;x,y) = \frac{\Gamma(a+a')\Gamma(c+d)}{\Gamma(a)\Gamma(a')\Gamma(c)\Gamma(d)}$$
$$\times \int_0^\infty \int_0^\infty (e^{-\alpha})^a (1-e^{-\alpha})^{a'-1} (e^{-\beta})^c (1-e^{-\beta})^{d-1}$$
$$\times H_6\left(1-a', a+a', b; \frac{e^{-(\alpha-\beta)}(1-e^{-\beta})x}{(1-e^{-\alpha})^2}, \frac{(e^{-\alpha}-1)e^{-\beta}y}{(1-e^{-\beta})}\right) d\alpha d\beta,$$
$$(\Re(a) > 0, \Re(a') > 0, \Re(c) > 0, \Re(d) > 0), \quad (3.9)$$

Theorem 3.3. *The integral representations are satisfied for G_3:*

$$G_3(a,b;x,y)=\frac{\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')(V_1-W_1)^{a+a'-1}(V_2-W_2)^{b+b'-1}}\int_{W_1}^{V_1}\int_{W_2}^{V_2}$$

$$\times(\alpha-W_1)^{a-1}(V_1-\alpha)^{a'-1}(\beta-W_2)^{b-1}(V_2-\beta)^{b'-1}$$

$$\times G_2\left(b+b',a+a',1-a',1-b';\frac{(V_1-\alpha)(\beta-W_2)^2x}{(S_2-W_2)(\alpha-W_1)(S_2-\beta)},\frac{(\alpha-W_1)^2(V_2-\beta)y}{(V_1-W_1)(V_1-\alpha)(\beta-W_2)}\right)$$

$$\times d\alpha d\beta,$$

$$(\Re(a)>0,\Re(a')>0,\Re(b)>0,\Re(b')>0,W_1<V_1,W_2<V_2), \quad (3.10)$$

$$G_3(a,b;x,y)=\frac{\Gamma(a+a')}{2^{a+a'-2}\Gamma(a)\Gamma(a')}\int_{-1}^{1}\frac{\left[(1+\alpha)^2\right]^{a'-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a-\frac{1}{2}}}{(1+\alpha^2)^{a+a'}}$$

$$\times G_3\left(1-a',b;-\left(\frac{1+\alpha}{1-\alpha}\right)^2x,\left(\frac{1-\alpha}{1+\alpha}\right)^4y\right)d\alpha,$$

$$(\Re(a)>0,\Re(a')>0). \quad (3.11)$$

$$G_3(a,b;x,y)=\frac{4(1+M_1)^a(1+M_2)^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{(sin^2\alpha)^{a-\frac{1}{2}}(cos^2\alpha)^{a'-\frac{1}{2}}}{(1+M_1sin^2\alpha)^{a+a'}}\frac{(sin^2\beta)^{b-\frac{1}{2}}(cos^2\beta)^{b'-\frac{1}{2}}}{(1+M_2sin^2\beta)^{b+b'}}$$

$$H_2\left(a+a',1-a',\frac{b+b'}{2},\frac{b+b'+1}{2};b;-\frac{(1+M_1)^2ysin^2\alpha tan^2\alpha cot^2\beta}{(1+M_2)(1+M_1sin^2\alpha)},\right.$$

$$\left.\frac{(1+M_2)^2xcsc^2\alpha sin^4\beta\,(1+M_1sin^2\alpha)}{(1+M_1)(1+M_2sin^2\beta)^2}\right)d\alpha d\beta,$$

$$(\Re(a)>0,\Re(a')>0,\Re(b)>0,\Re(b')>0,M_1>-1,M_2>-1). \quad (3.12)$$

$$G_3(a,b;x,y)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a'-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

$$\times H_6\left(b,1-a',a+a';-\left(\frac{1+2\alpha}{1-2\alpha}\right)x,-\frac{(1-2\alpha)^2y}{2(1+2\alpha)}\right)d\alpha,$$

$$(\Re(a)>0,\Re(a')>0), \quad (3.13)$$

4 Conclusion remarks

This paper aims is to obtain new transformations for Horn's double hypergeometric functions G_1, G_2 and G_3. Also, some new integral representations of Euler-type involving these functions in terms of hypergeometric functions of two variables have been discussed. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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