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DIFFERENT ORDER B-SPLINES WITH THE GALERKIN METHOD FOR A COUPLED SYSTEM OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract This paper introduces the finite element method (FEM) using the Galerkin method with cubic and quintic B-splines as the basis functions for solving coupled systems of nonlinear boundary value problems (BVPs). To linearize the nonlinear BVPs, the quasilinearization technique is employed to convert them into a sequence of linear BVPs. Quintic B-splines are used to approximate variables with fourth-order derivatives, while cubic B-splines are used for variables with second-order derivatives in the considered BVPs. The Galerkin method is then utilized to obtain the results with these approximations. The effectiveness of the proposed method was evaluated by applying it to a specific problem from the literature. The numerically obtained solutions with the proposed method were found to agree with those in the literature for the example tested. Additionally, an error analysis technique involving residual functions is employed to enhance the numerical solution.

1 Introduction

Several researchers have focused on the system of nonlinear BVPs that have arisen from the mathematical modeling of many physical systems over the past several decades. The system of BVPs with two or more unknowns is used in most mathematical modeling of physical systems. The study is interesting from a mathematical perspective. Sheikholeslami et al. [1] have studied the coupled system of nonlinear BVPs solved by the fourth-order method of Runge-Kutta. Bilal et al. [2] performed the partial differential equations (PDEs) corresponding to the forced balanced law, energy equations, and concentration equations, which were transformed into a system of nonlinear ordinary differential equations (ODEs) using appropriate techniques. The cash and carp coefficients were used to improve the numerical solutions of the transformed nonlinear systems. Furthermore, this problem was solved using the Runge-Kutta-Fehlberg (RKF) shooting technique.

Makinde [3] investigated the set of governing PDEs along with the respective boundary conditions, which were transformed into a system of nonlinear ODEs with the corresponding boundary conditions and then solved by the Runge-Kutta integration technique and the Newton-Raphson shooting method's modified version. Seddeek et al. [4] have reported that the similarity solution can be deployed to convert the set of PDEs into nonlinear ODEs together with boundary conditions, after which an efficient numerically modified fourth-order Runge-Kutta method is combined with the shooting algorithm. Pal [5] has used similarity transformations to transform the system of nonlinear equations under boundary conditions to a system of nonlinear ODEs. The implicit finite difference method is then used to integrate the differential equations. Pal and Mondal [6] have discussed the set of nonlinear ODEs with appropriate boundary conditions that were numerically solved by a shooting algorithm with the Runge-Kutta Fehlberg integration scheme. Hussain [7] found the solution by employing similarity transformations, which reduces the problems governing a set of PDEs to a coupled system of nonlinear ODEs. The resulting system of nonlinear ODEs is numerically solved using the shooting technique and continuous Galerkin-Petrov discretization.

In a study by El-Dabe et al. [8] computed, the governing PDEs are transformed into nonlinear ODEs by similarity transformation and resolved via the finite difference method (FDM). According to Idowu and Falodun [9], the governing coupled nonlinear PDEs were simplified into nonlinear ODEs using appropriate similarity transformations. The modeled equations were solved using the spectral homotopy analysis method (SHAM). Akinbo and Olajuwon [10] proposed that the method of similarity is used to transform nonlinear PDEs into a system of coupled nonlinear ODEs. Then, the resulting equation can then be solved using the homotopy analysis method (HAM). Nadeem et al. [11] calculated the governing PDEs into a set of nonlinear ODEs via similarity transformation. Then, it was solved numerically using the shooting method and the Runge-Kutta-Fehlberg method. Eid and Mahny [12] focused on the nonlinear governing equations, which were reduced to ODEs by a suitable similarity transform and then numerically solved using the Runge-Kutta-Fehlberg 4th–5th order (RKF45) along with the shooting technique. Beg et al. [13] have obtained the numerical solutions to nonlinear ODEs by using the technique of shooting with Runge-Kutta sixth-order. Viswanadham and Murali [14] depicted the coupled system of BVPs solved by the Galerkin method along with the basis functions of cubic B-splines. Dhivya and Murali [15] described the collocation method based on cubic and quartic basis functions for solving a system of third-order nonlinear BVPs. Murali and Dhivya [16], we investigated a Galerkin procedure with cubic and quartic B-splines for numerically solving highly coupled systems of nonlinear BVPs.

According to Makinde et al. [17], a similarity transformation converts the governing PDEs into ODEs, which are then solved numerically using the shooting quadrature. Vedavathi et al. [18] investigated the numerical solutions of nonlinear ODEs with boundary conditions using a shooting technique associated with the Runge-Kutta-4th order method. The equations of the boundary layer were reduced into a system of ordinary nonlinear differential equations using appropriate similarity transformations according to Sreedevi et al. [19]. Numerical solutions were also obtained and graphically illustrated using the shooting method and the Runge-Kutta fourth-order integration technique. The flow equations, which were solved by using the spectral quasi-linearization method, with residual errors less than 10^{-08}, are found in Dhlamini et al. [20]. Dewasurendra and Vajravelu [21] extended the Liaos method to the coupled nonlinear system from the three differential equations. Also, residual error versus the approximate values of the plot is found. Khan et al. [22] analyzed the appropriate similarity transformations used to transform the fluid flow velocity, temperature, and concentration into highly nonlinearly coupled differential equations under physical conditions. The problem was solved by the Homotopy Analysis Method (HAM). In addition, residual graphs and the residual error table demonstrate the work's authenticity.

Sithole et al. [23] examined and analyzed the method of spectral local linearization (SLLM) used to resolve the system of ODEs with boundary conditions. In addition, residual errors and error norms were evaluated to ensure the accuracy of the numerical scheme. The residual error indicates how close the numerical solution is to the real solution. Motsa and Makukula [24], who presented the bivariate spectral homotopy analysis method (BSHAM), which they used to solve the system of nonlinear PDEs that modeled heat and mass transfer applications. The residual error of the PDEs was used to observe convergence. As per Ferdows et al. [25], the specified problem of the governing equations is nonlinear and complex; therefore, similarity transformations are used to obtain the simplest mathematical model in which the differential equations are ordinary and asymmetric. This was performed using MATLAB software and the spectral relaxation technique (SRM). Residual errors versus number of iterations for several parameter rates were also observed. Shah et al. [26] have examined how governing PDEs can be converted into a system of ODEs. The HAM was used to solve the equation. The residual error was also found. Saleem et al. [27] enhanced the presented governing problems by using the optimal homotopy analysis method (OHAM). Moreover, as the order of approximation increases, the mean squared residuals and the total mean squared residuals become increasingly smaller. Issa [37] has studied how the Reconstruction of Variational Iteration Method (RVIM) can be implemented to solve a linear system of Volterra Integro-Differential Equations (VIDEs). Adibmanesh and Rashidinia [38] have applied the time-fractional-convection-diffusion problem using a numerical technique based on the Sinc function and B-spline scaling functions. Elomari et al. [39], examined the solutions for coupled systems of time-fractional differential problems.

In this paper, we propose the Galerkin method with cubic and quintic B-splines to resolve the following coupled system of fourth-order nonlinear BVPs.

$$NOL_i(\mathsf{g}, a, a^{'}, a^{''}, a^{'''}, a^{'v}, b, b^{'}, b^{''}, c, c^{'}, c^{''}, d, d^{'}, d^{''}) = 0 \tag{1.1}$$

along with the boundary conditions

$$a(\mathsf{g}_{l0}) = a_0,\ a(\mathsf{g}_{rn}) = a_1,\ a^{'}(\mathsf{g}_{l0}) = a_2,\ a^{'}(\mathsf{g}_{rn}) = a_3,\ b(\mathsf{g}_{l0}) = b_0,\ b(\mathsf{g}_{rn}) = b_1,$$
$$c(\mathsf{g}_{l0}) = c_0,\ c(\mathsf{g}_{rn}) = c_1,\ d(\mathsf{g}_{l0}) = d_0,\ d(\mathsf{g}_{rn}) = d_1. \tag{1.2}$$

Where NOL is denoted as the nonlinear operator, $i = 1, 2, 3, 4$, and a, b, c, and d are the unknown variables in the coupled system. We used the quasilinearization technique Bellman and Kalaba [28] to linearize the considered nonlinear BVPs, as given below.

$$\sum_{k=1}^{5} u_{ik}(\mathsf{g})a^{(5-k)} + \sum_{k=3}^{5} v_{ik}(\mathsf{g})b^{(5-k)} + \sum_{k=3}^{5} w_{ik}(\mathsf{g})c^{(5-k)}$$
$$+ \sum_{k=3}^{5} y_{ik}(\mathsf{g})d^{(5-k)} = B_i(\mathsf{g}), \quad i = 1, 2, 3, 4 \tag{1.3}$$

together with the boundary conditions

$$a(\mathsf{g}_{l0}) = a_0,\ a(\mathsf{g}_{rn}) = a_1,\ a^{'}(\mathsf{g}_{l0}) = a_2,\ a^{'}(\mathsf{g}_{rn}) = a_3,\ b(\mathsf{g}_{l0}) = b_0,\ b(\mathsf{g}_{rn}) = b_1,$$
$$c(\mathsf{g}_{l0}) = c_0,\ c(\mathsf{g}_{rn}) = c_1,\ d(\mathsf{g}_{l0}) = d_0,\ d(\mathsf{g}_{rn}) = d_1. \tag{1.4}$$

Where $a_i(i = 0, 1, 2, 3)$ and $b_i, c_i, d_i(i = 0, 1)$ are real constants. In addition, g_{l0} denotes the boundary points on the left, and g_{rn} denotes the boundary point on the right. Here $u_{ik}, v_{ik}, w_{ik}, y_{ik}$ are continuous functions in interval $[\mathsf{g}_{l0}, \mathsf{g}_{rn}]$. On the interval $[\mathsf{g}_{l0}, \mathsf{g}_{rn}]$, B_i is assumed to be continuous. This study presents a simple FEM for solving the coupled system of nonlinear BVPs using the Galerkin approach with B-splines as basis functions. The study structure is described below. Section 2 explains the implementation of the Galerkin method. Section 3 expresses the Galerkin technique and the basis functions of cubic and quintic B-splines. Section 4 evaluates the proposed method on a coupled system of fourth-order nonlinear BVPs. In addition, residual graphs are found. Section 5 summarizes the findings and discussion. Finally, the conclusions are presented in the last section.

2 Justification for the use of the Galerkin method

The approximate solution in the FEM is a linear combination of the basis functions, which form the basis for an approximation space. Using this method, a weak form of a solution of the approximation for a differential equation exists and is unique under suitable conditions, as discussed in Bers et al. [29] and Lions and Magenes [30] regardless of the properties of a given differential operator. Furthermore, if the boundary conditions are considered, the weak solution tends to be the classical solution of a given differential equation in Mitchel [31] and Reddy [32]. Where the Dirichlet-type boundary conditions are mentioned, the basis functions must vanish on the boundary; those details are presented in Murali and Dhivya [16].

3 Explanation of the methodology

The B-splines are defined in Cox [33], Carl [34], and Prenter [35]. The cubic B-splines are defined in Murali and Dhivya [16], introduces six additional knots $C_{3,-3}, C_{3,-2}, C_{3,-1}, C_{3,n+1}$,

$C_{3,n+2}, C_{3,n+3}$ in such a way that $\mathsf{g}_{-3} < \mathsf{g}_{-2} < \mathsf{g}_{-1} < \mathsf{g}_0$ and $\mathsf{g}_n < \mathsf{g}_{n+1} < \mathsf{g}_{n+2} < \mathsf{g}_{n+3}$. Schoenberg [36] demonstrated that cubic B-splines are only nonzero splines with the smallest compact support with the knots at $\mathsf{g}_{-3} < \mathsf{g}_{-2} < \mathsf{g}_{-1} < \mathsf{g}_0 < ... < \mathsf{g}_n < \mathsf{g}_{n+1} < \mathsf{g}_{n+2} < \mathsf{g}_{n+3}$. The cubic B-splines $C_{3,i}(\mathsf{g})$ are now defined by

$$C_{3,i}(\mathsf{g}) = \begin{cases} \sum\limits_{r=i-2}^{r=i+2} \left[\frac{(\mathsf{g}_r-\mathsf{g})_+^3}{\Pi'(\mathsf{g}_r)}\right] & \text{if } \mathsf{g} \in [\mathsf{g}_{i-2}, \mathsf{g}_{i+2}] \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

where $\Pi(\mathsf{g}) = (\mathsf{g}-\mathsf{g}_{i-2})(\mathsf{g}-\mathsf{g}_{i-1})(\mathsf{g}-\mathsf{g}_i)(\mathsf{g}-\mathsf{g}_{i+1})(\mathsf{g}-\mathsf{g}_{i+2})$ and $(\mathsf{g}_r-\mathsf{g})_+^3$ is the function of the positive part.
And $C_{3,-1}(\mathsf{g}), C_{3,0}(\mathsf{g}), C_{3,1}(\mathsf{g}), ..., C_{3,n-1}(\mathsf{g}), C_{3,n}(\mathsf{g}), C_{3,n+1}(\mathsf{g})$ forms the basis for a space of cubic polynomial splines defined on the given interval.

In a similar manner, quintic B-splines are defined by $C_{5,i}(\mathsf{g})$

$$C_{5,i}(\mathsf{g}) = \begin{cases} \sum\limits_{r=i-3}^{r=i+3} \left[\frac{(\mathsf{g}_r-\mathsf{g})_+^5}{\Pi'(\mathsf{g}_r)}\right] & \text{if } \mathsf{g} \in [\mathsf{g}_{i-3}, \mathsf{g}_{i+3}] \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where $\Pi(\mathsf{g}) = (\mathsf{g}-\mathsf{g}_{i-3})(\mathsf{g}-\mathsf{g}_{i-2})(\mathsf{g}-\mathsf{g}_{i-1})(\mathsf{g}-\mathsf{g}_i)(\mathsf{g}-\mathsf{g}_{i+1})(\mathsf{g}-\mathsf{g}_{i+2})(\mathsf{g}-\mathsf{g}_{i+3})$ and $(\mathsf{g}_r-\mathsf{g})_+^5$ is the function of the positive part.
And $C_{5,-2}(\mathsf{g}), C_{5,-1}(\mathsf{g}), C_{5,0}(\mathsf{g}), C_{5,1}(\mathsf{g}), ..., C_{5,n-1}(\mathsf{g}), C_{5,n}(\mathsf{g}), C_{5,n+1}(\mathsf{g}), C_{5,n+2}(\mathsf{g})$ forms the basis for a space of quintic polynomial splines with the addition of four additional knots are $\mathsf{g}_{-5}, \mathsf{g}_{-4}, \mathsf{g}_{n+4}, \mathsf{g}_{n+5}$. Schoenberg [36] demonstrated the quintic B-splines are only nonzero splines with the smallest compact support and the knots at $\mathsf{g}_{-5} < \mathsf{g}_{-4} < \mathsf{g}_{-3} < \mathsf{g}_{-2} < \mathsf{g}_{-1} < \mathsf{g}_0 < ... < \mathsf{g}_n < \mathsf{g}_{n+1} < \mathsf{g}_{n+2} < \mathsf{g}_{n+3} < \mathsf{g}_{n+4} < \mathsf{g}_{n+5}$.

We define the approximation for $a(\mathsf{g}), b(\mathsf{g}), c(\mathsf{g})$, and $d(\mathsf{g})$ to solve BVPs (1.1) - (1.2) using the Galerkin method with cubic, quintic B-splines.

$$a(\mathsf{g}) = \sum_{k=-2}^{n+2} a_k C_{5,k}(\mathsf{g}) \tag{3.3}$$

$$b(\mathsf{g}) = \sum_{k=-1}^{n+1} b_k C_{3,k}(\mathsf{g}) \tag{3.4}$$

$$c(\mathsf{g}) = \sum_{k=-1}^{n+1} c_k C_{3,k}(\mathsf{g}) \tag{3.5}$$

$$d(\mathsf{g}) = \sum_{k=-1}^{n+1} d_k C_{3,k}(\mathsf{g}) \tag{3.6}$$

where the parameters to be determined are a_k, b_k, c_k, d_k.

The basis functions in the Galerkin method must vanish on the boundary on which the type of Dirichlet boundary condition is defined. As a result, the basis functions must be redefined into a new form of a set of basis functions that vanish on the boundary, where the type of Dirichlet boundary condition is specified. Because cubic and quintic B-splines polynomials have been used to approximate the system of fourth-order BVPs, we redefine the basis functions into a new form of a set of basis functions that vanish on the boundary in which the type of Dirichlet boundary conditions are specified. We obtain an approximate solution at the boundary points using cubic and quintic B-spline definitions and the type of Dirichlet boundary conditions described in (1.2). We get

$$
\begin{aligned}
& a(\mathsf{g}_{l0}) = a_0 \\
& \Rightarrow a_{-2}C_{5,-2}(\mathsf{g}_{l0}) + a_{-1}C_{5,-1}(\mathsf{g}_{l0}) + a_0C_{5,0}(\mathsf{g}_{l0}) + a_1C_{5,1}(\mathsf{g}_{l0}) + a_2C_{5,2}(\mathsf{g}_{l0}) = a_0 \\
& \Rightarrow a_{-2} = \frac{1}{C_{5,-2}(\mathsf{g}_{l0})}\left\{a_0 - [a_{-1}C_{5,-1}(\mathsf{g}_{l0}) + a_0C_{5,0}(\mathsf{g}_{l0}) + a_1C_{5,1}(\mathsf{g}_{l0}) + a_2C_{5,2}(\mathsf{g}_{l0})]\right\}
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
& a(\mathsf{g}_{rn}) = a_1 \\
& \Rightarrow a_{n-2}C_{5,n-2}(\mathsf{g}_{rn}) + a_{n-1}C_{5,n-1}(\mathsf{g}_{rn}) + a_0C_{5,n}(\mathsf{g}_{rn}) + a_{n+1}C_{5,n+1}(\mathsf{g}_{rn}) + a_{n+2}C_{5,n+2}(\mathsf{g}_{rn}) = a_1 \\
& \Rightarrow a_{n+2} = \frac{1}{C_{5,n+2}(\mathsf{g}_{rn})}\left\{a_1 - [a_{n-2}C_{5,n-2}(\mathsf{g}_{rn}) + a_{n-1}C_{5,n-1}(\mathsf{g}_{rn}) + a_0C_{5,n}(\mathsf{g}_{rn}) + a_{n+1}C_{5,n+1}(\mathsf{g}_{rn})]\right\}
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
& b(\mathsf{g}_{l0}) = b_0 \\
& \Rightarrow b_{-1}C_{3,-1}(\mathsf{g}_{l0}) + b_0C_{3,0}(\mathsf{g}_{l0}) + b_1C_{3,1}(\mathsf{g}_{l0}) \\
& \Rightarrow b_{-1} = \frac{1}{C_{3,-1}(\mathsf{g}_{l0})}\left\{b_0 - [b_0C_{3,0}(\mathsf{g}_{l0}) + b_1C_{3,1}(\mathsf{g}_{l0})]\right\}
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
& b(\mathsf{g}_{rn}) = b_1 \\
& \Rightarrow b_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + b_nC_{3,n}(\mathsf{g}_{rn}) + b_{n+1}C_{3,n+1}(\mathsf{g}_{rn}) = b_1 \\
& \Rightarrow b_{n+1} = \frac{1}{C_{3,n+1}(\mathsf{g}_{rn})}\left\{b_1 - [b_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + b_nC_{3,n}(\mathsf{g}_{rn})]\right\}
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
& c(\mathsf{g}_{l0}) = c_0 \\
& \Rightarrow c_{-1}C_{3,-1}(\mathsf{g}_{l0}) + c_0C_{3,0}(\mathsf{g}_{l0}) + c_1C_{3,1}(\mathsf{g}_{l0}) = c_0 \\
& \Rightarrow c_{-1} = \frac{1}{C_{3,-1}(\mathsf{g}_{l0})}\left\{c_0 - [c_0C_{3,0}(\mathsf{g}_{l0}) + c_1C_{3,1}(\mathsf{g}_{l0})]\right\}
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
& c(\mathsf{g}_{rn}) = c_1 \\
& \Rightarrow c_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + c_nC_{3,n}(\mathsf{g}_{rn}) + c_{n+1}C_{3,n+1}(\mathsf{g}_{rn}) = c_1 \\
& \Rightarrow c_{n+1} = \frac{1}{C_{3,n+1}(\mathsf{g}_{rn})}\left\{c_1 - [c_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + c_nC_{3,n}(\mathsf{g}_{rn})]\right\}
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
& d(\mathsf{g}_{l0}) = d_0 \\
& \Rightarrow d_{-1}C_{3,-1}(\mathsf{g}_{l0}) + d_0C_{3,0}(\mathsf{g}_{l0}) + d_1C_{3,1}(\mathsf{g}_{l0}) = d_0 \\
& \Rightarrow d_{-1} = \frac{1}{C_{3,-1}(\mathsf{g}_{l0})}\left\{d_0 - [d_0C_{3,0}(\mathsf{g}_{l0}) + d_1C_{3,1}(\mathsf{g}_{l0})]\right\}
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
& d(\mathsf{g}_{rn}) = d_1 \\
& \Rightarrow d_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + d_nC_{3,n}(\mathsf{g}_{rn}) + d_{n+1}C_{3,n+1}(\mathsf{g}_{rn}) = d_1 \\
& \Rightarrow d_{n+1} = \frac{1}{C_{3,n+1}(\mathsf{g}_{rn})}\left\{d_1 - [d_{n-1}C_{3,n-1}(\mathsf{g}_{rn}) + d_nC_{3,n}(\mathsf{g}_{rn})]\right\}
\end{aligned} \tag{3.14}
$$

Substituting the values of each parameter in the approximations yields the revised approximations for our unknown variables as follows:

$$
a(\mathsf{g}) = aw(\mathsf{g}) + \sum_{k=-1}^{n+1} a_kP_{5,k}(\mathsf{g}) \tag{3.15}
$$

$$b(\mathsf{g}) = bw(\mathsf{g}) + \sum_{k=0}^{n} b_k Q_{3,k}(\mathsf{g}) \tag{3.16}$$

$$c(\mathsf{g}) = cw(\mathsf{g}) + \sum_{k=0}^{n} c_k Q_{3,k}(\mathsf{g}) \tag{3.17}$$

$$d(\mathsf{g}) = dw(\mathsf{g}) + \sum_{k=0}^{n} d_k Q_{3,k}(\mathsf{g}) \tag{3.18}$$

where

$$aw(\mathsf{g}) = \frac{a_0}{C_{5,-2}(\mathsf{g}_{l0})} C_{5,-2}(\mathsf{g}) + \frac{a_1}{C_{5,n+2}(\mathsf{g}_{rn})} C_{5,n+2}(\mathsf{g}) \tag{3.19}$$

$$bw(\mathsf{g}) = \frac{b_0}{C_{3,-1}(\mathsf{g}_{l0})} C_{3,-1}(\mathsf{g}) + \frac{b_1}{C_{3,n+1}(\mathsf{g}_{rn})} C_{3,n+1}(\mathsf{g}) \tag{3.20}$$

$$cw(\mathsf{g}) = \frac{c_0}{C_{3,-1}(\mathsf{g}_{l0})} C_{3,-1}(t) + \frac{c_1}{C_{3,n+1}(\mathsf{g}_{rn})} C_{3,n+1}(\mathsf{g}) \tag{3.21}$$

$$dw(\mathsf{g}) = \frac{d_0}{C_{3,-1}(\mathsf{g}_{l0})} C_{3,-1}(\mathsf{g}) + \frac{d_1}{C_{3,n+1}(\mathsf{g}_{rn})} C_{3,n+1}(\mathsf{g}) \tag{3.22}$$

and

$$P_{5,k}(\mathsf{g}) = \begin{cases} C_{5,k}(\mathsf{g}) - \left[\frac{C_{5,k}(\mathsf{g}_{l0})}{C_{5,-2}(\mathsf{g}_{l0})}\right] C_{5,-2}(\mathsf{g}), & \text{for } k = -1, 0, 1, 2; \\ C_{5,k}(\mathsf{g}), & \text{for } k = 3, ..., n-3; \\ C_{5,k}(\mathsf{g}) - \left[\frac{C_{5,k}(\mathsf{g}_{rn})}{C_{5,n+2}(\mathsf{g}_{rn})}\right] C_{5,n+2}(\mathsf{g}), & \text{for } k = n-2, n-1, n, n+1. \end{cases} \tag{3.23}$$

$$Q_{3,k}(\mathsf{g}) = \begin{cases} C_{3,k}(\mathsf{g}) - \left[\frac{C_{3,k}(\mathsf{g}_{l0})}{C_{3,-1}(\mathsf{g}_{l0})}\right] C_{3,-1}(\mathsf{g}), & \text{for } k = 0, 1; \\ C_{3,k}(\mathsf{g}), & \text{for } k = 2, 3, ..., n-2; \\ C_{3,k}(\mathsf{g}) - \left[\frac{C_{3,k}(\mathsf{g}_{rn})}{C_{3,n+1}(\mathsf{g}_{rn})}\right] C_{3,n+1}(\mathsf{g}), & \text{for } k = n-1, n. \end{cases} \tag{3.24}$$

We now have $n+3$ basis functions in a and $n+1$ functions in $b, c,$ and d. By applying the Galerkin method to the system of equations (1.3) with the new basis functions, we obtain the following system of algebraic equations with unknown parameters: a_k, b_k, c_k and d_k.

$$\begin{aligned} E_{11}a + E_{12}b + E_{13}c + E_{14}d &= K_1 \\ E_{21}a + E_{22}b + E_{23}c + E_{24}d &= K_2 \\ E_{31}a + E_{32}b + E_{33}c + E_{34}d &= K_3 \\ E_{41}a + E_{42}b + E_{43}c + E_{44}d &= K_4 \end{aligned} \tag{3.25}$$

where $a = [a_{-1}\ a_0\ a_1 ...\ a_{n+1}]^T$, $b = [b_0\ b_1 ...\ b_n]^T$, $c = [c_0\ c_1 ...\ c_n]^T$ and $d = [d_0\ d_1 ...\ d_n]^T$. Each entry of the matrices E_{ik} is shown below. For the first row, we have the following expression:

$$
\begin{aligned}
(e_{11})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[u_{m1}(\mathsf{g})P_{5,i}^{'''}(\mathsf{g})\right] - 3\left[u_{m1}^{'}(\mathsf{g})P_{5,i}^{''}(\mathsf{g})\right] - 3\left[u_{m1}^{''}(\mathsf{g})P_{5,i}^{'}(\mathsf{g})\right] \\
&- \left[u_{m1}^{'''}(\mathsf{g})P_{5,i}(\mathsf{g})\right] + \left[u_{m2}(\mathsf{g})P_{5,i}^{''}(\mathsf{g})\right] + 2\left[u_{m2}^{'}(\mathsf{g})P_{5,i}^{'}(\mathsf{g})\right] + \left[u_{m2}^{''}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \\
&- \left[u_{m3}(\mathsf{g})P_{5,i}^{'}(\mathsf{g})\right] - \left[u_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g})\right] + \left[u_{m4}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \Bigg\} P_{5,k}^{'}(\mathsf{g}) \\
&+ \left[u_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \Bigg\} P_{5,k}(\mathsf{g})d\mathsf{g} - \left[u_{m1}(\mathsf{g}_{rn})P_{5,i}^{'}(\mathsf{g}_{rn})\right] P_{5,k}^{''}(\mathsf{g}_{rn}) \\
&+ \left[u_{m1}(\mathsf{g}_{l0})P_{5,i}^{'}(\mathsf{g}_{l0})\right] P_{5,k}^{''}(\mathsf{g}_{l0}) \quad i = -1, ..., n+1;\ k = -1, ..., n+1
\end{aligned}
$$

$$
\begin{aligned}
(e_{12})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[v_{m3}(t)P_{5,i}{}^{'}(\mathsf{g})\right] - \left[v_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g})\right] + \left[v_{m4}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[v_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = -1, ..., n+1;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(e_{13})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[w_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g})\right] - \left[w_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g})\right] + \left[w_{m4}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[w_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = -1, ..., n+1;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(e_{14})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[y_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g})\right] - \left[y_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g})\right] + \left[y_{m4}(\mathsf{g})P_{5,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[y_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = -1, ..., n+1;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(k_1)_i = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ & k_m(\mathsf{g})P_{5,i}(\mathsf{g}) + u_{m1}(\mathsf{g})P_{5,i}{}^{'''}(\mathsf{g}) + 3u_{m1}^{'}(\mathsf{g})P_{5,i}{}^{''}(\mathsf{g}) + 3u_{m1}^{''}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ u_{m1}^{'''}(\mathsf{g})P_{5,i}(\mathsf{g}) - u_{m2}(\mathsf{g})P_{5,i}{}^{''}(\mathsf{g}) - 2u_{m2}^{'}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) - u_{m2}^{''}(\mathsf{g})P_{5,i}(\mathsf{g}) + u_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ u_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g}) - u_{m4}(\mathsf{g})P_{5,i}(\mathsf{g}) \Bigg\} aw^{'}(\mathsf{g}) - u_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})aw(\mathsf{g}) + \Bigg\{ v_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ v_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g}) - v_{m4}(\mathsf{g})P_{5,i}(\mathsf{g}) \Bigg\} bw^{'}(\mathsf{g}) - v_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})bw(\mathsf{g}) + \Bigg\{ w_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ w_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g}) - w_{m4}(\mathsf{g})P_{5,i}(\mathsf{g}) \Bigg\} cw^{'}(\mathsf{g}) - w_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})cw(\mathsf{g}) + \Bigg\{ y_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ y_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g}) - y_{m4}(\mathsf{g})P_{5,i}(\mathsf{g}) \Bigg\} dw^{'}(\mathsf{g}) - y_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})dw(\mathsf{g}) + \Bigg\{ z_{m3}(\mathsf{g})P_{5,i}{}^{'}(\mathsf{g}) \\
&+ z_{m3}^{'}(\mathsf{g})P_{5,i}(\mathsf{g}) - z_{m4}(\mathsf{g})P_{5,i}(\mathsf{g}) \Bigg\} ew^{'}(\mathsf{g}) - z_{m5}(\mathsf{g})P_{5,i}(\mathsf{g})ew(\mathsf{g}) \Bigg\} d\mathsf{g} \\
&+ \left[u_{m1}(\mathsf{g}_{rn})P_{5,i}{}^{'}(\mathsf{g}_{rn})\right] aw^{''}(\mathsf{g}_{rn}) - \left[u_{m1}(\mathsf{g}_{l0})P_{5,i}{}^{'}(\mathsf{g}_{l0})\right] aw^{''}(\mathsf{g}_{l0}) - \left[u_{m1}(\mathsf{g}_{rn})P_{5,i}{}^{''}(\mathsf{g}_{rn})\right] a_3 \\
&+ \left[u_{m1}(\mathsf{g}_{l0})P_{5,i}{}^{''}(\mathsf{g}_{l0})\right] a_2 - 2\left[u_{m1}^{'}(\mathsf{g}_{rn})P_{5,i}{}^{'}(\mathsf{g}_{rn})\right] a_3 + 2\left[u_{m1}^{'}(\mathsf{g}_{l0})P_{5,i}{}^{'}(\mathsf{g}_{l0})\right] a_2 \\
&+ \left[u_{m2}(\mathsf{g}_{rn})P_{5,i}{}^{'}(\mathsf{g}_{rn})\right] a_3 - \left[u_{m2}(\mathsf{g}_{l0})P_{5,i}{}^{'}(\mathsf{g}_{l0})\right] a_2 \quad i = -1, ..., n+1;
\end{aligned}
$$

The expressions for the second through fourth rows are as follows: For m = 2, 3, and 4, each matrix entry is given below.

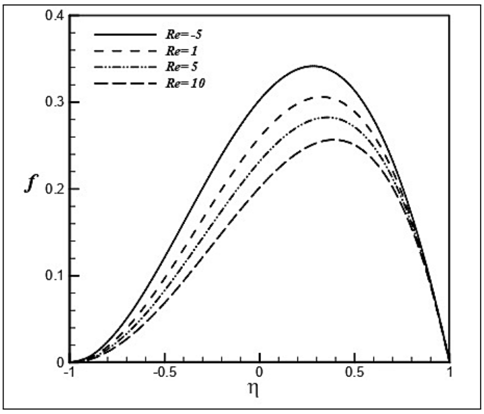
$$
\begin{aligned}
(e_{m1})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[u_{m1}(\mathsf{g})Q_{3,i}^{'''}(\mathsf{g})\right] - 3\left[u_{m1}^{'}(\mathsf{g})Q_{3,i}^{''}(\mathsf{g})\right] - 3\left[u_{m1}^{''}(\mathsf{g})Q_{3,i}^{'}(\mathsf{g})\right] - \left[u_{m1}^{'''}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \\
&+ \left[u_{m2}(\mathsf{g})Q_{3,i}^{''}(\mathsf{g})\right] + 2\left[u_{m2}^{'}(\mathsf{g})Q_{3,i}^{'}(\mathsf{g})\right] + \left[u_{m2}^{''}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] - \left[u_{m3}(\mathsf{g})Q_{3,i}^{'}(\mathsf{g})\right] \\
&- \left[u_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] + \left[u_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \Bigg\} P_{5,k}^{'}(\mathsf{g}) + \left[u_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \Bigg\} P_{5,k}(\mathsf{g})d\mathsf{g} \\
&- \left[u_{m1}(\mathsf{g}_{rn})Q_{3,i}^{'}(\mathsf{g}_{rn})\right] P_{5,k}^{''}(\mathsf{g}_{rn}) + \left[u_{m1}(\mathsf{g}_{l0})Q_{3,i}^{'}(\mathsf{g}_{l0})\right] P_{5,k}^{''}(\mathsf{g}_{l0}) \\
&i = 0, ..., n;\ k = -1, ..., n+1
\end{aligned}
$$

$$
\begin{aligned}
(e_{m2})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[v_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g})\right] - \left[v_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] + \left[v_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[v_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = 0, ..., n;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(e_{m3})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[w_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g})\right] - \left[w_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] + \left[w_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[w_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = 0, ..., n;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(e_{m4})_{ik} = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ &- \left[y_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g})\right] - \left[y_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] + \left[y_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] \Bigg\} Q_{3,k}{}^{'}(\mathsf{g}) \\
&+ \left[y_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})\right] Q_{3,k}(\mathsf{g}) \Bigg\} d\mathsf{g} \quad i = 0, ..., n;\ k = 0, ..., n
\end{aligned}
$$

$$
\begin{aligned}
(k_m)_i = \int_{\mathsf{g}_{l0}}^{\mathsf{g}_{rn}} \Bigg\{\Bigg\{ & k_m(\mathsf{g})Q_{3,i}(\mathsf{g}) + u_{m1}(\mathsf{g})P_{5,i}{}^{'''}(\mathsf{g}) + 3u_{m1}^{'}(\mathsf{g})Q_{3,i}{}^{''}(\mathsf{g}) + 3u_{m1}^{''}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) \\
&+ u_{m1}^{'''}(\mathsf{g})Q_{3,i}(\mathsf{g}) - u_{m2}(\mathsf{g})Q_{3,i}{}^{''}(\mathsf{g}) - 2u_{m2}^{'}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) - u_{m2}^{''}(\mathsf{g})Q_{3,i}(\mathsf{g}) \\
&+ u_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) + u_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g}) - u_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g}) \Bigg\} aw^{'}(\mathsf{g}) - u_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})aw(\mathsf{g}) \\
&+ \Bigg\{ v_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) + v_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g}) - v_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g}) \Bigg\} bw^{'}(\mathsf{g}) - v_{m5}(g)Q_{3,i}(\mathsf{g})bw(\mathsf{g}) \\
&+ \Bigg\{ w_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) + w_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g}) - w_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g}) \Bigg\} cw^{'}(\mathsf{g}) - w_{m5}(\mathsf{g})Q_{3,i}(g)cw(\mathsf{g}) \\
&+ \Bigg\{ y_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) + y_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g}) - y_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g}) \Bigg\} dw^{'}(\mathsf{g}) - y_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})dw(\mathsf{g}) \\
&+ \Bigg\{ z_{m3}(\mathsf{g})Q_{3,i}{}^{'}(\mathsf{g}) + z_{m3}^{'}(\mathsf{g})Q_{3,i}(\mathsf{g}) - z_{m4}(\mathsf{g})Q_{3,i}(\mathsf{g}) \Bigg\} ew^{'}(\mathsf{g}) - z_{m5}(\mathsf{g})Q_{3,i}(\mathsf{g})ew(\mathsf{g}) \Bigg\} d\mathsf{g} \\
&+ \left[u_{m1}(\mathsf{g}_{rn})Q_{3,i}{}^{'}(\mathsf{g}_{rn})\right] aw^{''}(\mathsf{g}_{rn}) - \left[u_{m1}(\mathsf{g}_{l0})Q_{3,i}{}^{'}(\mathsf{g}_{l0})\right] aw^{''}(\mathsf{g}_{l0}) \\
&- \left[u_{m1}(\mathsf{g}_{rn})Q_{3,i}{}^{''}(\mathsf{g}_{rn})\right] a_3 + \left[u_{m1}(\mathsf{g}_{l0})Q_{3,i}{}^{''}(\mathsf{g}_{l0})\right] a_2 - 2\left[u_{m1}^{'}(\mathsf{g}_{rn})Q_{3,i}{}^{'}(\mathsf{g}_{rn})\right] a_3
\end{aligned}
$$



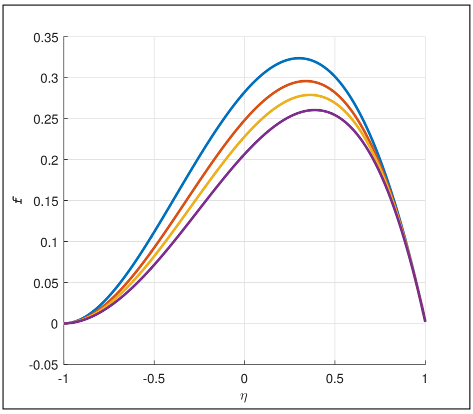


Figure 1. Variation in f with increasing values of Reynolds number in Sheikholeslami et al. [1] and proposed method.

$$+2\left[u_{m1}^{'}(\mathsf{g}_{l0})Q_{3,i}^{'}(\mathsf{g}_{l0})\right]a_2+\left[u_{m2}(\mathsf{g}_{rn})Q_{3,i}^{'}(\mathsf{g}_{rn})\right]a_3-\left[u_{m2}(\mathsf{g}_{l0})Q_{3,i}^{'}(\mathsf{g}_{l0})\right]a_2$$
$$i=0,...,n;$$

Evaluation of each integration from $(e_{11})_{ik},(e_{12})_{ik},(e_{13})_{ik},(e_{14})_{ik}$, and $(k_1)_i$, as well as for m = 2 to 4, the integration $(e_{m1})_{ik},(e_{m2})_{ik},(e_{m3})_{ik},(e_{m4})_{ik}$ and $(k_m)_i$. We use the Gauss-Legendre quadrature formula to obtain the nodal parameter vectors a_k, b_k, c_k and d_k. After determining the nodal parameters a_k, b_k, c_k, and d_k's, we can use the approximation formula to approximate each unknown variable. The residual error was also developed to test accuracy. The error is obtained after approximate solutions of the proposed method's approximate solutions. The proposed method was used to solve the system of fourth-order BVPs (1.3) and (1.4) using a computer program written in MATLAB.

4 Numerical Example

Here, we demonstrate how well the proposed method can be used to solve coupled systems of fourth-order nonlinear BVPs of types (1.1) and (1.2). The numerical solutions for the examples are represented graphically and are compared to existing published work by Sheikholeslami et al. [1]. Consider the following example:

$$\begin{aligned}(1+N_1)f^{IV}-N_1g-Re(ff^{'''}-f^{'}f^{''})&=0,\\ N_2g^{''}+N_1(f^{''}-2g)-N_3Re(fg^{'}-f^{'}g)&=0,\\ \theta^{''}+Pe_hf^{'}\theta-Pe_hf\theta^{'}&=0,\\ \varphi^{''}+Pe_mf^{'}\varphi-Pe_mf\varphi^{'}&=0,\end{aligned}\tag{4.1}$$

with the boundary conditions

$$\begin{aligned}f^{'}=0,f=0,g=0,\theta=1,\varphi=1\quad at\quad \eta=-1,\\ f^{'}=-1,f=0,g=1,\theta=0,\varphi=0\quad at\quad \eta=+1.\end{aligned}\tag{4.2}$$

The maximum value of η_{+1} is determined for each parameter N_1, N_2, N_3, Re, Pe_h and Pe_m. Using the quasilinearization technique Bellman and Kalaba [28], the nonlinear system of equations (4.1) can be converted to a sequence of linear systems of differential equations. The proposed method solved this example, and we obtained solutions for each unknown part of the example.

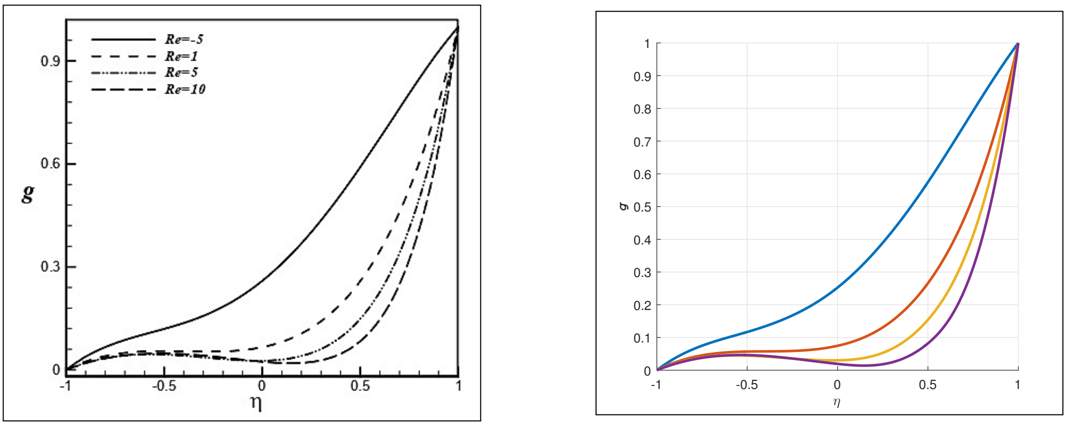


Figure 2. Variation in g with increasing values of Reynolds number in Sheikholeslami et al. [1] and proposed method.

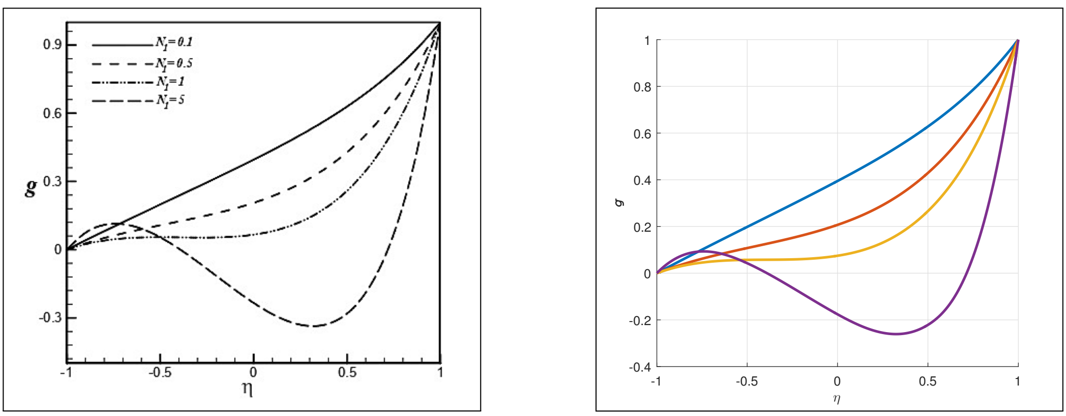


Figure 3. Variation in g with increasing values of coupling parameter in Sheikholeslami et al. [1] and proposed method.

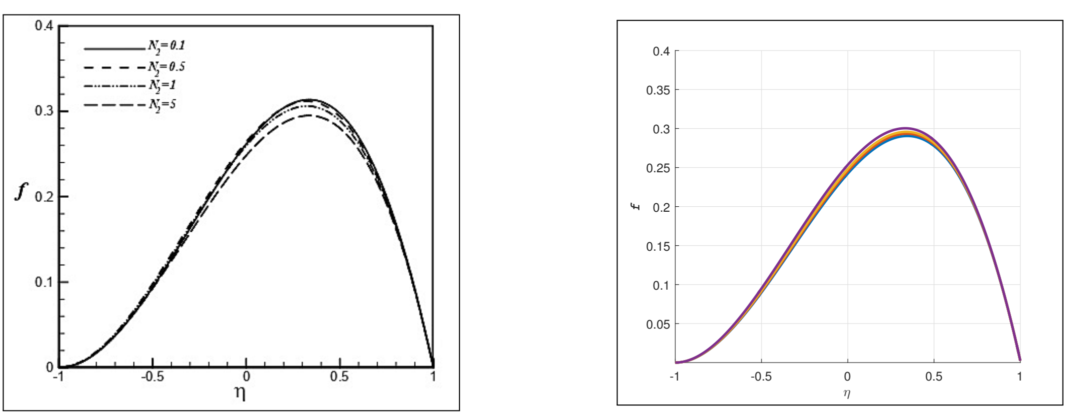


Figure 4. Variation in f with the increasing values of spin gradient viscosity parameter in Sheikholeslami et al. [1] and proposed method.

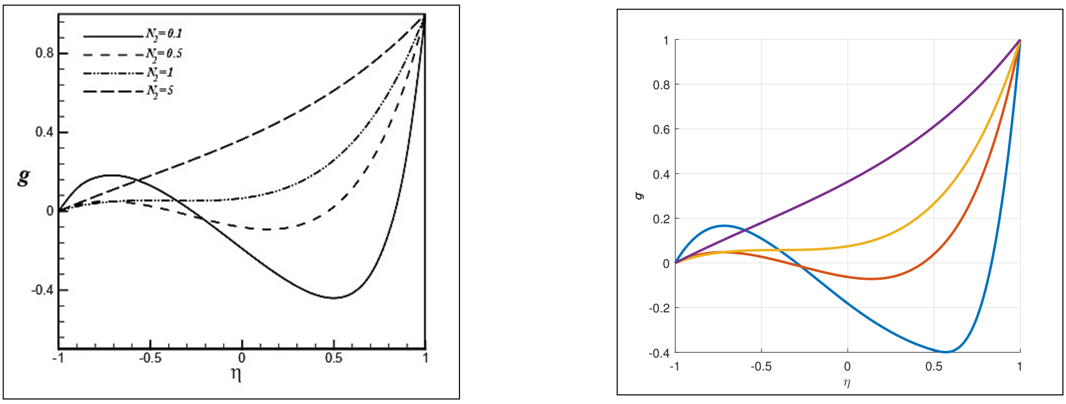


Figure 5. Variation in g with the increasing values of spin gradient viscosity parameter in Sheikholeslami et al. [1] and proposed method.

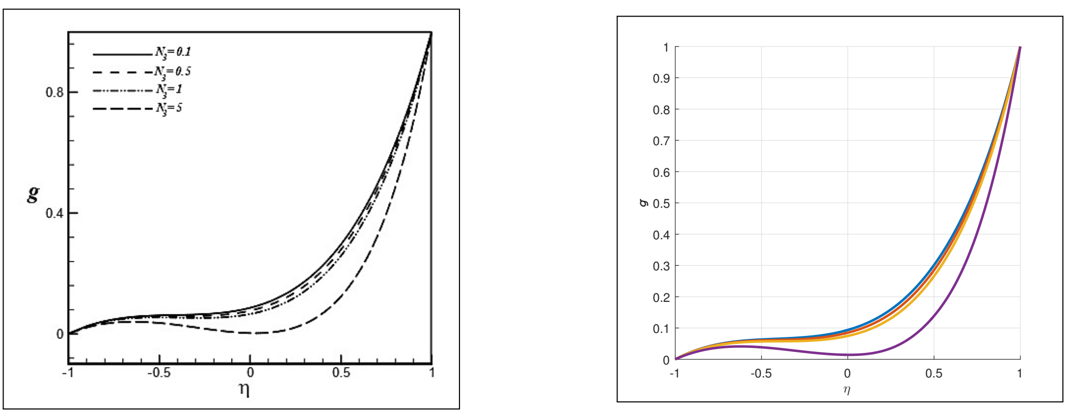


Figure 6. Variation in g with increasing values of angular velocity in Sheikholeslami et al. [1] and proposed method.

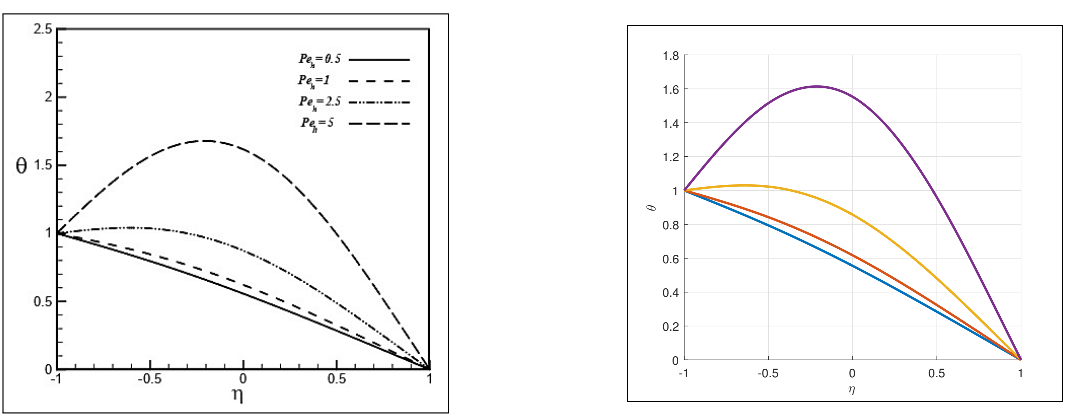


Figure 7. Variation in θ with increasing values of the Peclet number for the heat diffusion in Sheikholeslami et al. [1] and proposed method.

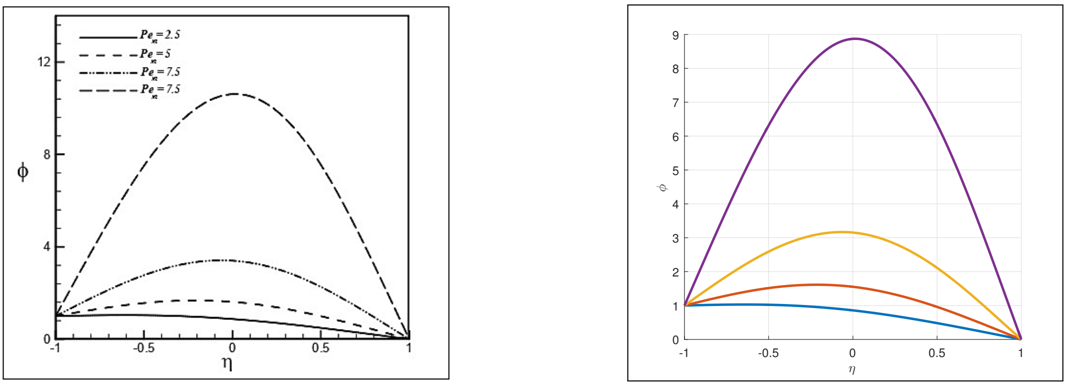


Figure 8. Variation in φ with increasing values of the Peclet number for the mass diffusion in Sheikholeslami et al. [1] and proposed method.

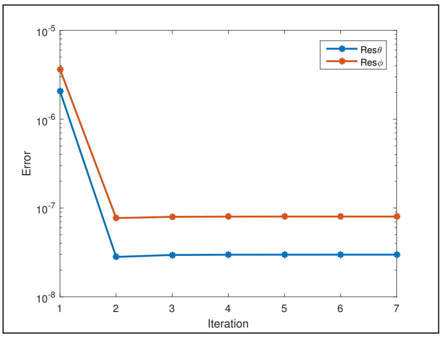


Figure 9. Residual errors for fluid temperature and concentration profile.

5 Results and Discussion

The coupled system of ODEs (4.1) with boundary conditions (4.2) was solved using the Galerkin technique together with the cubic and quintic B-splines schemes. The proposed method generates numerical solutions for unknown parameters with high accuracy and few computational steps. The behaviour of f as Reynolds number increases is illustrated in Figure 1. Figure 2 illustrates the variation in g with increasing Reynolds number. Figure 3 illustrates the variation in g as the coupling parameter increases. Figure 4 elucidates the stimulus of the variation of f as the spin-gradient viscosity parameter increases. Figure 5 presents the behaviour of g with an increasing spin gradient viscosity parameter. Figure 6 depicts the impact of g as the angular velocity increases. Figure 7 illustrates the influence of θ as the liquid temperature increases. Figure 8 outlines the effect of φ on increasing the concentration profile. Figure 9 presents residual errors in the fluid temperature and concentration profiles with residual errors less than 10^{-8}.

6 Conclusions

This study has developed Galerkin's method using cubic and quintic B-splines to solve the coupled system of fourth-order nonlinear BVPs. The cubic and quintic B-splines basis functions were reformulated into a new form of the set of basis functions that vanish at the boundary where the type of Dirichlet boundary conditions are given. This method was applied to solve the numerical example and then compared to the literature. The obtained results are in good agreement with the literature results. Furthermore, residual error was computed to demonstrate the validity of the study. This paper presents an accurate technique for solving the coupled system of fourth-order nonlinear BVPs.

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