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Some Graphical Properties in Terms of Eccentric Connectivity Coindex

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Abstract The eccentric connectivity coindex $\overline{\xi}^c(\Gamma)$ is defined as the total eccentricity sum of all non-adjacent vertex pairs in a connected graph Γ. This paper presents the sufficient conditions for a Hamiltonian path, Hamiltonian cycle, Hamiltonian connectedness, α-edge Hamiltonicity, and α-path coverable of graph Γ in terms of first Zagreb index and eccentric connectivity coindex.

1 Introduction

The topological indices are numerical quantities associated with molecular graphs of the compounds. One of the applications of topological indices is to predict the physicochemical properties of chemical compounds. QSPR/QSAR studies have been performed for various drugs related to many diseases using topological indices [14, 15]. In 1947, the first topological index was introduced by Wiener [28]. Subsequently, various topological indices were introduced, namely the Zagreb index [13], Schultz Index [26], and Harary Index [24], etc. However, some other graphical properties have also been discussed in terms of topological indices in [12, 25]. For instance, Hua [17] discussed sufficient conditions for traceability in terms of the Harary index. An et al. [3] derived sufficient conditions for α-connected, β-deficient, and Hamiltonian cycle in terms of the first Zagreb index. The topological indices were initially defined based on the degree of adjacent vertices. Further, the concept of topological coindex based on non-adjacent vertices was introduced by Došlić [9] called Zagreb coindex. In 2010, Ashrafi et al. [4] investigated the properties of the Zagreb coindices and obtained explicit formulae for their values under several graph operations. In subsequent studies, Gutman et al. [11] established the relation between Zagreb indices and Zagreb coindices. In 2020, Azari [1] presented sharp lower bounds for the eccentric connectivity coindex of several graph operations. Motivated by these studies, we establish sufficient conditions for a Hamiltonian path, Hamiltonian cycle, Hamiltonian connectedness, α-edge Hamiltonicity, and α-path coverable of graphs in terms of eccentric connectivity coindex and first Zagreb index.

Let Γ be a simple connected graph of order n and size m. Throughout the paper, Γ represents a graph with vertex set $V(\Gamma)$ and the set of edges $E(\Gamma)$. Two vertices u and v of the graph Γ are said to be adjacent if an edge, denoted by uv, is connecting them. The degree of vertex v denoted by $d(v)$ is the number of vertices adjacent to v. The distance $d(u,v)$ is the length of the shortest path joining vertices u and v. The eccentricity of the vertex v, denoted by $\epsilon_\Gamma(v)$, is the maximum distance from v to any other vertex of the graph Γ. The complement graph of Γ, denoted by $\overline{\Gamma}$, is the graph with vertex set $V(\Gamma)$ and there is an edge $uv \in E(\overline{\Gamma})$ if and only if edge $uv \notin E(\Gamma)$. We denote a complete graph of order n by K_n. If Γ_1 and Γ_2 are two vertex disjoint graphs, then *join* $\Gamma_1 \vee \Gamma_2$ is the graph obtained from disjoint union of Γ_1 and Γ_2 by adding all edges between $V(\Gamma_1)$ and $V(\Gamma_2)$. For two vertex disjoint graphs Γ_1 and Γ_2, we use $\Gamma_1 + \Gamma_2$ to denote their

union. For more notations and terminologies, the reader may refer to [5].
A cycle in the graph Γ which contains all the vertices is called a Hamiltonian (briefly, HM) cycle. A graph is a Hamiltonian if it contains a HM cycle. A HM path in the graph Γ is the path that passes through all the vertices of the graph Γ. A graph that contains a HM path is called a traceable graph. A graph Γ is considered HM connected if a HM path exists between every pair of vertices. If graph Γ remains connected by removing fewer than α vertices, then for such maximum α graph Γ is called α-connected. A graph Γ is called α-HM, if $\Gamma[V(\Gamma) \setminus Y]$ is HM for all $|Y| \leq \alpha$. For $\alpha = 0$, the graph Γ is HM. A graph Γ is α-edge-HM, if every path of length equal to or lesser than α, $1 \leq \alpha \leq n-2$, is part of a HM cycle in Γ. For $\alpha = 1$, the graph is HM connected. If $V(\Gamma)$ can be covered by α vertex-disjoint paths, then graph Γ is called α-path-coverable. Note that for $\alpha = 1$, the graph is traceable.
This paper is organized as follows: Section 2 contains basic definitions for topological indices and coindices. Subsequently, we present the relevant findings used in the paper. Section 3 provides the main results of this work. Section 4 concludes the work.

2 Preliminaries

This section outlines the relevant basic definitions and results utilized throughout this paper. Particularly, we revisit the definitions of the topological indices and coindices, which will serve as the foundation for the subsequent sections.

The Wiener index, named after H. Wiener [28], is the first topological index, defined by

$$W(\Gamma) = \sum_{\{u,v\} \subseteq V(\Gamma)} d(u,v).$$

For more information and details, we refer [18, 19, 27, 29].

The First and second Zagreb indices proposed by Gutman et al. [13] are defined by

$$M_1(\Gamma) = \sum_{uv \in E(\Gamma)} (d(u) + d(v)) = \sum_{v \in V(\Gamma)} d(v)^2;$$
$$M_2(\Gamma) = \sum_{uv \in E(\Gamma)} d(u)d(v).$$

We refer [10, 23] for more information on Zagreb indices.

The First and Second Zagreb coindices proposed by Došlić [9] are defined by

$$\overline{M}_1(\Gamma) = \sum_{uv \notin E(\Gamma)} (d(u) + d(v));$$
$$\overline{M}_2(\Gamma) = \sum_{uv \notin E(\Gamma)} d(u)d(v).$$

The reader may see [4, 11] for more information on Zagreb coindices.

Motivated by the definition of Zagreb coindices and eccentric connectivity index [22], Hua et al. [16] introduced an eccentricity version of the first Zagreb coindex called eccentric connectivity coindex of Γ defined by

$$\overline{\xi}^c(\Gamma) = \sum_{uv \notin E(\Gamma)} (\epsilon_\Gamma(u) + \epsilon_\Gamma(v)).$$

The eccentric connectivity coindex of an n-vertex graph Γ can also be expressed as a sum over vertices of Γ [16],

$$\overline{\xi}^c(\Gamma) = \sum_{u \in V(\Gamma)} \epsilon_\Gamma(u)(n - 1 - d(u)).$$

Since $\epsilon_\Gamma(u) \leq n - d(u)$, the above equation can be written as

$$\begin{aligned}
\overline{\xi}^{c}(\Gamma) &= \sum_{u\in V(\Gamma)} \epsilon_\Gamma(u)(n-1-d(u)) \\
&\leq \sum_{u\in V(\Gamma)} (n-d(u))(n-1-d(u)) \\
&= \sum_{u\in V(\Gamma)} \big(n(n-1)-(2n-1)d(u)+d(u)^2\big) \\
&= n^2(n-1) + \sum_{u\in V(\Gamma)} d(u)^2 - (2n-1)\sum_{u\in V(\Gamma)} d(u). \qquad (2.1)
\end{aligned}$$

Now, we state degree conditions for a graph to be traceable, HM, and HM connected. In the following Propositions, we assume that the graph satisfies the degree sequence $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$.

Proposition 2.1. *[7] Let π be the degree sequence of the graph Γ of order $n \geq 3$. If*

$$d_i \leq i-1 < \frac{1}{2}(n-1) \Longrightarrow d_{n-i+1} \geq n-i,$$

then Γ is traceable.

Proposition 2.2. *[7] Let π be the degree sequence of the graph Γ of order $n \geq 3$. If*

$$d_i \leq i < \frac{n}{2} \Longrightarrow d_{n-i} \geq n-i,$$

then Γ is HM graph.

Proposition 2.3. *[8] Let π be the degree sequence of the graph Γ of order $n \geq 3$. If*

$$d_{i-1} \leq i \Longrightarrow d_{n-i} \geq n-i+1, \text{ for } 2 \leq i \leq \frac{n}{2},$$

then Γ is HM connected graph.

Proposition 2.4. *[20] Let π be the degree sequence of the graph Γ of order $n \geq 3$ and $0 \leq \alpha \leq n-3$. If*

$$d_{i-\alpha} \leq i \Longrightarrow d_{n-i} \geq n-i+\alpha, \ \text{ for } \alpha+1 \leq i < \frac{n+\alpha}{2},$$

then Γ is α-edge HM graph.

Proposition 2.5. *[6, 21] If $\alpha \geq 1$ and the degree sequence π of the graph Γ satisfies*

$$d_{i+\alpha} \leq i \Longrightarrow d_{n-i} \geq n-i-\alpha, \text{ for } 1 \leq i < \frac{1}{2}(n-\alpha),$$

then Γ is α-path coverable.

3 Main Results

This section contains the main results of the paper.

Theorem 3.1. *Let Γ be a graph of order n. If*

$$\overline{\xi}^{c}(\Gamma) > n^2(n-1) + f(x_1),$$

where $f(x) = -x^3 - (n+2)x^2 + (n^2+2n+1)x$ and $x_1 = \frac{-2(n+2)+\sqrt{16n^2+40n+28}}{6}$, then the graph Γ is traceable.

Proof. Let Γ be not traceable. By Proposition 2.1, there exists an integer $2 \le i < \frac{n+1}{2}$ such that $d_i \le i-1$ and $d_{n-i+1} \le n-i-1$. Then, by Equation 2.1. We have

$$\begin{aligned}\overline{\xi}^c(\Gamma) &\le n^2(n-1) + \sum_{u\in V(\Gamma)} d(u)^2 - (2n-1)\sum_{u\in V(\Gamma)} d(u)\\ &\le n^2(n-1) + [i(i-1)^2 + (n-2i+1)(n-i-1)^2 + (i-1)(n-1)^2]\\ &- (2n-1)[i(i-1) + (n-2i+1)(n-i-1) + (i-1)(n-1)]\\ &= -i^3 - (n+2)i^2 + (n^2+2n+1)i.\end{aligned}$$

Let $f(x) = -x^3 - (n+2)x^2 + (n^2+2n+1)x$ with $2 \le x \le \frac{n}{2}$. The first derivatives of $f(x)$ is

$$f'(x) = -3x^2 - 2(n+2)x + (n^2+2n+1) \text{ and the second derivative of } f(x) \text{ is}$$
$$f''(x) = -6x - 2(n+2) < 0 \text{ when } 2 \le x \le \frac{n}{2}.$$

This shows that the function $f(x)$ is concave down. Therefore, $f(x)$ has the maximum value at some points of the interval. The critical points of the function $f(x)$ are $x_1 = \frac{-(n+2)+\sqrt{4n^2+10n+7}}{3}$ and $x_2 = \frac{-(n+2)-\sqrt{4n^2+10n+7}}{3}$. Since $2 \le x \le \frac{n}{2}$, but $x_2 < 0$, hence maximum value occurs at x_1. Thus, $\overline{\xi}^c(\Gamma) \le n^2(n-1) + f(x_1)$.

□

Theorem 3.2. *Let Γ be a graph of order n. If*

$$\overline{\xi}^c(\Gamma) > n^2(n-1) + f(x_1),$$

where $f(x) = -x^3 - (n+1)x^2 + n^2x$ and $x_1 = \frac{-(n+1)+\sqrt{4n^2+2n+1}}{3}$, then Γ is an HM graph.

Proof. Let Γ be not the HM graph. By Proposition 2.2, there exists an integer $2 \le i < \frac{n}{2}$ such that $d_i \le i$ and $d_{n-i} \le n-i-1$. Then, by Equation 2.1, we get the following.

$$\begin{aligned}\overline{\xi}^c(\Gamma) &\le n^2(n-1) + \sum_{u\in V(\Gamma)} d(u)^2 - (2n-1)\sum_{u\in V(\Gamma)} d(u)\\ &\le n^2(n-1) + [ii^2 + (n-2i)(n-i-1)^2 + i(n-1)^2]\\ &- (2n-1)[ii + (n-2i)(n-i-1) + i(n-1)]\\ &= -i^3 - (n+1)i^2 + n^2 i.\end{aligned}$$

Let $f(x) = -x^3 - (n+1)x^2 + n^2x$ with $2 \le x < \frac{n}{2}$. The first derivative of $f(x)$ is

$$f'(x) = -3x^2 - 2(n+1)x + n^2 \text{ and the second derivative of } f(x) \text{ is}$$
$$f''(x) = -6x - 2(n+1) < 0 \text{ when } 2 \le x < \frac{n}{2}.$$

Therefore, the function $f(x)$ is concave down. Hence, $f(x)$ has the maximum value at some point in the interval. The critical points of $f(x)$ are $x_1 = \frac{-(n+1)+\sqrt{4n^2+2n+1}}{3}$ and $x_2 = \frac{-(n+1)-\sqrt{4n^2+2n+1}}{3}$. Since $2 \le x < \frac{n}{2}$, but $x_2 < 0$, the maximum occurs at x_1. Thus, $\overline{\xi}^c(\Gamma) \le n^2(n-1) + f(x_1)$. □

Theorem 3.3. *Let Γ be a graph of order n. If*

$$\overline{\xi}^c(\Gamma) > n^2(n-1) + f(x_1),$$

where $f(x) = -x^3 - (n-3)x^2 + (n^2-2)x - n^3 + n$ and $x_1 = \frac{-(n-3)+\sqrt{4n^2-6n+3}}{3}$, then Γ is an HM connected graph.

Proof. Suppose Γ is not an HM-connected graph. By Proposition 2.3, there exists an integer $2 \leq i \leq \frac{n}{2}$ such that $d_{i-1} \leq i$ and $d_{n-i} \leq n-i$. Then, by Equation 2.1, we get the following.

$$\begin{aligned}
\overline{\xi}^c(\Gamma) &\leq n^2(n-1) + \sum_{u \in V(\Gamma)} d(u)^2 - (2n-1) \sum_{u \in V(\Gamma)} d(u) \\
&\leq n^2(n-1) + [(i-1)i^2 + (n-2i+1)(n-i)^2 + i(n-1)^2] \\
&- (2n-1)[(i-1)i + (n-2i+1)(n-i) + i(n-1)] \\
&= n^2(n-1) - i^3 - (n-3)i^2 + (n^2-2)i - n^3 + n.
\end{aligned}$$

Let $f(x) = -x^3 - (n-3)x^2 + (n^2-2)x - n^3 + n$ with $2 \leq x \leq \frac{n}{2}$. The first derivative of $f(x)$ is

$$f'(x) = -3x^2 - 2(n-3)x + (n^2-2) \text{ and the second derivative of } f(x) \text{ is}$$

$$f''(x) = -6x - 2(n-3) < 0 \text{ when } 2 \leq x \leq \frac{n}{2}.$$

Hence, the function $f(x)$ is concave down. The maximum of $f(x)$ occurs at some critical point in the interval. The critical points of $f(x)$ are $x_1 = \frac{-(n-3)+\sqrt{4n^2-6n+3}}{3}$ and $x_2 = \frac{-(n-3)-\sqrt{4n^2-6n+3}}{3}$. Since $2 \leq x \leq \frac{n}{2}$, but $x_2 < 0$, the maximum occurs at x_1. Thus, $\overline{\xi}^c(\Gamma) \leq n^2(n-1) + f(x_1)$. □

3.1 Conditions with the size of the graph

In this section, we state sufficient conditions for Hamiltonicity and graphical properties in terms of eccentric connectivity coindex involving the size of the graph.
Let Γ be a graph of order n and size m. Then, by Equation 2.1, we have

$$\begin{aligned}
\overline{\xi}^c(\Gamma) &\leq n^2(n-1) + \sum_{u \in V(\Gamma)} d(u)^2 - (2n-1) \sum_{u \in V(\Gamma)} d(u) \\
&= n^2(n-1) - 2m(2n-1) + \sum_{u \in V(\Gamma)} d(u)^2 \\
&= n^2(n-1) - 2m(2n-1) + M_1(\Gamma). \qquad (3.1)
\end{aligned}$$

Proposition 3.4. *[2] Let Γ be a graph of order $n \geq 6$. If*

$$M_1 \geq n^3 - 5n^2 + 12n - 6,$$

then Γ is either HM-connected graph or $K_2 \vee (K_1 + K_{n-3})$.

Theorem 3.5. *Let Γ be a graph of order $n \geq 6$, and size m. If*

$$\overline{\xi}^c(\Gamma) \geq n^2(n-1) - 2m(2n-1) + n^3 - 5n^2 + 12n - 6,$$

then Γ is either HM-connected graph or $K_2 \vee (K_1 + K_{n-3})$.

Proof. The proof follows from Equation 3.1 and Proposition 3.4. □

Proposition 3.6. *[3] Let Γ be a graph of order $n \geq \alpha + 1$, If*

$$M_1(\Gamma) \geq (\alpha-1)^2 + (n-\alpha)(n-2)^2 + (\alpha-1)(n-1)^2,$$

then Γ is either α-connected or $K_{\alpha-1} \vee (K_1 + K_{n-\alpha})$.

Theorem 3.7. *Let Γ be a graph of order $n \geq \alpha + 1$, and size m. If*

$$\overline{\xi}^c(\Gamma) \geq n^2(n-1) - 2m(2n-1) + (\alpha-1)^2 + (n-\alpha)(n-2)^2 + (\alpha-1)(n-1)^2,$$

then Γ is either α-connected or $K_{\alpha-1} \vee (K_1 + K_{n-\alpha})$.

Proof. The proof follows from Equation 3.1 and Proposition 3.6. □

Proposition 3.8. *[3] Let Γ be a graph of order $n \geq 5$, and $0 \leq \alpha \leq n-3$. If*

$$M_1(\Gamma) \geq (\alpha+1)^2 + (n-\alpha-2)(n-2)^2 + (\alpha+1)(n-1)^2,$$

then Γ is either α-HM graph or $K_{\alpha+1} \vee (K_1 + K_{n-\alpha-2})$.

Theorem 3.9. *Let Γ be a graph of order $n \geq 5$, size m, and $0 \leq \alpha \leq n-3$. If*

$$\overline{\xi}^c(\Gamma) \geq n^2(n-1) - 2m(2n-1) + (\alpha+1)^2 + (n-\alpha-2)(n-2)^2 + (\alpha+1)(n-1)^2,$$

then Γ is either $\alpha{-}HM$ graph or $K_{\alpha+1} \vee (K_1 + K_{n-\alpha-2})$.

Proof. The proof follows from Equation 3.1 and Proposition 3.8. □

Proposition 3.10. *[3] Let Γ be a graph of order $n \geq 5$. If*

$$M_1(\Gamma) \geq n^3 - 5n^2 + 10n - 6,$$

then Γ is either HM graph or $K_1 \vee (K_1 + K_{n-2})$.

Theorem 3.11. *Let Γ be a graph of order $n \geq 5$, and size m. If*

$$\overline{\xi}^c(\Gamma) \geq n^2(n-1) - 2m(2n-1) + n^3 - 5n^2 + 10n - 6,$$

then Γ is either HM graph or $K_1 \vee (K_1 + K_{n-2})$.

Proof. The proof follows from Equation 3.1 and Proposition 3.10. □

Theorem 3.12. *Let Γ be a graph of order n, and α be an integer such that $0 \leq \alpha \leq n-3$. If*

$$M_1(\Gamma) \geq \alpha^3 + 3\alpha^2 n - 6\alpha^2 + 3\alpha n^2 - 12\alpha n + 11\alpha + n^3 - 5n^2 + 10n - 6,$$

then Γ is α-edge HM graph or $\Gamma = K_1 \vee (K_1 + K_{n-2})$.

Proof. Let Γ be not α-edge HM graph. By Proposition 2.4, there exists an integer $1 \leq i < \frac{n+\alpha}{2}$ such that $d_{i-\alpha} \leq i$ and $d_{n-i} \leq n-i+\alpha-1$. The first Zagreb index of Γ is

$$\begin{aligned}
M_1(\Gamma) &\leq \sum_{u \in V(\Gamma)} d(u)^2 \\
&\leq (i-\alpha)i^2 + (n-2i+\alpha)(n-i+\alpha-1)^2 + i(n-1)^2 \\
&= -i^3 + (5n+4\alpha-4)i^2 - (3n^2+4\alpha^2+8\alpha n-4n-6\alpha+1)i \\
&+ \alpha^3 + 3\alpha^2 n + 3\alpha n^2 - 2\alpha^2 - 4\alpha n + \alpha + n^3 - 2n^2 + n \\
&= (i-1)(-4\alpha^2 - 8\alpha n + 10\alpha - 3n^2 + 9n - i^2 + i(4\alpha+5n-5) - 6) \\
&+ \alpha^3 + 3\alpha^2 n - 6\alpha^2 + 3\alpha n^2 - 12\alpha n + 11\alpha + n^3 - 5n^2 + 10n - 6,
\end{aligned}$$

Following the condition of the theorem, we obtain that $(i-1)(4\alpha^2+8\alpha n-10\alpha+3n^2-9n+i^2-i(4\alpha+5n-5)+6) \leq 0$. Now, the following cases arise.

Case 1: $(i-1)(4\alpha^2+8\alpha n-10\alpha+3n^2-9n+i^2-i(4\alpha+5n-5)+6) = 0$.
In this case, $M_1(\Gamma) = \alpha^3+3\alpha^2 n-6\alpha^2+3\alpha n^2-12\alpha n+11\alpha+n^3-5n^2+10n-6$, and then all the inequalities above should be equalities. Hence, we obtain $d_1 = \cdots = d_{i-\alpha} = i$, $d_{i-\alpha+1} = \cdots = d_{n-i} = n-i+\alpha-1$, $d_{n-i+1} = \cdots = d_n = n-1$. Now, the following subcases arise.

Subcase 1.1: For $i-1=0$, we get $d_1 = 1$, $d_2 = \cdots = d_{n-1} = n-2$, $d_n = n-1$. Which implies $\Gamma = K_1 \vee (K_1 + K_{n-2})$.

Subcases 1.2: For $4\alpha^2+8\alpha n-10\alpha+3n^2-9n+i^2-i(4\alpha+5n-5)+6 = 0$, we have $4\alpha^2+8\alpha n-10\alpha+3n^2-9n+i^2-i(4\alpha+5n-5)+6 = 0$. Since $i < \frac{n+\alpha}{2}$, we get

$0 = 4\alpha^2 + 8\alpha n - 10\alpha + 3n^2 - 9n + i^2 - i(4\alpha + 5n - 5) + 6 = 4\alpha^2 + 8\alpha n - 10\alpha + 3n^2 - 9n - i(4\alpha + 5n - 5 - i) + 6 > 4\alpha^2 + 8\alpha n - 10\alpha + 3n^2 - 9n - \frac{1}{2}(n+\alpha)(4\alpha + 5n - 5 - \frac{1}{2}(n+\alpha)) + 6 = \frac{1}{4}(9\alpha^2 + 16\alpha n + 3n^2 - 30\alpha - 26n + 24)$. Through straightforward computation and a combination of $n \geq \alpha+3$, we get $\alpha = 0$. Therefore, $3n^2 + i^2 - 5in + 5i - 9n + 6 = 0$. Since $i < \frac{n}{2}$, we get $n < 0$.

Subcase 2: $(i-1)(4\alpha^2 + 8\alpha n - 10\alpha + 3n^2 - 9n + i^2 - i(4\alpha + 5n - 5) + 6) < 0$.
In this case, we have $i \geq 2$ and $4\alpha^2 + 8\alpha n - 10\alpha + 3n^2 - 9n + i^2 - i(4\alpha + 5n - 5) + 6 < 0$. From the Subcase 1.2, we have $n < 0$. This completes the proof. □

Theorem 3.13. *Let Γ be a graph of order n, size m, and α be an integer such that $0 \leq \alpha \leq n-3$. If*

$$\overline{\xi}^c(\Gamma) \geq n^2(n-1) - 2m(2n-1)\alpha^3 + 3\alpha^2 n - 6\alpha^2 + 3\alpha n^2 - 12\alpha n + 11\alpha + n^3 - 5n^2 + 10n - 6,$$

then Γ be α-edge HM graph or $\Gamma = K_1 \vee (K_1 + K_{n-2})$.

Proof. The proof follows from Equation 3.1 and Theorem 3.12. □

Theorem 3.14. *Let Γ be a graph of order n, and α be an integer such that $1 \leq \alpha \leq n-1$.*

(i) When $n - \alpha - 1$ is even, for $n \geq 5\alpha + 4$ or $n - \alpha - 1$ is odd, for $n \geq 5\alpha + 2$, and if $M_1(\Gamma) \geq n^3 - (3\alpha+5)n^2 + (3\alpha^2 + 12\alpha + 10)n - \alpha^3 - 6\alpha^2 - 11\alpha - 6$, then Γ be α-path coverable or $\Gamma = K_1 \vee (\overline{K_{\alpha+1}} + K_{n-\alpha-2})$.

(ii) When $n - \alpha - 1$ is even, for $\alpha + 3 \leq n \leq 5\alpha + 3$, and if $M_1(\Gamma) \geq \frac{1}{8}(5n^3 - (5\alpha+13)n^2 - (\alpha^2 - 6\alpha - 11)n + \alpha^3 + 3\alpha^2 - \alpha - 3)$, then Γ be α-path coverable or $\Gamma = K_{\frac{n-\alpha-1}{2}} \vee \frac{n+\alpha+1}{2} K_1$.

(iii) When $n - \alpha - 1$ is odd, for $\alpha + 4 \leq n \leq 5\alpha + 1$, if $M_1(\Gamma) \geq f(\frac{n-\alpha-2}{2}) \geq 0$. Therefore, $\max f(x) = f(\frac{n-\alpha-2}{2}) = \frac{1}{8}(5n^3 - (5\alpha+18)n^2 - (\alpha^2 - 4\alpha - 32)n + \alpha^3 + 6\alpha^2 - 8\alpha - 16)$, and then Γ be α-path coverable or $\Gamma = K_{\frac{n-\alpha-2}{2}} \vee (K_2 + \frac{n+\alpha-2}{2} K_1)$.

Proof. Let Γ be not α-path coverable. By Proposition 2.5, there exists an integer $1 \leq i < \frac{1}{2}(n-\alpha)$ such that $d_{i+\alpha} \leq i$ and $d_{n-i} \leq n - i - \alpha - 1$. The first Zagreb index is

$$\begin{aligned} M_1(\Gamma) &\leq \sum_{u \in V(\Gamma)} d(u)^2 \\ &\leq (i+\alpha)i^2 + (n - 2i - \alpha)(n - i - \alpha - 1)^2 + i(n-1)^2 \\ &= -i^3 + (5n - 4\alpha - 4)i^2 - (4\alpha^2 - 8\alpha n + 3n^2 + 6\alpha - 4n + 1)i \\ &- \alpha^3 + 3\alpha^2 n - 3\alpha n^2 + n^3 - 2\alpha^2 + 4\alpha n - 2n^2 - \alpha + n. \end{aligned}$$

Let $f(x) = -x^3 + (-4\alpha + 5n - 4)x^2 + (-4\alpha^2 + 8\alpha n - 3n^2 - 6\alpha + 4n - 1)x - \alpha^3 + 3\alpha^2 n - 3\alpha n^2 + n^3 - 2\alpha^2 + 4\alpha n - 2n^2 - \alpha + n$ with $1 \leq x < \frac{1}{2}(n-\alpha)$ and $1 \leq \alpha \leq n-1$. The first derivative of $f(x)$ is

$$f'(x) = -3x^2 + 2(5n - 4\alpha - 4)x - (4\alpha^2 - 8\alpha n + 3n^2 + 6\alpha - 4n + 1)),$$

and the second derivative of $f(x)$ is $f''(x) = -6x + 2(5n - 4\alpha - 4)$.

Since $1 \leq x < \frac{1}{2}(n-\alpha)$, we can assume $1 \leq x \leq \frac{1}{2}(n - \alpha - 1)$. Then $f''(x) = -6x + 2(5n - 4\alpha - 4) = 3(n - \alpha - 1 - 2x) + 7n - 5\alpha - 5 > 0$, hence $f(x)$ is convex function. Therefore, $\max f(x) \in \{f(1), f(\frac{1}{2}(n - \alpha - 1))\}$. By simple calculation $f(1) = n^3 - (3\alpha+5)n^2 + (3\alpha^2 + 12\alpha + 10)n - \alpha^3 - 6\alpha^2 - 11\alpha - 6$.
We consider the following two cases.
Case 1: When $n - \alpha - 1$ is even.

$$f(\frac{n-\alpha-1}{2}) = \frac{1}{8}(5n^3 - (5\alpha+13)n^2 - (\alpha^2 - 6\alpha - 11)n + \alpha^3 + 3\alpha^2 - \alpha - 3).$$

Consider the difference

$$f(1)-f(\frac{n-\alpha-1}{2}) = \frac{1}{8}(3n^3-(19\alpha+27)n^2+(25\alpha^2+90\alpha+69)n-9\alpha^3-51\alpha^2 -87\alpha-45) = \frac{1}{8}(n-\alpha-3)(3n^2-(16\alpha+18)n+9\alpha^2+24\alpha+15).$$

Hence, the following subcases arise.
Subcase 1.1 For $\alpha+3\leq n\leq 5\alpha+3$:
From $n-\alpha-3\geq 0$ and $n-5\alpha-3\leq 0$, we have $f(1)-f(\frac{n-\alpha-1}{2})\leq 0$. Therefore, $\max f(x)=f(\frac{n-\alpha-1}{2})=\frac{1}{8}(5n^3-(5\alpha+13)n^2-(\alpha^2-6\alpha-11)n+\alpha^3+3\alpha^2-\alpha-3)$.
Hence, by the given condition of the theorem, if equality holds, then $i=\frac{n-\alpha-1}{2}$ and this implies that $d_1=d_2=\cdots=d_{\frac{n+\alpha-1}{2}}=\frac{n-\alpha-1}{2}$, $d_{\frac{n+\alpha+1}{2}}=\frac{n-\alpha-1}{2}$, and $d_{\frac{n+\alpha+3}{2}}=\cdots=d_n=n-1$, this means $\Gamma=K_{\frac{n-\alpha-1}{2}}\vee\frac{n+\alpha+1}{2}K_1$.

Subcase 1.2 For $n\geq 5\alpha+4$:
From $n-\alpha-3\geq 0$ and $n-5\alpha-4\geq 0$, we have $f(1)-f(\frac{n-\alpha-1}{2})\geq 0$. As, $3n^2-(16\alpha+18)n+9\alpha^2+24\alpha+15\geq 4\alpha^2-10\alpha-9>0\ \forall\ \alpha\geq 4, n\geq 5\alpha+4$, therefore, $\max f(x)=f(1)=-\alpha^3+(3n-6)\alpha^2-(3n^2-12n+11)\alpha+n^3-5n^2+10n-6$.
If equality holds, then $i=1$, and this implies that $d_1=d_2=\cdots=d_{\alpha+1}=1$, $d_{\alpha+2}=d_{n-1}=n-\alpha-2$, and $d_n=n-1$, hence $\Gamma=K_1\vee(\overline{K_{\alpha+1}}+K_{n-\alpha-2})$.

Case 2: When $n-\alpha-1$ is odd.

$$f(\frac{n-\alpha-2}{2})=\frac{1}{8}(5n^3-(5\alpha+18)n^2-(\alpha^2-4\alpha-32)n+\alpha^3+6\alpha^2-8\alpha-16).$$

Consider the difference

$$f(1)-f(\frac{n-\alpha-2}{2}) = \frac{1}{8}(-9\alpha^3+(25n-54)\alpha^2-(19n^2-92n+80)\alpha+3n^3-22n^2 +48n-32) = \frac{1}{8}(n-\alpha-4)(3n^2-(16\alpha+10)n+9\alpha^2+18\alpha+8).$$

Now, consider the following subcases.
Subcase 2.1 For $\alpha+4\leq n\leq 5\alpha+1$:
From $n-\alpha-4\geq 0$ and $n-5\alpha-1\leq 0$, we have $f(1)-f(\frac{n-\alpha-2}{2})\leq 0$, therefore, $\max f(x)=f(\frac{n-\alpha-2}{2})=\frac{1}{8}(5n^3-(5\alpha+18)n^2-(\alpha^2-4\alpha-32)n+\alpha^3+6\alpha^2-8\alpha-16)$.
Hence, by the given condition of the theorem, if equality holds, then $i=\frac{n-\alpha-2}{2}$ and we have $d_1=d_2=\cdots=d_{\frac{n+\alpha-2}{2}}=\frac{n-\alpha-2}{2}$, $d_{\frac{n+\alpha}{2}}=d_{\frac{n+\alpha+2}{2}}=\frac{n-\alpha}{2}$, and $d_{\frac{n+\alpha+4}{2}}=\cdots=d_n=n-1$. Hence, $\Gamma=K_{\frac{n-\alpha-2}{2}}\vee(K_2+\frac{n+\alpha-2}{2}K_1)$.

Subcase 2.2 For $n\geq 5\alpha+2$:
From $n-\alpha-3\geq 0$ and $n-5\alpha-2\geq 0$, we have $f(1)-f(\frac{n-\alpha-2}{2})>0$. As, $3n^2-(16\alpha+10)n+9\alpha^2+18\alpha+8\geq 4\alpha^2-4\alpha>0\ \forall\alpha\geq 4,\ n\geq 5\alpha+2$, therefore, $\max f(x)=f(1)=-\alpha^3+(3n-6)\alpha^2-(3n^2-12n+11)\alpha+n^3-5n^2+10n-6$. This completes the proof. □

Theorem 3.15. *Let Γ be a graph of order n, size m, and α be an integer such that $1\leq\alpha\leq n-1$.*

(i) When $n-\alpha-1$ is even, for $n\geq 5\alpha+4$ or $n-\alpha-1$ is odd, for $n\geq 5\alpha+2$, if $\overline{\xi}^c(\Gamma)\geq n^2(n-1)-2m(2n-1)n^3-(3\alpha+5)n^2+(3\alpha^2+12\alpha+10)n-\alpha^3-6\alpha^2-11\alpha-6$, then Γ be α-path coverable or $\Gamma=K_1\vee(\overline{K_{\alpha+1}}+K_{n-\alpha-2})$.

(ii) When $n-\alpha-1$ is even, for $\alpha+3\leq n\leq 5\alpha+3$, if $\overline{\xi}^c(\Gamma)\geq n^2(n-1)-2m(2n-1)\frac{1}{8}(5n^3-(5\alpha+13)n^2-(\alpha^2-6\alpha-11)n+\alpha^3+3\alpha^2-\alpha-3)$, then Γ be α-path coverable or $\Gamma=K_{\frac{n-\alpha-1}{2}}\vee\frac{n+\alpha+1}{2}K_1$.

(iii) *When $n-\alpha-1$ is odd, for $\alpha+4 \leq n \leq 5\alpha+1$, if $\overline{\xi}^c(\Gamma) \geq n^2(n-1)-2m(2n-1)f(\frac{n-\alpha-2}{2}) \leq 0$. Therefore,* $\max f(x) = f(\frac{n-\alpha-2}{2}) = \frac{1}{8}(5n^3-(5\alpha+18)n^2-(\alpha^2-4\alpha-32)n+\alpha^3+6\alpha^2-8\alpha-16)$*, and then Γ be α-path coverable or $\Gamma = K_{\frac{n-\alpha-2}{2}} \vee (K_2 + \frac{n+\alpha-2}{2}K_1)$.*

Proof. The proof follows from Equation 3.1 and Theorem 3.14. □

4 Conclusion

This work has provided sufficient conditions for the traceable, Hamiltonian, Hamiltonian connected graph in terms of the eccentric connectivity coindex. Further, the sufficient conditions for Hamiltonian, Hamiltonian-connected, α-connected, and α-Hamiltonian graphs involving the number of edges in terms of eccentric Zagreb coindex have been presented. Moreover, we have proved the conditions for α-path coverable and α-edge Hamiltonian for First Zagreb indices. There is a high scope that study can be extended to eccentricity-based Harmonic coindex, symmetric division coindex, Inverse Randić coindex, etc.

Declarations

Data Availability Statement: The authors declare that [the/all other] data supporting the findings of this study are available within the article. Any clarification may be requested from the corresponding author, provided it is essential.

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