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On Topological and Topoline Numbers of Stars, Paths and Some Related Graphs

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Abstract This paper investigates the topological and topoline numbers of some graphs and settles certain conjectures related to the topological numbers of star graphs. The topoline numbers of some stars, paths, and certain related graphs are also obtained.

1 Introduction

The concept of set-valuation in graph theory involves assigning subsets of a given nonempty set to the vertices and/or edges of a graph. B. D. Acharya introduced the notions of set-valuations, set-indexers, and set-indexing numbers of graphs in [1]. Moreover, he established a connection between graph theory and point set topology by introducing the concept of topological set-indexers (t-set-indexers) in [2]. The topological number (t-number) of a graph refers to the minimum cardinality among its topological indexing sets, serving as a significant characteristic of the graph. This concept gained attention due to its universality; every graph possesses a topological indexing set.

In [12], U. Thomas and S. C. Mathew expanded upon this by introducing the notion of topoline set-indexers, which involve admitting a topology on the edge set of a graph. They defined the topoline number for a topoline graph in a manner analogous to the topological number of a graph.

[13] discusses the topological numbers of certain classes of graphs and presents conjectures. In this context, we offer results related to these conjectures. Specifically, we calculate the topoline numbers of trees with orders up to seven, along with some paths. Additionally, we determine the topoline numbers of special graphs such as complete bipartite graphs, n-sun graphs, and double star graphs. For recent trends in graph labeling, see [5], [6], [9], [14] and [16].

2 Preliminaries

B. D. Acharya introduced the notion of a set-indexer of a graph as follows: Let $G = (V, E)$ be a graph and X be a nonempty set. Then a mapping $f : V \to 2^X$ or $f : E \to 2^X$ or $f : V \cup E \to 2^X$ is called a set-valuation or set-assignment of the vertices or edges or both. It is proved in [1] that every graph G has a set-valuation.

Definition 2.1. Let $G = (V, E)$ be a graph and X be a nonempty set. Then a set-valuation $f : V \cup E \to 2^X$ is called a set-indexer of G if

(i) $f(uv) = f(u) \oplus f(v)$ where $\oplus$ is the symmetric difference and

(ii) the restrictive maps $f|_V$ and $f|_E$ are both injective.

In this case, X is called an indexing set of G. A graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the set-indexing number of G, denoted by $\gamma(G)$. It is also proved that every graph G has a set-indexer[8]. A graph G is set-graceful if $\gamma(G) = \log_2(|E| + 1)$ and the corresponding set-indexer is called set-graceful labeling and is an optimal set-indexer of G. A set-indexer f of a graph G with indexing set X is said to be a topological set-indexer (t-set-indexer) if $f(V)$ is a topology on X and X is called the topological indexing set (t-indexing set) of G. The minimum number among the cardinalities of such topological indexing sets is said to be the topological number (t-number) of G and is denoted by $\tau(G)$.

Definition 2.2. [11] A set-indexer $f : V \cup E \to 2^X$ of a nonempty graph G is a topoline set-indexer, if $f(E) \cup \{\emptyset\}$ is a topology on X.

The set X is called the topoline indexing set of G. The minimum among the cardinalities of such topoline indexing sets is called the topoline number of G and is denoted by $\tau_e(G)$. A nonempty graph G is said to be topoline if it has a topoline set-indexer. A graph is said to be topoline set-graceful if it is topoline and $\gamma(G) = \tau_e(G)$.

Theorem 2.3. [2] For a graph $G = (V, E)$, $\gamma(G) \geq \lceil \log_2(|E| + 1) \rceil$.

Theorem 2.4. [2] For a graph G with at least 2 vertices, $\tau(G) \geq \gamma(G)$.

Theorem 2.5. [2] For every integer $n \geq 2$, P_{2^n} is not set-graceful.

Theorem 2.6. [10] For a path graph P_n with at least 3 vertices, $\gamma(P_n) = \lfloor \log_2 n \rfloor + 1$.

Theorem 2.7. [11] If $G = ST(m, n)$, $|V(G)| = 2^l, l \geq 2$ and m is odd, then $\gamma(G) = l + 1$.

Theorem 2.8. [12] For a star graph G, $\tau(G) = \tau_e(G)$.

Theorem 2.9. [12] For a graph G with at least 2 vertices, $\tau_e(G) \geq \gamma(G)$.

Theorem 2.10. [4] For $n \geq 3$, there is no topology on n elements having k open sets, where $3 \cdot 2^{(n-2)} < k < 2^n$.

Theorem 2.11. [7] $m(4) = 2$, $m(5) = m(6) = 3$, $m(7) = 4$, $m(8) = 3$, $m(9) = m(10) = 4$, $m(11) = m(13) = m(14) = 5$, $m(19) = m(21) = m(22) = 6$ and $m(35) = 7$ where $m(k)$ denote the minimum number of points needed to make a topology having k open sets.

Definition 2.12. [3] An n-sun graph is obtained by joining a vertex of degree 1 to each vertex of a cycle C_n with a bridge.

Definition 2.13. [15] A double star graph $ST(m, n)$ is obtained from two stars $K_{1,m}$ and $K_{1,n}$ by joining their centers by an edge.

3 Topological numbers of stars

In [13], we see the following conjectures on topological numbers of star graphs.

Conjecture 3.1. $\tau(K_{1,2^n+2}) = n + 2; n \geq 2$.

Conjecture 3.2. $\tau(K_{1,2^n+3}) = n + 1; n \geq 2$.

Conjecture 3.3. $\tau(K_{1,3 \cdot 2^{(n-2)}}) = n; n \geq 2$.

With regard to the conjecture 3.1, we prove the following two theorems.

Theorem 3.4. $n + 1 \leq \tau(K_{1,2^n+2}) \leq n + 2; n \geq 1$.

Proof. A topological set-indexer for $G = K_{1,2^n+2}$ with indexing set $X = \{x_1, x_2, \ldots, x_{n+2}\}$ can be obtained as follows: Assign $\emptyset$ to the vertex of degree $2^n + 2$ and assign X, $X \setminus \{x_{n+1}\}$, $X \setminus \{x_{n+2}\}$ and all the $(2^n - 1)$ nonempty subsets of the set $X \setminus \{x_{n+1}, x_{n+2}\}$ to the other vertices of G. Thus $\tau(G) \leq n + 2$ for $n \geq 1$. Also by Theorem 2.3 and Theorem 2.4, $\tau(G) \geq n + 1$ for $n \geq 1$. Combining, $n + 1 \leq \tau(G) \leq n + 2; n \geq 1$. □

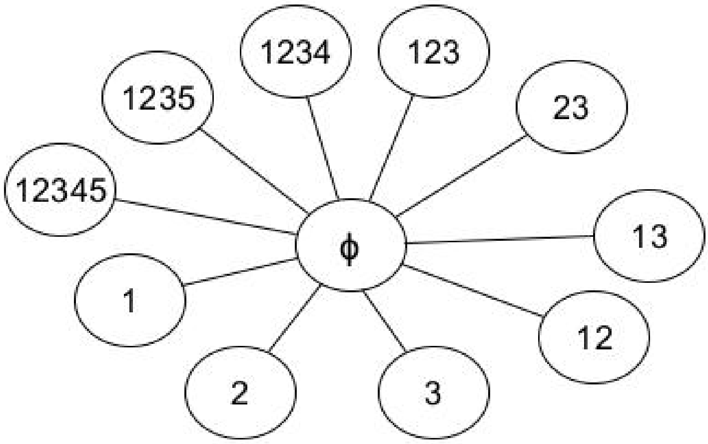

Figure 1. Labeling of $K_{1,10}$

Example 3.5. The Figure 1 illustrates the labeling technique described in Theorem 3.4 for the case $n = 3$. Here, the graph is $G = K_{1,10}$ and the indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$ is indicated as 12345. By the labeling, $\tau(G) \le 5$. Also by Theorem 2.3 and Theorem 2.4, we have $\tau(G) \ge 4$. Combining, $\tau(G)$ is either 4 or 5. However, the following theorem shows that it is actually 5!

Theorem 3.6. $\tau(K_{1,2^n+2}) = n + 2$; $1 \le n \le 5$.

Proof. By Theorem 3.4, we have for $G = K_{1,2^n+2}$, $n + 1 \le \tau(G) \le n + 2$ for $n \ge 1$. Further, we have by Theorem 2.11, $m(5) = 3$, $m(7) = 4$, $m(11) = 5$, $m(19) = 6$ and $m(35) = 7$. Thus $\tau(G) \ge n + 2$ for $1 \le n \le 5$, and so $\tau(G) = n + 2$; $1 \le n \le 5$. □

In the following theorem, we settle the Conjecture 3.2 in the affirmative.

Theorem 3.7. $\tau(K_{1,2^n+3}) = n + 1$; $n \ge 2$.

Proof. A topological set-indexer for the graph $G = K_{1,2^n+3}$ can be obtained as follows:

Consider $X = \{x_1, x_2, \ldots, x_{n+1}\}$. Assign $\emptyset$ to the vertex of degree $2^n + 3$ and assign X, $X \setminus \{x_1\}$, $X \setminus \{x_2\}$, $X \setminus \{x_1, x_2\}$ and all the $2^n - 1$ nonempty subsets of $X \setminus \{x_{n+1}\}$ to the other vertices of G having degree one.

Thus $\tau(G) \le n+1$ for $n \ge 2$. Also by Theorem 2.3 and Theorem 2.4, we have $\tau(G) \ge n+1$ for $n \ge 2$. Combining, we get $\tau(G) = n + 1$; $n \ge 2$. □

Example 3.8. The Figure 2 is the illustration of the labeling technique described in Theorem 3.7 for the case $n = 3$. In this case, the graph is $G = K_{1,11}$ and the indexing set $X = \{x_1, x_2, x_3, x_4\}$ is indicated in the figure as 1234. By the labeling, $\tau(G) \le 4$. Also by using Theorem 2.3 and Theorem 2.4, we have $\tau(G) \ge 4$. Combining, we get $\tau(G) = 4$.

Some related results are presented below:

Theorem 3.9. $n + 1 \le \tau(K_{1,2^n+4}) \le n + 2$; $n \ge 2$.

Proof. A topological set-indexer for $G = K_{1,2^n+4}$ with the indexing set $X = \{x_1, x_2, \ldots, x_{n+2}\}$ can be obtained as follows: Let $Y = X \setminus \{x_{n+2}\}$. Assign $\emptyset$ to the vertex of degree $2^n + 4$ and assign $X, Y, Y \setminus \{x_1\}, Y \setminus \{x_2\}, Y \setminus \{x_1, x_2\}$ and all the $2^n - 1$ nonempty subsets of $Y \setminus \{x_{n+1}\}$ to the other vertices of G having degree one.

Thus $\tau(G) \le n + 2$ for $n \ge 2$. Also by Theorem 2.3 and Theorem 2.4, $\tau(G) \ge n + 1$ for $n \ge 2$. Combining, $n + 1 \le \tau(G) \le n + 2$; $n \ge 2$. □

Theorem 3.10. $\tau(K_{1,2^n+4}) = n + 2$; $2 \le n \le 4$.

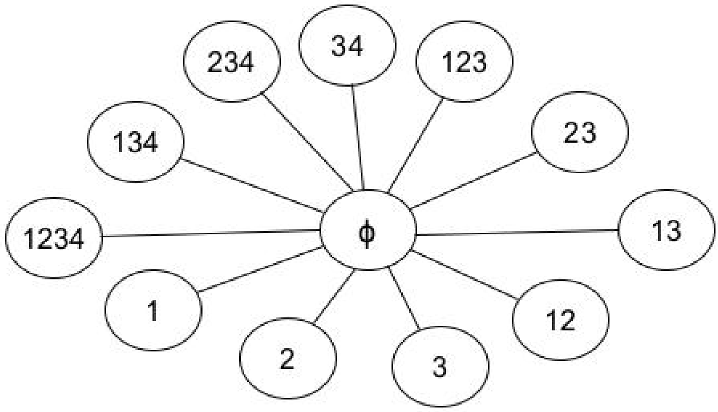

Figure 2. Labeling of $K_{1,11}$

Proof. By Theorem 3.9, we have for $G = K_{1,2^n+4}$, $n+1 \le \tau(G) \le n+2$; $n \ge 2$. Further, we have by Theorem 2.11 $m(13) = 5$ and $m(21) = 6$. Thus $\tau(G) \ge n+2$ for $2 \le n \le 4$ and so $\tau(G) = n+2$; $2 \le n \le 4$. □

Theorem 3.11. $n+1 \le \tau(K_{1,2^n+5}) \le n+2$; $n \ge 2$.

Proof. A topological set-indexer for $G = K_{1,2^n+5}$ with indexing set $X = \{x_1, x_2, \ldots, x_{n+2}\}$ can be obtained as follows:
Assign $\emptyset$ to the vertex of degree 2^n+5 and assign X, $X \setminus \{x_{n+2}\}$, $X \setminus \{x_{n+1}\}$, $X \setminus \{x_1, x_{n+2}\}$, $X \setminus \{x_2, x_{n+2}\}$, $X \setminus \{x_1, x_2, x_{n+2}\}$ and all the $2^n - 1$ nonempty subsets of $X \setminus \{x_{n+1}, x_{n+2}\}$ to the other vertices of G having degree one.

Thus $\tau(G) \le n+2$ for $n \ge 2$. Also by Theorem 2.3 and Theorem 2.4, $\tau(G) \ge n+1$ for $n \ge 1$. Then $n+1 \le \tau(G) \le n+2$; $n \ge 2$. □

Theorem 3.12. $\tau(K_{1,2^n+5}) = n+2$; $1 \le n \le 4$.

Proof. By Theorem 3.11, we have for $G = K_{1,2^n+5}$, $n+1 \le \tau(G) \le n+2$ for $n \ge 2$. For $n = 1$, $G = K_{1,7}$ and a topoline set-indexer for G with the indexing set $X = \{x_1, x_2, x_3\}$ is obtained by assigning $\emptyset$ to the vertex of order 7 and all the nonempty subsets of X to the other vertices. Thus $\tau(G) \le n+2$ for $n = 1$. Further, we have by Theorem 2.11 $m(8) = 3$, $m(10) = 4$, $m(14) = 5$, and $m(22) = 6$. Thus $\tau(G) \ge n+2$ for $1 \le n \le 4$. Combining, $\tau(G) = n+2$; $1 \le n \le 4$. □

Example 3.13. The Figure 3 illustrates the procedure described in Theorem 3.11 for the case $n = 2$. The graph in this instance is $G = K_{1,9}$ and 1234 reprsents the indexing set $X = \{x_1, x_2, x_3, x_4\}$. Obviously, $\tau(G) \le 4$. Also by Theorem 2.3 and Theorem 2.4, we have $\tau(G) \ge 3$. Combining, we get $3 \le \tau(G) \le 4$. But, by Theorem 3.12 we conclude that $\tau(G) = 4$.

The following theorem shows that the Conjecture 3.3 is false at least in the case of $n = 3, 4$ and 5.

Theorem 3.14. $\tau(K_{1,3\cdot 2^{(n-2)}}) = n+1$; $3 \le n \le 5$.

Proof. For the graph $G = K_{1,3\cdot 2^{(n-2)}}$, we have $\tau(G) \ge n$; $n \ge 3$ by Theorem 2.3 and Theorem 2.4. But since $3 \cdot 2^{(n-2)} < 3 \cdot 2^{(n-2)} + 1 < 2^n$ for $n \ge 3$, we get $\tau(G) \ge n+1$; $n \ge 3$ by Theorem 2.10.

For the case $n = 3$, a topological set-indexer for G with indexing set $X = \{x_1, x_2, x_3, x_4\}$ can be obtained as follows: Assign $\emptyset$ to the vertex of degree 6 and assign $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, X and all non-empty subsets of $\{x_1, x_2\}$ to the other vertices.

For the case $n = 4$, to obtain a topological set-indexer for G with indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$, we assign $\emptyset$ to the vertex of degree 12 and assign the subsets $\{x_2\}$, $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, $\{x_1, x_2, x_5\}$, $\{x_2, x_3, x_4\}$, $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_2, x_3, x_5\}$, $\{x_1, x_2, x_4, x_5\}$ and X to the other vertices.

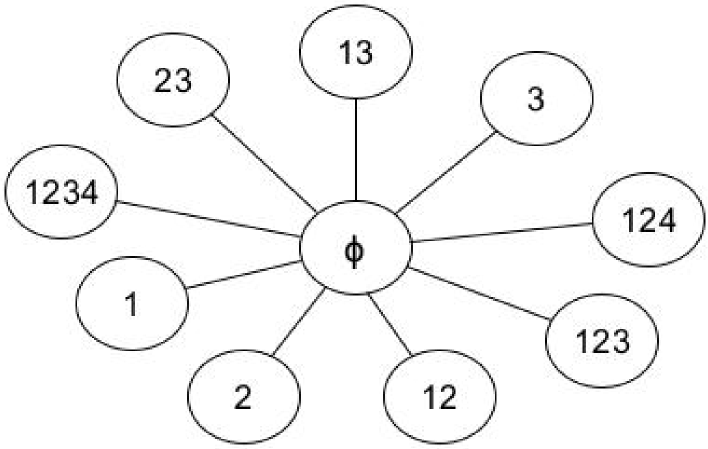

Figure 3. Labeling of $K_{1,9}$

Similarly for $n = 5$, a topological set-indexer of G can be obtained with indexing set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ as follows:
Assign $\emptyset$ to the vertex of degree 24 and assign the subsets $\{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}$, $\{x_2, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_1, x_2, x_6\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_2, x_4, x_5\}$, $\{x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}, \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_4, x_6\}$, $\{x_1, x_2, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_6\}, \{x_1, x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_4, x_5, x_6\}$ and X to the other vertices. Thus $\tau(G) \leq n + 1$, $3 \leq n \leq 5$. Combining, $\tau(G) = n + 1$; $3 \leq n \leq 5$. □

4 Topoline numbers of certain paths and trees

It is known that all set-graceful graphs are topoline set-graceful. In [11] it is proved that P_{2^n+1} and P_{2^n+2} are topoline set-graceful. By Theorem 2.5, P_{2^n} is not set-graceful for $n \geq 2$. We show that P_8 is an example of a topoline graph which is not set-graceful, but topoline set-graceful. The topoline numbers of some related graphs are also obtained.

Theorem 4.1. $\tau_e(P_7) = 4$.

Proof. Since $m(7) = 4$ by Theorem 2.11, $\tau_e(P_7) \geq 4$. A topoline set-indexer for P_7 with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained by assigning $\{x_1\}$, $\emptyset$, $\{x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_1, x_2\}$ and $\{x_3, x_4\}$ to the vertices of P_7 in sequential order. Thus $\tau_e(P_7) \leq 4$. Combining, $\tau_e(P_7) = 4$. □

Theorem 4.2. P_8 is topoline set-graceful.

Proof. By Theorem 2.6 $\gamma(P_8) = 4$ and by Theorem 2.9, $\tau_e(P_8) \geq 4$. A topoline set-indexer for P_8 with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained by assigning $\{x_1\}$, $\emptyset$, $\{x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_1, x_4\}$, $\{x_3, x_4\}$ and $\{x_1, x_2, x_3\}$ to the vertices of P_8 in sequential order as shown in Figure 4. Thus $\tau_e(P_8) \leq 4$. Combining $\tau_e(P_8) = 4$. It follows that P_8 is topoline set-graceful. □

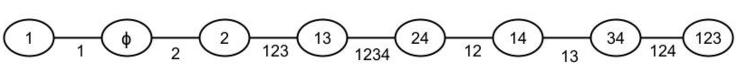

Figure 4. Labeling of P_8

Theorem 4.3. $\tau_e(P_{11}) = 5$.

Proof. Since $m(11) = 5$ by Theorem 2.11, $\tau_e(P_{11}) \geq 5$. A topoline set-indexer for P_11 with indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$ is obtained as follows: Assign $\{x_1\}$, $\{x_1, x_2\}$, $\{x_3\}$, $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_1, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_4\}$, $\{x_2, x_3, x_4\}$, $\emptyset$ and X to the vertices of P_{11} in sequential order. Thus $\tau_e(P_{11}) \leq 5$. Combining, $\tau_e(P_{11}) = 5$. □

From [11] we know that all trees are topoline. The topoline number of trees of order 2 and 3 are clearly 1 and 2 respectively. The topoline numbers of trees of order upto seven are determined below:

Theorem 4.4. If G is a tree of order 4, then $2 \leq \tau_e(G) \leq 3$.

Proof. Since $m(4) = 2$ by Theorem 2.11, $\tau_e(G) \geq 2$. Then the following cases arise.
Case I: A vertex has degree 3.

In this case, a topoline set-indexer for G with the indexing set $X = \{x_1, x_2\}$ is obtained by assigning $\emptyset$ to the vertex of degree 3 and assigning $\{x_1\}$, $\{x_2\}$ and X to the other vertices. Thus $\tau_e(G) \leq 2$ and so $\tau_e(G) = 2$.
Case II: Degree of all vertices ≤ 2. ie, $G = P_4$.

By Theorem 2.6 $\gamma(P_4) = 3$ and by Theorem 2.9, $\tau_e(G) \geq 3$. A topoline set-indexer for G with the indexing set $X = \{x_1, x_2, x_3\}$ is obtained as follows: Assign $\{x_1\}$, $\emptyset$, $\{x_1, x_2\}$ and $\{x_3\}$ to the vertices in sequential order. Thus $\tau_e(G) \leq 3$. Combining, we get $\tau_e(G) = 3$. Thus $2 \leq \tau_e(G) \leq 3$. □

Theorem 4.5. All trees of order 5 have topoline number 3.

Proof. Let G be a tree of order 5 and let $X = \{x_1, x_2, x_3\}$. Then the following cases arise.
Case I: A vertex has degree 4.

In this case, a topoline set-indexer for G with the indexing set X is obtained by assigning $\emptyset$ to the vertex of degree 4 and assigning $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$ and X to the other vertices.
Case II: A vertex has degree 3.

In this instance, a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to the vertex u of degree 3, assign X to the vertex w of degree 2, assign $\{x_3\}$ to the pendant vertex adjacent to w and assign $\{x_1\}$ and $\{x_2\}$ to the pendant vertices adjacent to u.
Case III: Degree of all vertices ≤ 2.

Here G is a path and a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\{x_1\}$, $\emptyset$, $\{x_2\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$ to the vertices in sequential order.
In all the possible cases, the cardinality of the indexing set is 3 and so $\tau_e(G) \leq 3$. Also Since $m(5) = 3$ by Theorem 2.11, $\tau_e(G) \geq 3$. Combining, $\tau_e(G) = 3$. □

Theorem 4.6. All trees of order 6 have topoline number 3.

Proof. Let G be a tree of order 6 and let $X = \{x_1, x_2, x_3\}$. Then the following cases arise.
Case I: A vertex has degree 5.

In this case, a topoline set-indexer for G with the indexing set X is obtained by assigning $\emptyset$ to the vertex of degree 5 and assigning $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$ and X to the other vertices.
Case II: A vertex has degree 4.

A topoline set-indexer for G with the indexing set X is obtained as follows: Let u be the vertex of degree 4 and v be the vertex of degree 2. Assign $\{x_1, x_3\}$ to u, assign $\{x_2, x_3\}$ to v, assign $\{x_1\}$ to the vertex adjacent to v and assign $\{x_3\}$, $\{x_1, x_2\}$ and X to the remaining vertices.
Case III: Two vertices have degree 3 each.

In this case, a topoline set-indexer for G with the indexing set X is obtained as follows: Let u and v be the vertices of degree 3 each. Assign $\{x_1, x_3\}$ to u, assign $\{x_2, x_3\}$ to v, assign $\{x_1\}$ and $\emptyset$ to the vertices adjacent to v, assign $\{x_3\}$ and X to the vertices adjacent to u.
Case IV: Exactly one vertex, say u has degree 3.

Case IV(A): Two vertices say v and w of degree 2 are adjacent to u.

Then a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_2\}$ to v, assign $\{x_1, x_2\}$ to w, assign $\{x_1\}$, $\{x_1, x_3\}$ and $\{x_2, x_3\}$ to the vertices adjacent to u, v and w respectively.
Case IV(B): Exactly one vertex, say v of degree 2 is adjacent to u.

In this case, there must be a vertex w of degree 2 adjacent to v. A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\{x_1, x_3\}$ to u, assign $\{x_2, x_3\}$ to v, assign

$\{x_1\}$ to w, assign $\emptyset$ to the vertex adjacent to w, assign $\{x_1, x_2\}$ and X to the vertices adjacent to u.

Case V: Degree of all vertices ≤ 2. ie, G is a path.

A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\{x_2\}$, $\emptyset$, $\{x_1\}$, $\{x_2, x_3\}$, $\{x_1, x_3\}$ and $\{x_1, x_2\}$ to the vertices in sequential order.

In all the possible cases, the cardinality of the indexing set is 3 and so $\tau_e(G) \leq 3$. Also Since $m(6) = 3$ by Theorem 2.11, $\tau_e(G) \geq 3$. Combining, $\tau_e(G) = 3$. □

Theorem 4.7. All trees of order 7 have topoline number 4.

Proof. Let G be a tree of order 7. Since $m(7) = 4$, $\tau_e(G) \geq 4$. Let $X = \{x_1, x_2, x_3, x_4\}$. Then the following cases arise.

Case I: A vertex has degree 6.

A topoline set-indexer for G with the indexing set X for this case is obtained as follows: Assign $\emptyset$ to the vertex of degree 6 and assign $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_1, x_2, x_3\}$ and X to the other vertices.

Case II: A vertex has degree 5.

Here, a topoline set-indexer for G with the indexing set X is obtained as follows: Let u be the vertex of degree 5 and v be the vertex of degree 2. Assign $\emptyset$ to u, assign $\{x_1, x_2, x_3\}$ to v, assign $\{x_4\}$ to the vertex adjacent to v and assign $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2\}$ and $\{x_1, x_3\}$ to the remaining vertices.

Case III: A vertex, say u has degree 4.

Case III(A): A vertex, say v of degree 3 is adjacent to u.

In this case, a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2\}$ to the pendant vertices adjacent to u, assign X to v, assign $\{x_4\}$ and $\{x_2, x_4\}$ to the vertices adjacent to v.

Case III(B): Two vertices, say v and w of degree 2 each are adjacent to u.

In this case, a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$ and $\{x_2\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2\}$ to v, assign $\{x_1, x_2, x_3\}$ to w, assign $\{x_4\}$ to the vertex adjacent to w and assign $\{x_2, x_3\}$ to the vertex adjacent to v.

Case III(C): Only one vertex, say v of degree 2 is adjacent to u.

Then there must be a vertex say w of degree 2 adjacent to v. A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2, x_3\}$ to v, assign $\{x_4\}$ to w, assign $\{x_1, x_3, x_4\}$ to the vertex adjacent to w.

Case IV : The maximum degree of vertices of G is 3.

Case IV (A) : Only one vertex, say u has degree 3 and a vertex v of degree 2 and two pendant vertices are adjacent to u.

In this case, there must be a vertex w of degree 2 adjacent to v and a vertex s of degree 2 adjacent to w. A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$ and $\{x_2\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2\}$ to v, assign $\{x_3\}$ to w, assign $\{x_1, x_2, x_4\}$ to S and assign $\{x_2, x_3, x_4\}$ to the pendant vertex adjacent to s.

Case IV (B) : Only one vertex, say u has degree 3 and two vertices v and w of degree 2 each and a pendant vertex are adjacent to u.

In this case, there must be a vertex s of degree 2 adjacent to v. A topoline set-indexer for G with indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1, x_2\}$ to the pendant vertex adjacent to u, assign $\{x_2\}$ to v, assign $\{x_1\}$ to w, assign $\{x_1, x_3\}$ to S, assign $\{x_2, x_4\}$ and $\{x_3\}$ to the pendant vertices adjacent to s and w respectively.

Case IV (C) : Two vertices u and v each of degree 3 are adjacent.

In this case, there must be a vertex w of degree 2 adjacent to v. A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$ and $\{x_2\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2\}$ to v, assign $\{x_3, x_4\}$ to w, assign $\{x_3\}$ and $\{x_1, x_4\}$ to the pendant vertices adjacent to v and w respectively.

Case IV (D) : Two vertices u and v each of degree 3 are not adjacent.

In this case, there must be a vertex w of degree 2 adjacent to both u and v. A topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\emptyset$ to u, assign $\{x_1\}$ and

$\{x_2\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2, x_3\}$ to w, assign $\{x_4\}$ to v, assign $\{x_1, x_2, x_4\}$ and $\{x_1, x_3, x_4\}$ to the pendant vertices adjacent to v.
Case V: Degree of all vertices ≤ 2. ie, G is a path.

Then, a topoline set-indexer for G with the indexing set X is obtained as follows: Assign $\{x_1\}$, $\emptyset$, $\{x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_1, x_2\}$ and $\{x_3, x_4\}$ to the vertices in sequential order.

In all the possible cases, the cardinality of the indexing set is 4 and so $\tau_e(G) \leq 4$. Also since $m(7) = 4$ by Theorem 2.11, $\tau_e(G) \geq 4$. Combining, $\tau_e(G) = 4$. □

5 Topoline numbers of some other graphs

Theorem 5.1. $\tau_e(K_{2^m,2^n}) = m + n + 1$.

Proof. By Theorem 2.3, Theorem 2.4 and Theorem 2.8, we have $\tau_e(G) \geq m+n+1$ where $G = K_{2^m,2^n}$. Let $X = \{x_1, x_2, \ldots x_{m+n+1}\}$, $Y = \{x_1, x_2, \ldots x_m\}$ and $Z = \{x_{m+1}, x_{m+2}, \ldots x_{m+n}\}$. A topoline set-indexer for G for with indexing set X is obtained as follows: Assign all subsets of Y to the vertices of degree 2^n and assign the subsets obtained by taking union of $\{x_{m+n+1}\}$ with all subsets of Z to the vertices of degree 2^m. Thus $\tau_e(G) \leq m + n + 1$. Combining, $\tau_e(G) = m + n + 1$. □

Example 5.2. The Figure 5 illustrates the labeling technique described in Theorem 5.1 for the case $m = n = 2$. In this case, the graph is $G = K_{4,4}$ and the indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$. Note that $\{x_1, x_2\}$ is indicated in the figure as 12.

The edgelabels are the symmetric differences of the labels of end vertices. For the vertex

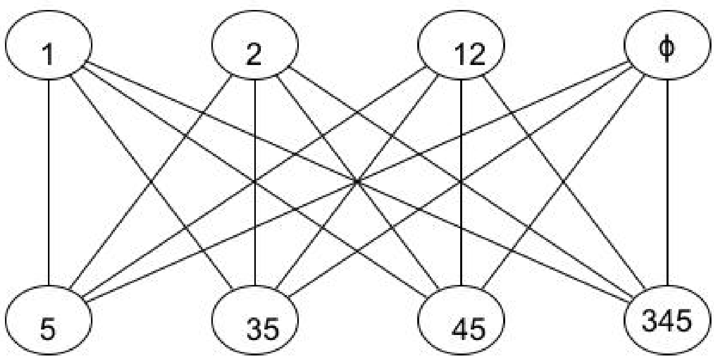

Figure 5. Labeling of $K_{2^2,2^2}$

labeling of the graph shown in Figure 5, the edgelabels are $\{x_1, x_5\}$, $\{x_1, x_3, x_5\}$, $\{x_1, x_4, x_5\}$, $\{x_1, x_3, x_4, x_5\}$, $\{x_2, x_5\}$, $\{x_2, x_3, x_5\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_1, x_2, x_5\}$, $\{x_1, x_2, x_3, x_5\}$, $\{x_1, x_2, x_4, x_5\}$, $\{x_1, x_2, x_3, x_4, x_5\}$,$\{x_5\}$, $\{x_3, x_5\}$, $\{x_4, x_5\}$, and $\{x_3, x_4, x_5\}$. By the labeling $\tau_e(G) \leq 5$. Also by Theorem 2.3, Theorem 2.4 and Theorem 2.8, we have $\tau_e(G) \geq 5$. Combining, we get $\tau_e(G) = 5$.

Since the topologial number and topoline number of star graphs are equal by Theorem 2.8, the results obtained in section 3 gives the following:

Theorem 5.3. (i) $n + 1 \leq \tau_e(K_{1,2^n+2}) \leq n + 2$; $n \geq 1$.

(ii) $\tau_e(K_{1,2^n+2}) = n + 2$; $1 \leq n \leq 5$.

(iii) $\tau_e(K_{1,2^n+3}) = n + 1$; $n \geq 2$.

(iv) $n + 1 \leq \tau_e(K_{1,2^n+4}) \leq n + 2$; $n \geq 2$.

(v) $\tau_e(K_{1,2^n+4}) = n + 2$; $3 \leq n \leq 4$.

(vi) $n + 1 \leq \tau_e(K_{1,2^n+5}) \leq n + 2$; $n \geq 2$.

(vii) $\tau_e(K_{1,2^n+5}) = n + 2$; $1 \leq n \leq 4$.

(viii) $\tau_e(K_{1,3\cdot 2^{(n-2)}}) = n + 1; 3 \leq n \leq 5$.

Theorem 5.4. The 3-sun graph is topoline with topoline number 4.

Proof. A topoline set-indexer for the 3-sun graph G with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained as follows: Let u_1, u_2 and u_3 be the vertices of C_3. Assign $\{x_1\}$, $\{x_3\}$ and $\{x_1, x_2\}$ to u_1, u_2 and u_3 respectively. Assign $\{x_2\}$, $\{x_1, x_3\}$ and $\{x_3, x_4\}$ to the vertices adjacent to u_1, u_2 and u_3 in sequential order. Thus G is topoline and $\tau_e(G) \leq 4$. Since $m(7) = 4$ by Theorem 2.11, $\tau_e(G) \geq 4$. Combining, $\tau_e(G) = 4$. □

Theorem 5.5. The 4-sun graph is topoline with topoline number 4.

Proof. A topoline set-indexer for the 4-sun graph G with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained as follows: Let u_1, u_2, u_3 and u_4 be the vertices of C_4. Label u_1, u_2, u_3, and u_4 respectively by $\{x_1\}$, $\{x_2, x_3\}$, $\{x_1, x_4\}$ and $\{x_1, x_2\}$. Assign $\emptyset$, $\{x_1, x_3\}$, $\{x_3, x_4\}$ and $\{x_4\}$ to the vertices adjacent to u_1, u_2, u_3, and u_4 in sequential order. Thus G is topoline and $\tau_e(G) \leq 4$. Since $m(9) = 4$ by Theorem 2.11, $\tau_e(G) \geq 4$. Combining, $\tau_e(G) = 4$. □

Theorem 5.6. For the double star graph $G = ST(2, 4)$, $\tau(G) = \tau_e(G) = 3$.

Proof. A topological as well as topoline set-indexer for G with indexing set $X = \{x_1, x_2, x_3\}$ is obtained as follows: Assign $\{x_1, x_2, x_3\}$ to the vertex of degree 2 and assign $\{x_1, x_3\}$ and $\{x_1, x_2\}$ to the pendant vertices attached to it. Assign $\{x_1\}$ to the vertex of degree 4 and label the four pendant vertices attached to it using $\emptyset$, $\{x_2\}$, $\{x_3\}$ and $\{x_2, x_3\}$. It follows that $\tau(G) \leq 3$ and $\tau_e(G) \leq 3$. By Theorem 2.3, Theorem 2.4 and Theorem 2.9, we have $\tau(G) \geq 3$ and $\tau_e(G) \geq 3$. Thus $\tau(G) = \tau_e(G) = 3$. □

Theorem 5.7. $\tau_e(ST(2, 3)) = 4$.

Proof. A topoline set-indexer for $G = ST(2, 3)$ with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained as follows: Assign $\emptyset$ to the vertex of degree 2 and assign $\{x_1\}$ and $\{x_2\}$ to the pendant vertices attached to it. Assign $\{x_1, x_2\}$ to the vertex of degree 3 and label the three pendant vertices attached to it using $\{x_2\}$, $\{x_2, x_3\}$ and $\{x_3, x_4\}$. It follows that $\tau_e(G) \leq 4$. Since $m(7) = 4$ by Theorem 2.11, $\tau_e(G) \geq 4$. Thus $\tau_e(G) = 4$. □

Theorem 5.8. $\tau_e(ST(3, 3)) = 4$.

Proof. Let $G = ST(3, 3)$. By Theorem 2.7, $\gamma(G) = 4$ and by Theorem 2.9, $\tau_e(G) \geq 4$. Let u and v be the vertices of degree 3 each. A topoline set-indexer for G with indexing set $X = \{x_1, x_2, x_3, x_4\}$ is obtained as follows: Assign $\emptyset$ to u and assign $\{x_1\}$, $\{x_2\}$ and $\{x_1, x_2, x_3, x_4\}$ to the pendant vertices adjacent to u, assign $\{x_1, x_2\}$ to v and assign $\{x_3\}$, $\{x_4\}$ and $\{x_2, x_3\}$ to the vertices adjacent to v. It follows that $\tau_e(G) \leq 4$. Thus $\tau_e(G) = 4$. □

6 Conclusion Remarks

This paper resolves several conjectures regarding the topological numbers of stars and derives the topological and topoline numbers of certain related graphs. Specifically, the investigation delves into the topological numbers of stars and extends the analysis to explore the topological and topoline numbers of graphs closely connected to star graphs. By addressing these conjectures, this study contributes to a deeper understanding of the structural properties of these graph classes and their implications in various applications.

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