Palestine Journal of Mathematics

Vol 13(4)(2024) , 1005–1008

SOLUTION OF NEGATIVE PELL'S EQUATION USING SELF PRIMES

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11D09; Secondary 11B37,11D72.

Keywords and phrases: Diophantine equation, Integral solution, Pell's equation, Brahma Gupta lemma,Self Prime.

The authors would like to thank the reviewers and editor for their remarkable suggestions to improve the paper in its present form.

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Abstract A Diophantine equation of the form $x^2 - Dy^2 = \pm N$, where D is a positive non-square integer and N is any fixed positive integer, is referred to as Pell's equation. In this article, we search for a non-trivial integer solution to the equation $x^2 - 97y^2 = -53^t, \forall t \in \mathbb{N}$. Here we choose D and N to be the Self primes 97 and 53 respectively and then look for solutions to the equation for various values of (i) $t = 1$, (ii) $t = 3$, (iii) $t = 5$, (iv) $t = 2k$, (v) $t = 2k+5, \forall k \in \mathbb{N}$. Finally the recurrence relations on the solutions are discovered.

1 Introduction

Number theory is a branch of pure mathematics devoted primarily to the study of the integers and integer-valued functions. A Diophantine equation is an equation, typically a polynomial equation in two or more unknowns with integer coefficients, such that the only solutions of interest are the integer ones. The sum of two or more degree one monomials is a constant in a linear Diophantine equation.

The Pell's equation, a kind of Diophantine equation, which takes the form $x^2 - Dy^2 = 1$, where x and y are integers, D is a positive non-square integer. The negative Pell's equation is a specific type of Diophantine equation that has the form $x^2 - Dy^2 = -1$, where D is a positive non-square integer, and x and y are integers [3]. Many researchers have studied about such kind of Pell's equation [3, 4, 5, 6, 7, 8, 12] and still work is undergoing in this type of equations.

In this paper, we search for a non-trivial integer solution to the equation of the form $x^2 - Dy^2 = -N$, where D and N to be the Self primes 97 and 53 respectively. That is we have the Pell's equation $x^2 - 97y^2 = -53^t, \forall t \in \mathbb{N}$ and then look for solutions to the equation for various values of (i) $t = 1$, (ii) $t = 3$, (iii) $t = 5$, (iv) $t = 2k$, (v) $t = 2k + 5$, $\forall k \in \mathbb{N}$.

Self prime number is a self number that is a prime. A Self number in a given number base b is a natural number that cannot be written as the sum of any other natural number n and the individual digits of n. The first few self primes in base 10 are $3, 5, 7, 31, 53, 97, 211, 233, 277, 367, 389, 457, 479, 547, 569, 613, 659, 727, ...$

2 Preliminaries

Theorem 2.1. *[2] If (x_1, y_1) is the fundamental solution of $x^2 - Dy^2 = 1$, then every positive solutions of the equation is given by (x_n, y_n) where x_n and y_n are the integers determined from*

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n.,n = 1, 2, 3, ...$$

Theorem 2.2. *[5] Let p be a prime. The negative Pell's equation $x^2 + py^2 = -1$ is solvable if and only if $p = 2$ or $p \equiv 1 \pmod 4$.*

Testing the solubility of the negative Pell's equation:

Suppose D is a positive integer, not a perfect square. Then the negative Pell equation $x^2 - Dy^2 = -1$ is soluble if and only if D is expressible as $D = a^2 + b^2$; $gcd(a; b) = 1$: a and b are positive integers, b odd and the Diophantine equation $-bV^2 + 2aVW + bW^2 = 1$ has a solution(The case of solubility occurs for exactly one such $(a; b)$).
The Algorithm

(i) Find all expressions of D as a sum of two relatively-prime squaresusing Cornacchia's method. If none, exist - the negative Pell equation is not solvable.

(ii) For each representation , $D = a^2 + b^2; gcd(a; b) = 1$: a and b is positive, b odd , test the solubility of $-bV^2 + 2aVW + bW^2 = 1$ using the Lagrange-Matthews algorithm. If solution exist then the negative Pell equation is solvable.

(iii) If each representation yields no solution, then the negative Pell equation is insolvable.

This paper deals with a negative Pell's equation

$$x^2 - 97y^2 = -53^t, \forall t \in \mathbb{N}$$

For this particular equation, we consider the prime $p = 97$, which satisfies the conditions of Theorem 2.2. Therefore, we can substantiate the proof that the negative Pell's equation $x^2 - 97y^2 = -53^t, \forall t \in \mathbb{N}$ is solvable in integers.
Using the Algorithm as in 2.2 and testing $(a, b) = (4, 9)$ and $-bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (20, 13)$, so $x^2 - 97y^2 = -1$ is solvable.

3 Main Results

Choice 1 : $t = 1$ Consider the equation

$$x^2 = 97y^2 - 53 \tag{3.1}$$

Let (x_0, y_0) be the initial solution of (3.1) given by $x_0 = 7298, y_0 = 741$.
To find the other solutions of (3.1) consider the Pell equation,

$$x^2 = 97y^2 + 1$$

whose intial solution $(\tilde{x}_n, \tilde{y}_n)$ is given by $\tilde{x}_n = \frac{1}{2} f_n$, $\tilde{y}_n = \frac{1}{2\sqrt{97}} g_n$ where

$$f_n = [(62809633 + 6377352\sqrt{97})^{n+1} + (62809633 - 6377352\sqrt{97})^{n+1}]$$
$$g_n = [(62809633 + 6377352\sqrt{97})^{n+1} - (62809633 - 6377352\sqrt{97})^{n+1}]$$

Applying Brahmaguta Lemma between (x_0, y_0) and (x_n, y_n), the sequence of non zero distinct integer solutions to (3.1) are obtained as

$$x_{n+1} = \frac{1}{2}[7298 f_n + 741\sqrt{97} g_n] ,$$
$$y_{n+1} = \frac{1}{2\sqrt{97}}[741\sqrt{97} f_n + 7298 g_n] .$$

The recurrence relations satisfied by the solutions of (3.1) are given by

$$x_{n+2} - 125619266 x_{n+1} + x_n = 0 ,$$
$$y_{n+2} - 125619266 y_{n+1} + y_n = 0 .$$

Choice 2 : $t = 3$ Consider the equation

$$x^2 = 97y^2 - 53^3 \tag{3.2}$$

Let (x_0, y_0) be the initial solution of (3.2) given by $x_0 = 12374, y_0 = 1257$.
Applying Brahmaguta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non zero distinct integer solutions to (3.2) are obtained as

$$x_{n+1} = \frac{1}{2}[12374f_n + 1257\sqrt{97}g_n],$$
$$y_{n+1} = \frac{1}{2\sqrt{97}}[1257\sqrt{97}f_n + 12374g_n].$$

The recurrence relations satisfied by the solutions of (3.2) are given by

$$x_{n+2} - 125619266x_{n+1} + x_n = 0,$$
$$y_{n+2} - 125619266y_{n+1} + y_n = 0.$$

Choice 3 : $t = 5$ Consider the equation

$$x^2 = 97y^2 - 53^5 \tag{3.3}$$

Let (x_0, y_0) be the initial solution of (3.3) given by $x_0 = 16010, y_0 = 2637$.
Applying Brahmaguta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non zero distinct integer solutions to (3.3) are obtained as

$$x_{n+1} = \frac{1}{2}[16010f_n + 2637\sqrt{97}g_n],$$
$$y_{n+1} = \frac{1}{2\sqrt{97}}[2637\sqrt{97}f_n + 16010g_n].$$

The recurrence relations satisfied by the solutions of (3.3) are given by

$$x_{n+2} - 125619266x_{n+1} + x_n = 0,$$
$$y_{n+2} - 125619266y_{n+1} + y_n = 0.$$

Choice 4 : $t = 2k, k > 0$ Consider the equation

$$x^2 = 97y^2 - 53^{2k}, k > 0 \tag{3.4}$$

Let (x_0, y_0) be the initial solution of (3.4) given by $x_0 = 5604(53)^k, y_0 = 569(53)^k$.
Applying Brahmaguta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non zero distinct integer solutions to (3.4) are obtained as

$$x_{n+1} = \frac{53^k}{2}[5604f_n + 569\sqrt{97}g_n],$$
$$y_{n+1} = \frac{53^k}{2\sqrt{97}}[569\sqrt{97}f_n + 5604g_n].$$

The recurrence relations satisfied by the solutions of (3.4) are given by

$$x_{n+2} - 125619266x_{n+1} + x_n = 0,$$
$$y_{n+2} - 125619266y_{n+1} + y_n = 0.$$

Choice 5 : $t = 2k + 5, k > 0$ Consider the equation

$$x^2 = 97y^2 - 53^{2k+5}, k > 0 \tag{3.5}$$

Let (x_0, y_0) be the initial solution of (3.5) given by $x_0 = 53^{k-1}[8213986], y_0 = 53^{k-1}[841233]$.
Applying Brahmaguta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non zero distinct integer solutions to (3.4) are obtained as

$$x_{n+1} = \frac{53^{k-1}}{2}[8213986f_n + 841233\sqrt{97}g_n],$$
$$y_{n+1} = \frac{53^{k-1}}{2\sqrt{97}}[841233\sqrt{97}f_n + 8213986g_n].$$

The recurrence relations satisfied by the solutions of (3.4) are given by

$$x_{n+2} - 125619266x_{n+1} + x_n = 0,$$
$$y_{n+2} - 125619266y_{n+1} + y_n = 0.$$

4 Conclusion

As seen from the study presented above, solving a negative Pell's equation involving the Self primes has led to the development of a more fundamental and dynamic theory for solving equations of a similar sort.

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Received: 2023-09-25
Accepted: 2024-01-22